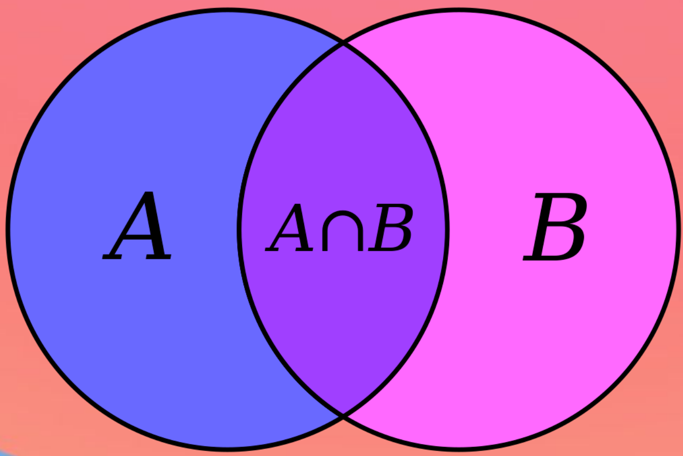
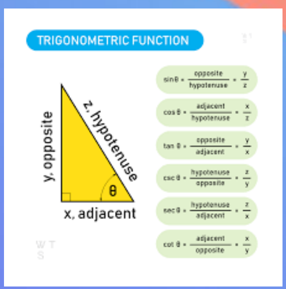
TEXT BOOK FOR

SR. SECONDARY COURSE

MATHEMATICS





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CONTENT-A

NO. **CHAPTER** **PAGE NO.**

MODULE-I
Sets, Relations and Function

MODULE-II
Sequences and Series

MODULE-III
Algebra-I

CONTENT-B

NO. **CHAPTER** **PAGE NO.**

CONTENT -C

1

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SETS

Let us consider the following situation : One day Mrs. and Mr. Mehta went to the market. Mr. Mehta purchased the following objects/items. "a toy, one kg sweets and a magazine". Where as Mrs. Mehta purchased the following objects/items. "Lady fingers, Potatoes and Tomatoes".

In both the examples, objects in each collection are well defined. What can you say about the collection of students who speak the truth ? Is it well defined? Perhaps not. A set is a collection of well defined objects. For a collection to be a set it is necessary that it should be well defined.

The word well defined was used by the German Mathematician George Cantor (1845- 1918 A.D) to define a set. He is known as father of set theory. Now-a-days set theory has become basic to most of the concepts in Mathematics. In this lesson we shall discuss some basic definitions and operations involving sets.

OBJECTIVES

After studying this lesson, you will be able to :

- define a set and represent the same in different forms;
- define different types of sets such as, finite and infinite sets, empty set, singleton set, equivalent sets, equal sets, sub sets and cite examples thereof;
- define and cite examples of universal set, complement of a set and difference between two sets;
- define union and intersection of two sets;
- represent union and intersection of two sets, universal set, complement of a set, difference between two sets by Venn Diagram;

EXPECTED BACKGROUND KNOWLEDGE

- Number systems,

1.1 SOME STANDARD NOTATIONS

Before defining different terms of this lesson let us consider the following examples:

(i) collection of tall students in your school.	(i) collection of those students of your school whose height is more than 180 cm.
(ii) collection of honest persons in your colony.	(ii) collection of those people in your colony who have never been found involved in any theft case.
(iii) collection of interesting books in your school library.	(iii) collection of Mathematics books in your school library.
(iv) collection of intelligent students in your school.	(iv) collection of those students in your school who have secured more than 80% marks in annual examination.

In all collections written on left hand side of the vertical line the term tallness, interesting, honesty, intelligence are not well defined. In fact these notions vary from individual to individual. Hence these collections can not be considered as sets.

While in all collections written on right hand side of the vertical line, 'height' 'more than 180 cm.' 'mathematics books' 'never been found involved in theft case,' ' marks more than 80%' are well defined properties. Therefore, these collections can be considered as sets.

If a collection is a set then each object of this collection is said to be an element of this set. A set is usually denoted by capital letters of English alphabet and its elements are denoted by small letters.

For example, A = {toy elephant, packet of sweets, magazines.}

Some standard notations to represent sets :

N : the set of natural numbers

W : the set of whole numbers

Z : the set of integers

Z^+: the set of positve integers

Z^-: the set of negative integers

Q : the set of rational numbers

I : the set of irrational numbers

R : the set of real numbers

C : the set of complex numbers

Other frequently used symbols are :

$\in$: 'belongs to'

$\notin$: 'does not belong to'

$\exists$: There exists, $\nexists$: There does not exist.

For example N is the set of natural numbers and we know that 2 is a natural number but -2 is not a natural number. It can be written in the symbolic form as $2 \in N$ and $-2 \notin N$.

1. 2 REPRESENTATION OF A SET

There are two methods to represent a set.

1.2.1 (i) Roster method (Tabular form)

In this method a set is represented by listing all its elements, separating them by commas and enclosing them in curly bracket.

If V be the set of vowels of English alphabet, it can be written in Roster form as :

$$V = \{ a, e, i, o, u\}$$

(ii) If A be the set of natural numbers less than 7. then

A=$\{1, 2, 3, 4, 5, 6\}$, is in the Roster form.

Note : To write a set in Roster form elements are not to be repeated i.e. all elements are taken as distinct. For example if A be the set of letters used in the word mathematics, then

$$A = \{m, a, t, h, e, i, c, s\}$$

1.2.2 Set-builder form

In this form elements of the set are not listed but these are represented by some common property.

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

$$V = \{x : x \text{ is a vowel of English alphabet}\}$$

(ii) Let A be the set of natural numbers less than 7. then $A = \{x : x \in N \text{ and } x < 7\}$

Note : Symbol ':' read as 'such that'

Example: 1.1 Write the following in set -builder form :

(a) $A=\{-3,-2,-1,0,1,2,3\}$ (b) $B = \{3,6,9,12\}$

Solution : (a) $A = \{x : x \in Z \text{ and } -3 \le x \le 3\}$

(b) $B = \{x : x = 3n \text{ and } n \in N, n \le 4\}$

Example: 1.2 Write the following in Roster form.

(a) $C = \{x : x \in N \text{ and } 50 \le x \le 60\}$

(b) $D = \left\{x : x \in R \text{ and } x^2 - 5x + 6 = 0\right\}$

Solution : (a) $C = \{50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60\}$

(b) $x^2 - 5x + 6 = 0$

$$\Rightarrow \quad (x-3)(x-2)=0 \quad \Rightarrow \quad x=3,2.$$

$$\therefore \quad D=\{2,3\}$$

1.3 CLASSIFICATION OF SETS

1.3.1 Finite and infinite sets

Let A and B be two sets where

A = {x : x is a natural number}

B = {x : x is a student of your school}

As it is clear that the number of elements in set A is not finite while number of elements in set B is finite. A is said to be an infinite set and B is said to be a finite set.

A set is said to be finite if its elements can be counted and it is said to be infinite if it is not possible to count upto its last element.

1.3.2 Empty (Null) Set : Consider the following sets.

$$A=\{x: x \in R \text{ and } x^2+1=0\}$$

B = {x : x is number which is greater than 7 and less than 5}

Set A consists of real numbers but there is no real number whose square is -1. Therefore this set consists of no element. Similiarly there is no such number which is less than 5 and greater than 7. Such a set is said to be a null (empty) set. It is denoted by the symbol ϕ or { }

A set which has no element is said to be a null/empty/void set, and is denoted by ϕ. or {}

1.3.3 Singleton Set : Consider the following set :

A = {x : x is an even prime number}

As there is only one even prime number namely 2, so set A will have only one element. Such a set is said to be singleton. Here A = $\{2\}$.

A set which has only one element is known as singleton.

1.3.4 Equal and equivalent sets : Consider the following examples.

(i) $A=\{1,2,3\}$, $B=\{2,1,3\}$ (ii) $D=\{1,2,3\}$, $E=\{a,b,c\}$.

In example (i) Sets A and B have the same elements. Such sets are said to be equal sets and it is written as A = B. In example (ii) sets D and E have the same number of elements but elements are different. Such sets are said to be equivalent sets and are written as A $\approx$ B.

Two sets A and B are said to be equivalent sets if they have same number of elements but they are said to be equal if they have not only the same number of elements but elements are also the same.

1.3.5 Disjoint Sets : Two sets are said to be disjoint if they do not have any common element. For example, sets A= { 1,3,5} and B = { 2,4,6 } are disjoint sets.

Example 1.3 Given that $A = \{2, 4\}$ and $B = \{x : x \text{ is a solution of } x^2 + 6x + 8 = 0\}$

Are A and B disjoint sets ?

Solution : If we solve $x^2 + 6x + 8 = 0$, we get

$x = -4, -2$. $\therefore$ $B = \{-4, -2\}$ and $A = \{2, 4\}$

Clearly ,A and B are disjoint sets as they do not have any common element.

Example 1.4 If $A = \{x : x \text{ is a vowel of English alphabet}\}$

and $B = \{y : y \in N \text{ and } y \le 5\}$ Is (i) $A = B$ (ii) $A \approx B$?

Solution : $A = \{a, e, i, o, u\}$, $b = \{1, 2, 3, 4, 5\}$.

Each set is having five elements but elements are different

$\therefore$ $A \ne B$ but $A \approx B$.

Example 1.5 Which of the following sets

$A = \{x : x \text{ is a point on a line}\}$, $B = \{y : y \in N \text{ and } y \le 50\}$ are finite or infinite ?

Solution : As the number of points on a line is uncountable (cannot be counted) so A is an infinite set while the number of natural numbers upto fifty can be counted so B is a finite set.

Example 1.6 Which of the following sets

$A = \{x : x \text{ is irrational and } x^2 - 1 = 0\}$.

$B = \{x : x \in z \text{ and } -2 \le x \le 2\}$ are empty?

Solution : Set A consists of those irrational numbers which satisfy $x^2 - 1 = 0$. If we solve $x^2 - 1 = 0$ we get $x = \pm 1$. Clearly ± 1 are not irrational numbers. Therefore A is an empty set.

But $B = \{-2, -1, 0, 1, 2\}$. B is not an empty set as it has five elements.

Example 1.7 Which of the following sets are singleton ?

$A = \{x : x \in Z \text{ and } x - 2 = 0\}$ $\quad B = \{y : y \in R \text{ and } y^2 - 2 = 0\}$.

Solution : Set A contains those integers which are the solution of $x - 2 = 0$ or $x = 2$. $\therefore$ $A = \{2\}$.

$\Rightarrow$ A is a singleton set.

B is a set of those real numbers which are solutions of $y^2 - 2 = 0$ or $y = \pm\sqrt{2}$

$\therefore$ $B = \{-\sqrt{2}, \sqrt{2}\}$ Thus, B is not a singleton set.

CHECK YOUR PROGRESS 1.1

1. Which of the following collections are sets ?
   (i) The collection of days in a week starting with S.
   (ii) The collection of natural numbers upto fifty.
   (iii) The collection of poems written by Tulsidas.
   (iv) The collection of fat students of your school.

2. Insert the appropriate symbol in blank spaces. If A = $\{1, 2, 3\}$.
   (i) 1...........A (ii) 4........A.

3. Write each of the following sets in the Roster form :
   (i) $A = \{x : x \in z \text{ and } -5 \le x \le 0\}$.
   (ii) $B = \{x : x \in R \text{ and } x^2 - 1 = 0\}$.
   (iii) C = {x : x is a letter of the word banana}.
   (iv) D = {x : x is a prime number and exact divisor of 60}.

4. Write each of the following sets in the set builder form ?
   (i) A = {2, 4, 6, 8, 10} (ii) $B = \{3, 6, 9, ... \infty\}$
   (iii) C = {2, 3, 5, 7} (iv) $D = \{-\sqrt{2}, \sqrt{2}\}$

   Are A and B disjoints sets ?

5. Which of the following sets are finite and which are infinite ?
   (i) Set of lines which are parallel to a given line.
   (ii) Set of animals on the earth.
   (iii) Set of Natural numbers less than or equal to fifty.
   (iv) Set of points on a circle.

6. Which of the following are null set or singleton ?
   (i) $A = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 + 2 = 0\}$.
   (ii) $B = \{x : x \in Z \text{ and } x \text{ is a solution of } x - 3 = 0\}$.
   (iii) $C = \{x : x \in Z \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
   (iv) D = {x : x is a student of your school studying in both the classes XI and XII }

7. In the following check whether A = B or $A \approx B$.
   (i) A = {a}, B = {x : x is an even prime number}.
   (ii) A = {1, 2, 3, 4}, B = {x : x is a letter of the word guava}.
   (iii) $A = \{x : x \text{ is a solution of } x^2 - 5x + 6 = 0\}, B = \{2, 3\}$.

1.4 SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as $B \subseteq A$

Consider $D = \{1, 2, 3, 4, ...\}$, $E = \{..... -3 -2, -1, 0, 1, 2, 3, ...\}$

Clearly each element of set D is an element of set E also $\therefore$ $D \subseteq E$

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. $A \subseteq A$.

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iv) If $A \subseteq B$ and $A \neq B$ then A is said to be a proper subset of B and B is said to be a super set of A. i.e. $A \subset B$ or $B \supset A$.

Example 1.8 If $A = \{x : x$ is a prime number less than $5\}$ and

$B = \{y : y$ is an even prime number$\}$, then is B a proper subset of A ?

Solution : It is given that

$A = \{2, 3\}$, $B = \{2\}$.

Clearly $B \subseteq A$ and $B \neq A$

We write $B \subset A$ and say that B is a proper subset of A.

Example 1.9 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$. is $A \subseteq B$ or $B \subseteq A$?

Solution : Here $1 \in A$ but $1 \notin B \Rightarrow A \not\subseteq B$. Also $5 \in B$ but $5 \notin A \Rightarrow B \not\subseteq A$.

Hence neither A is a subset of B nor B is a subset of A.

Example 1.10 If $A = \{a, e, i, o, u\}$, $B = \{e, i, o, u, a\}$

Is $A \subseteq B$ or $B \subseteq A$ or both ?

Solution : Here in the given sets each element of set A is an element of set B also

$\therefore$ $A \subseteq B$ .... (i)

and each element of set B is an element of set A also. $\therefore$ $B \subseteq A$ ......(ii)

From (i) and (ii) $A = B$

1.4.1 Number of Subsets of a Set :

Let A = {x}, then the subsets of A are ϕ, A.

Note that $n(A) = 1$, number of subsets of A = 2 = 2^1

Let A = {2, 4}, then the subsets of A are ϕ, {4}, {2}, {2, 4}.

Note that $n(A) = 2$, number of subsets of A = 4 = 2^2

Let A = {1, 3, 5}, then subsets of A are ϕ, {1}, {3}, {5}, {1, 3}, {1, 5}, {3, 5}, {1, 3, 5}.

Note that $n(A) = 3$, number of subsets of A = 8 = 2^3

> If A is a set with $n(A) = p$, then the number of subsets of A = 2^p and number of proper subsets of A = $2^p - 1$.

Subsets of real Numbers :

We know some standard sets of numbers as-

The set of natural numbers N = {1, 2, 3, 4, ...}

The set of whole numbers W = {0, 1, 2, 3, 4, ...}

The set of Integers Z = {........., –4, –3, –2, –1, 0, 1, 2, 3, 4, ...}

The set of Rational numbers $Q = \left\{ x : x = \frac{p}{q}, p, q \in Z \text{ and } q \neq 0 \right\}$

The set of irrational numbers denoted by I.

$I = \{ x : x \in R \text{ and } x \notin Q \}$ i.e. all real numbers that are not rational

These sets are subsets of the set of real numbers. Some of the obvious relations among these subsets are

$N \subset W \subset Z \subset Q,\ Q \subset R,\ I \subset R,\ N \not\subset I$

1.4.2 INTERVALS AS SUBSETS OF REAL NUMBERS

An interval I is a subset of R such that if $x, y \in$ I and z is any real numbers between x and y then $z \in$ I.

> Any real number lying between two different elements of an interval must be contained in the interval.

If $a, b \in R$ and $a < b$, then we have the following types of intervals :

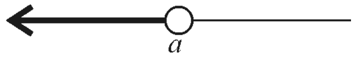
(i) The set $\{x \in R : a < x < b\}$ is called an <u>open interval</u> and is denoted by (a, b). On the number line it is shown as :

(ii) The set $\{x \in R : a \leq x \leq b\}$ is called a <u>closed interval</u> and is denoted by $[a, b]$. On the number line it is shown as :

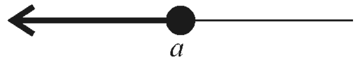
(iii) The set $\{x \in R : a < x \le b\}$ is an interval, open on left and closed on right. It is denoted by $(a, b]$. On the number line it is shown as :

(iv) The set $\{x \in R : a \le x < b\}$ is an interval, closed on left and open on right. It is denoted by $[a, b)$. On the number line it is shown as :

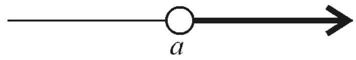
(v) The set $\{x \in R : x < a\}$ is an interval, which is dentoed by $(-\infty, a)$. It is open on both sides. On the number line it is shown as :



(vi) The set $\{x \in R : x \le a\}$ is an interval which is denoted by $(-\infty, a]$. It is closed on the right. On the number line it is shown as :



(vii) The set $\{x \in R : x > a\}$ is an interval which is denoted by (a, ∞). It is open on the both sides. On the number line it is shown as :



(viii) The set $\{x \in R : x \ge a\}$ is an interval which is denoted by $[a, \infty)$. It is closed on left. On the number line it is shown as :

First four intervals are called finite intervals and the number $b - a$ (which is always positive) is called the length of each of these four intervals (a, b), $[a, b]$, $(a, b]$ and $[a, b)$.

The last four intervals are called infinite intervals and length of these intervals is not defined.

1.5 POWER SET

Let A = {a, b} then, Subset of A are ϕ, {a}, {b} and {a, b}.

If we consider these subsets as elements of a new set B (say) then, B = $\{\phi, \{a\}, \{b\}, \{a, b\}\}$

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).
and it is the set of all subsets of the given set.

Example 1.11 Write the power set of each of the following sets :

(i) $A = \{x : x \in R \text{ and } x^2 + 7 = 0\}$.

(ii) $B = \{y : y \in N \text{ and } 1 \le y \le 3\}$.

Solution :

(i) Clearly A = ϕ (Null set), $\therefore$ ϕ is the only subset of given set, $\therefore$ P (A)={ ϕ }

(ii) The set B can be written as {1, 2, 3}

Subsets of B are ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}.

$\therefore$ P (B) = { ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }.

Example 1.12 Write each of the following sets as intervals :

(i) $\{x \in R : -1 < x \le 2\}$ (ii) $\{x \in R : 1 \ge 2x - 3 \ge 0\}$

Solution : (i) The given set = $\{x \in R : -1 < x \le 2\}$

Hence, Interval of the given set = (–1, 2]

(ii) The given set = $\{x \in R : 1 \ge 2x - 3 \ge 0\}$

$\Rightarrow \{x \in R : 4 \ge 2x \ge 3\}$, $\Rightarrow \left\{x \in R : 2 \ge x \ge \frac{3}{2}\right\}$

$\Rightarrow \left\{x \in R : \frac{3}{2} \le x \le 2\right\}$, Hence, Interval of the given set = $\left[\frac{3}{2}, 2\right]$

1.6 UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}

B = {y : y is a male student of your school}

C = {z : z is a female student of your school}

D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A. A can be considered as the universal set for this particular example. Universal set is generally denoted by U. In a particular problem a set U is said to be a universal set if all the sets in that problem are subsets of U.

Remarks

(i) Universal set does not mean a set containing all objects of the universe.

(ii) A set which is a universal set for one problem may not be a universal set for another problem.

Example 1.13 Which of the following sets can be considered as a universal set ?

X = {x : x is a real number}

Y = {y : y is a negative integer}

Z = {z : z is a natural number}

Solution : As it is clear that both sets **Y** and **Z** are subset of **X**.

$\therefore$ **X** is the universal set for this problem.

1.7 VENN DIAGRAM

British mathematician John Venn (1834 – 1883 AD) introduced the concept of diagrams to represent sets. According to him universal set is represented by the interior of a rectangle and other sets are represented by interior of circles.

For example if U= {1, 2, 3, 4, 5}, A = {2, 4} and B = {1,3}, then these sets can be represented as

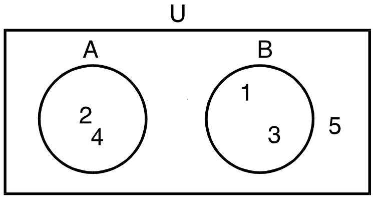


Fig. 1.1

Diagramatical representation of sets is known as a Venn diagram.

1.8 DIFFERENCE OF SETS

Consider the sets

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{2, 4, 6\}.$$

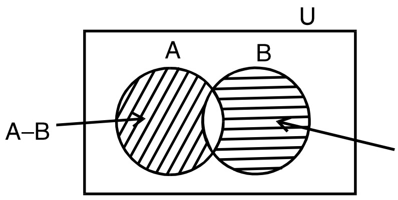
A new set having those elements which are in A but not in B is said to be the difference of sets A and B and it is denoted by $A - B$. $\therefore$ $A - B = \{1, 3, 5\}$

Similiarly a set of those elements which are in B but not in A is said to be the difference of B and A and it is devoted by $B - A$. $\therefore$ $B - A = \{6\}$

In general, if A and B are two sets then

$$A - B = \{ x : x \in A \text{ and } x \notin B \} \text{ and } B - A = \{ x : x \in B \text{ and } x \notin A \}$$

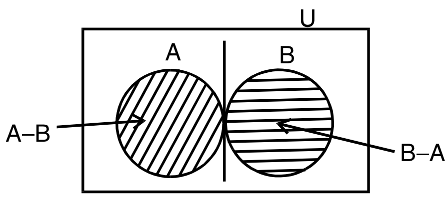
Difference of two sets can be represented using Venn diagram as :



When A and B are not **disjoint** sets

Fig. 1.2

or



When A and B are **disjoint** sets

Fig. 1.3

1.9. COMPLEMENT OF A SET

Let **X** denote the universal set and **Y, Z** its sub sets where

X = {x : x is any member of a family}

Y = {x : x is a male member of the family}

Z = {x : x is a female member of the family}

X – Y is a set having female members of the family.

X – Z is a set having male members of the family.

X – Y is said to be the complement of Y and is usally denoted by Y' or Y^c.

X – Z is said to be complement of Z and denoted by Z' or Z^c.

If U is the universal set and A is its subset then the complement of A is a set of those elements which are in U but not in A. It is denoted by A' or A^c.

$A' = U - A = \{x : x \in U \text{ and } x \notin A\}$

The complement of a set can be represented using Venn diagram as :

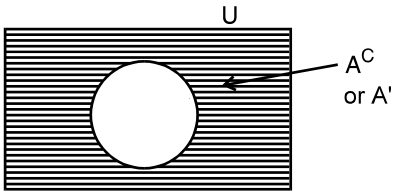


Fig. 1.4

Remarks

(i) Difference of two sets can be found even if none is a subset of the other but complement of a set can be found only when the set is a subset of some universal set.

(ii) $\phi^c = U$. (iii) $U^c = \phi$.

Example 1.14 Given that

$A = \{x : x \text{ is a even natural number less than or equal to } 10\}$

and $B = \{x : x \text{ is an odd natural number less than or equal to } 10\}$

Find (i) A – B (ii) B – A (iii) is A – B = B – A ?

Solution : It is given that

$A = \{2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9\}$

Therefore, (i) $A - B = \{2, 4, 6, 8, 10\}$, (ii) $B - A = \{1, 3, 5, 7, 9\}$

(iii) Clearly from (i) and (ii) $A - B \neq B - A$.

Example 1.15 Let U be the universal set and A its subset where

$U = \{x : x \in N \text{ and } x \leq 10\}$

$A = \{y : y \text{ is a prime number less than } 10\}$

Find (i) A^c (ii) Represent A^c in Venn diagram.

Solution : It is given

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. and $A = \{2, 3, 5, 7\}$

(i) $A^c = U - A = \{1, 4, 6, 8, 9, 10\}$

(ii)

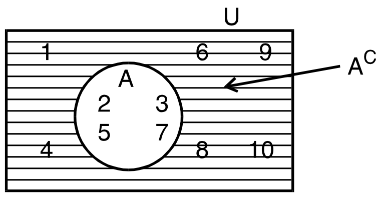


Fig. 1.5

1.9.1 Properties of complement of sets

1. Complement Law's
   (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$
2. De Morgan's Law
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
3. Law of double complementation $(A')' = A$
4. Law of empty set and universal set i.e $\phi' = U$ and $U' = \phi$

1. Verification of Complement Law

Let $U = \{1, 2, 3, ... 10\}$ and $A = \{2, 4, 6, 8, 10\}$

Then $A' = \{1, 3, 5, 7, 9\}$

Now, $A \cup A' = \{1, 2, 3, 4, ..., 10\} = U$ and $A \cap A' = \phi$

Hence, $A \cup A' = U$ and $A \cap A' = \phi$

2. Verification of De Morgan's Law

Let $U = \{1, 2, 3, ..., 9\}$ and $A = \{2, 4, 6, 8\}$, $B = \{2, 3, 5, 7\}$

Hence, $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

and $(A \cup B)' = U - (A \cup B) = \{1, 9\}$ ...(1)

Now $A' = U - A = \{1, 3, 5, 7, 9\}$ and $B' = U - B = \{1, 4, 6, 8, 9\}$

$\therefore$ $A' \cap B' = \{1, 9\}$ ...(2)

From (1) and (2), $(A \cup B)' = A' \cap B'$

Also $A \cap B = \{2\}$

$\therefore$ $(A \cap B)' = U - (A \cap B) = \{1, 3, 4, 5, 6, 7, 8, 9\}$ ...(3)

and $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9\}$ ...(4)

From (3) and (4), we get $(A \cap B)' = A' \cup B'$

Verification of $(A')' = A$

Let $U = \{1, 2, 3, ..., 10\}$ and $A = \{1, 2, 3, 5, 7, 9\}$

Then $A' = U - A = \{4, 6, 8, 10\}$ $(A')' = U - A' = \{1, 2, 3, 5, 7, 9\} = A$

MODULE - I
Sets, Relations and Functions

Notes

Thus from the definition of the complement of a subset A of the universal set U it follows that $(A')' = A$

CHECK YOUR PROGRESS 1.2

1. Insert the appropriate symbol in the blank spaces, given that $A=\{1, 3, 5, 7, 9\}$

   (i) ϕ ....A (ii) $\{2, 3, 9\}$..........A

   (iii) 3...........A (iv) 10..................A

2. Given that $A = \{a, b\}$, how many elements P(A) has ?

3. Let $A = \{ \phi, \{1\}, \{2\}, \{1,2\}\}$. Which of the following is true or false ?

   (i) $\{1,2\} \subset A$ (ii) $\phi \in A$.

4. Which of the following statements are true or false ?

   (i) Set of all boys, is contained in the set of all students of your school.

   (ii) Set of all boy students of your school, is contained in the set of all students of your school.

   (iii) Set of all rectangles, is contained in the set of all quadrilaterals.

   (iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.

5. If $A = \{ 1, 2, 3, 4, 5 \}$, $B= \{5, 6, 7\}$ find (i) $A - B$ (ii) $B - A$.

6. Let N be the universal set and A, B, C, D be its subsets given by

   $A = \{x : x$ is a even natural number$\}$, $B = \{x : x \in N$ and x is a multiple of 3$\}$

   $C = \{x : x \in N$ and $x \geq 5\}$, $D = \{x : x \in N$ and $x \leq 10\}$

   Find complements of A, B, C and D respectively.

7. Write the following sets in the interval form.

   (a) $\{x \in R: -8 < x < 3\}$ (b) $\{x \in R: 3 \leq 2x \angle 7\}$

8. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then verify the following

   (i) $(A')' = A$ (ii) $(B')' = B$

   (iii) $A \cap A' = \phi$ (iv) $(A \cup B)' = A' \cap B'$

1.10. INTERSECTION OF SETS

Consider the sets

$A = \{1, 2, 3, 4\}$ and $B = \{ 2, 4, 6\}$

It is clear, that there are some elements which are common to both the sets A and B. Set of these common elements is said to be interesection of A and B and is denoted by $A \cap B$.

Here $A \cap B = \{2, 4\}$

If A and B are two sets then the set of those elements which belong to both the sets is said to be the intersection of A and B. It is denoted by $A \cap B$. $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ can be represented using Venn diagram as :

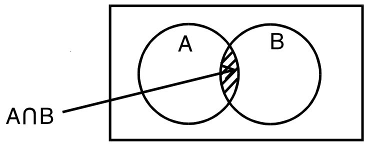


Fig. 1.6

Remarks

If $A \cap B = \phi$ then A and B are said to be **disjoint sets**. In Venn diagram disjoint sets can be represented as

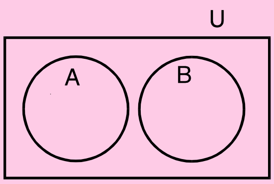


Fig.1.7

Example 1.16 Given that

$A = \{x : x$ is a king out of 52 playing cards$\}$

and $B = \{ y : y$ is a spade out of 52 playing cards$\}$

Find (i) $A \cap B$ (ii) Represent $A \cap B$ using Venn diagram .

Solution : (i) As there are only four kings out of 52 playing cards, therefore the set A has only four elements. The set B has 13 elements as there are 13 spade cards but out of these 13 spade cards there is one king also. Therefore there is one common element in A and B.

$\therefore A \cap B = \{$ King of spade$\}$.

(ii)

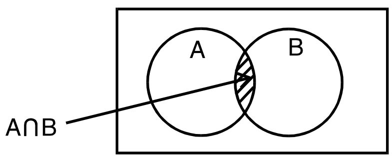


Fig. 1.8

MODULE - I
Sets, Relations and Functions

Notes

1.11 UNION OF SETS

Consider the following examples :

(i) A is a set having all players of Indian men cricket team and B is a set having all players of Indian women cricket team. Clearly A and B are disjoint sets. Union of these two sets is a set having all players of both teams and it is denoted by $A \cup B$.

(ii) D is a set having all players of cricket team and E is the set having all players of Hockey team, of your school. Suppose three players are common to both the teams then union of D and E is a set of all players of both the teams but three common players to be written once only.

If A and B are any two sets then union of A and B is the set of those elements which belong to A or B.

In set builder form : $A \cup B = \{x : x \in A \text{ or } x \in B\}$

OR

$$A \cup B = \{x : x \in A - B \text{ or } x \in B - A \text{ or } x \in A \cap B\}$$

$A \cup B$ can be represented using Venn diagram as :

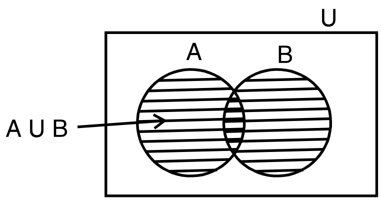


Fig. 1.9

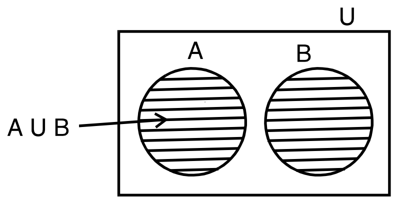


Fig. 1.10

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B).$$

or $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where n $A \cup B$ stands for number of elements in $A \cup B$.

Example 1.17 $A = \{x : x \in Z^+ \text{ and } x \le 5\}$, B = {y : y is a prime number less than 10}

Find (1) $A \cup B$ (ii) represent $A \cup B$ using Venn diagram.

Solution : We have,

A={1, 2, 3, 4, 5} B = {2, 3, 5, 7}. $\therefore$ $A \cup B$ = {1, 2, 3, 4, 5, 7}.

(ii)

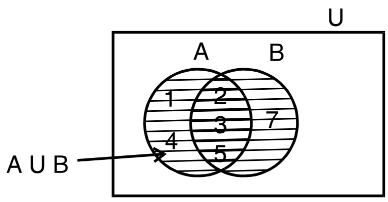


Fig.1.11

CHECK YOUR PROGRESS 1.3

1. Which of the following pairs of sets are disjoint and which are not ?

   (i) {x : x is an even natural number}, {y : y is an odd natural number}

   (ii) {x : x is a prime number and divisor of 12}, $\{ y : y \in N$ and $3 \le y \le 5\}$

   (iii) {x :x is a king of 52 playing cards}, { y : y is a diamond of 52 playing cards}

   (iv) {1, 2, 3, 4, 5}, {a, e, i, o, u}

2. Find the intersection of A and B in each of the following :

   (i) $A = \{ x : x \in Z \}$, $B = \{ x : x \in N \}$ (ii) A = {Ram, Rahim, Govind, Gautam}

   B = {Sita, Meera, Fatima, Manprit}

3. Given that A = {1, 2, 3, 4, 5}, B={5, 6, 7, 8, 9, 10}

   find (i) $A \cup B$ (ii) $A \cap B$.

4. If $A = \{ x : x \in N \}$, $B = \{ y : y \in z$ and $-10 \le y \le 0 \}$, find $A \cup B$ and write your answer in the Roster form as well as in set-builder form.

5. If A ={2, 4, 6, 8, 10}, B {8, 10, 12, 14}, C ={14, 16, 18, 20}.

   Find (i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$.

6. Let U = {1, 2, 3,.......10}, A = {2, 4, 6, 8, 10}, B = { 1, 3, 5, 7, 9, 10}

   Find (i) $(A \cup B)'$ (ii) $(A \cap B)'$ (iii) (B')' (iv) $(B - A)'$.

7. Draw Venn diagram for each of the following :

   (i) $A \cap B$ when $B \subset A$ (ii) $A \cap B$ when A and B are disjoint sets.

   (iii) $A \cap B$ when A and B are neither subsets of each other nor disjoint sets.

8. Draw Venn diagram for each of the following :

   (i) $A \cup B$ when $A \subset B$. (ii) $A \cup B$ when A and B are disjoint sets.

   (iii) $A \cup B$ when A and B are neither subsets of each other nor disjoint sets.

9. Draw Venn diagram for each the following :

   (i) $A - B$ and $B - A$ when $A \subset B$.

   (ii) $A - B$ and $B - A$ when A and B are disjoint sets.

   (iii) $A - B$ and $B - A$ when A and B are neither subsets of each other nor disjoint sets.

LET US SUM UP

- Set is a well defined collection of objects.
- To represent a set in Roster form all elements are to be written but in set builder form a set is represented by the common property of its elements.

- If the elements of a set can be counted then it is called a finite set and if the elements cannot be counted, it is infinite.
- If each element of set A is an element of set B, then A is called sub set of B.
- For two sets A and B, $A - B$ is a set of those elements which are in A but not in B.
- Complement of a set A is a set of those elements which are in the universal set but not in A. i.e. $A^c = U - A$
- Intersection of two sets is a set of those elements which belong to both the sets.
- Union of two sets is a set of those elements which belong to either of the two sets.
- Any set 'A' is said to be a subset of a set 'B' if every element of A is contained in B.
- Empty set is a subset of every set.
- Every set is a subset of itself.
- The set 'A' is a proper subset of set 'B' iff $A \subseteq B$ and $A \neq B$
- The set of all subsets of a given set 'A' is called power set of A.
- Two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$
- If $n(A) = p$ then number of subsets of $A = (2)^p$
- (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are finite intervals as their length $b - a$ is real and finite.
- Complement of a set A with respect to U is denoted by A' and defined as $A' = \{x : x \in U \text{ and } x \notin A\}$
- $A' = U - A$
- If $A \subset U$, then $A' \subset U$
- Properties of complement of set A with respect to U
  - $A \cup A' = U$ and $A \cap A' = \phi$
  - $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
  - $(A')' = A$
  - $\phi' = U$ and $U' = \phi$

SUPPORTIVE WEB SITES

http://www.mathresource.iitb.ac.in/project/indexproject.html

http://mathworld.wolfram.com/SetTheory.html

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Beginnings_of_set_theory.html

TERMINAL EXERCISE

1. Which of the following statements are true or false :

(i) $\{1, 2, 3\} = \{1, \{2\}, 3\}$.   (ii) $\{1, 2, 3\} = \{3, 1, 2\}$.

(iii) $\{a, e, o\} = \{a, b, c\}$.   (iv) $\{\phi\} = \{\ \}$

2. Write the set in Roster form represented by the shaded portion in the following.

(i) $A = \{1, 2, 3, 4, 5\}$

$B = \{5, 6, 7, 8, 9\}$

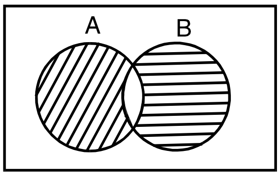


Fig. 1.12

(ii) $A = \{1, 2, 3, 4, 5, 6\}$

$B = \{2, 6, 8, 10, 12\}$

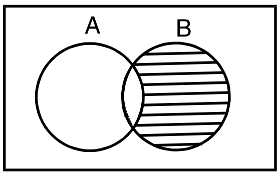


Fig. 1.13

3. Represent the follwoing using Venn diagram.

(i) $(A \cup B)'$ provided A and B are not disjoint sets.

(ii) $(A \cap B)'$ provided A and B are disjoint sets.

(iii) $(A - B)'$ provided A and B are not disjoint sets.

4. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A\{2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$

Verify that

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$A = \{1, 3, 5, 7, 9\}$.

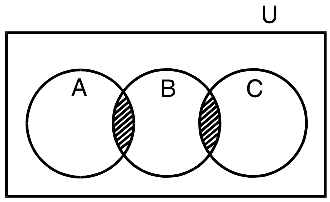
$B = \{2, 4, 6, 8, 10\}$, $C = \{1, 2, 3\}$.

Find (i) $A' \cap (B - C)$.   (ii) $A \cup (B \cup C)$

(iii) $A' \cap (B \cup C)'$ (iv) $(A \cap B)' \cup C'$

6. What does the shaded portion represent in each of the following Venn diagrams :

(i)



(ii)

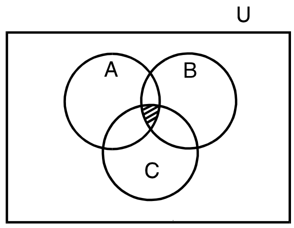
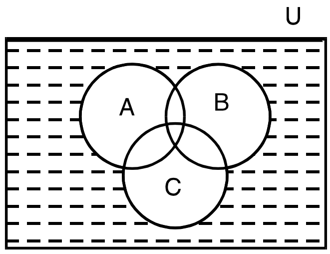


Fig. 1.14

Fig. 1.15

(iii)



(iv)

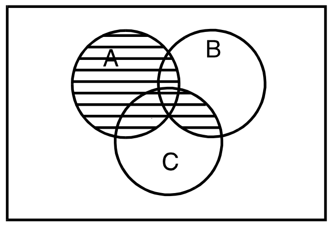


Fig. 1.16

Fig. 1.17

7. Draw Venn diagram for the following :

(i) $A' \cap (B \cup C)$ (ii) $A' \cap (C - B)$

Where A,B,C are not disjoint sets and are subsets of the universal set U.

8. Verify De Morgan's Law if $U = \{x : x \in N \text{ and } x \leq 10\}$

$A = \{x: x \in U \text{ and } x \text{ is a prime number}\}$and

$B = \{x : x \in U \text{ and } x \text{ is a factor of } 24\}$

9. Examine whether the following statements are true or false :

(*a*) $\{a, e\} \subset \{x : x$ is a vowel in the English alphabet]

(*b*) $\{1, 2, 3\} \subset \{1, 3, 5\}$ (*c*)$\{a, b, c\} \subset \{a, b, c\}$ (*d*) $\phi \subset \{1, 3, 5\}$

10. Write down all the subsets of the following sets :

(*a*) $\{a\}$ (*b*) $\{1, 2, 3\}$ (*c*) ϕ

11. Write down the following as intervals :

(*a*) $\{x : x \in R, -4 < x \leq 6\}$ (*b*) $\{x : x \in R, -12 < x < -10\}$

(*c*) $\{x : x \in R, 0 \leq x < 7\}$ (*d*) $\{x : x \in R, 3 \leq x \leq 4\}$

ANSWERS

CHECK YOUR PROGRESS 1.1

1. (i), (ii), (iii) are sets.
2. (i) $\in$ (ii) $\notin$
3. (i) $A = \{-5, -4, -3, -2, -1, 0\}$ (ii) $B = \{-1, 1\}$,
   (iii) $C = \{a, b, n\}$ (iv) $D = \{2, 3, 5\}$.
4. (i) $A = \{x : x \text{ is a even natural number less than or equal to ten}\}$.
   (ii) $B = \{x : x \in N \text{ and } x \text{ is a multible of } 3\}$.
   (iii) $C = \{x : x \text{ is a prime number less than } 10\}$.
   (iv) $D = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
5. (i) Infinite (ii) Finite (iii) Finite (iv) Infinite
6. (i) Null (ii) Singleton (iii) Null (iv) Null
7. (i) $A \approx B$ (ii) $A \approx B$ (iii) $A = B$.

CHECK YOUR PROGRESS 1.2

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\in$ (iv) $\notin$
2. 4 3. (i) False (ii) True
4. (i) False (ii) True (iii) True (iv) False
5. (i) $\{1, 2, 3, 4\}$ (ii) $\{6, 7\}$.
6. $A^c = \{x : x \text{ is an odd natural number}\}$

   $B^c = \{x : x \in N \text{ and } x \text{ is not a multiple of } 3\}$

   $C^c = \{1, 2, 3, 4\}$. $D^c = \{11, 12, 13, ...\}$
7. (a) (-8,3) (b) $\left[\frac{3}{2}, \frac{7}{2}\right)$

CHECK YOUR PROGRESS 1.3

1. (i) Disjoint (ii) Not disjoint (iii) Not disjoint (iv) Disjoint
2. (i) $\{x : x \in N\}$ (ii) ϕ
3. (i) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (ii) $\{5\}$

4. Roster from $\{-10, -9, -8, ...0, 1, 2, 3, ...\}$

Set builder from $\{x : x \in Z \text{ and } -10 \le x \le \infty\}$

5. (i) $\{x : x \text{ is a even natural number less than equal to } 20\}$. (ii) ϕ

6. (i) ϕ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(iii) $\{1, 3, 5, 7, 9, 10\}$ (iv) $\{2, 4, 6, 8, 10\}$

7. (i)

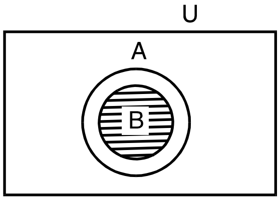


Fig. 1.18

(ii)

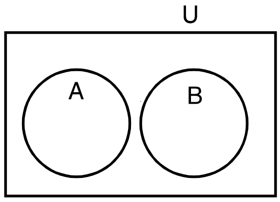


Fig. 1.19

(iii)

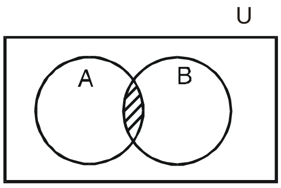


Fig. 1.20

8. (i)

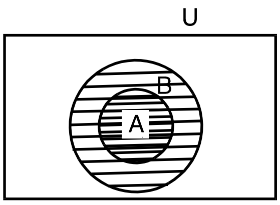


Fig. 1.21

(ii)

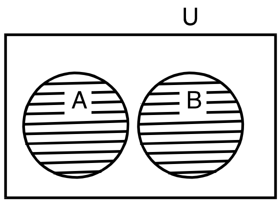


Fig. 1.22

(iii)

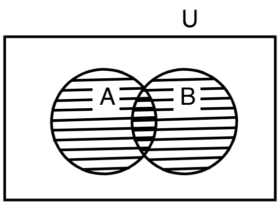
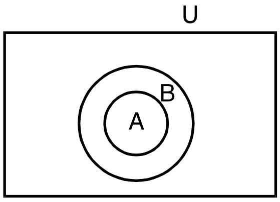
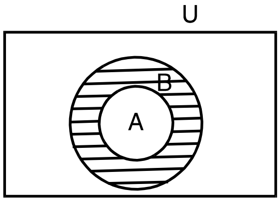


Fig. 1.23

9. (i)



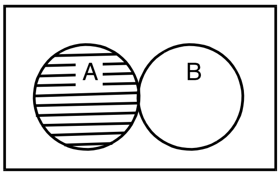
$A - B = \phi$



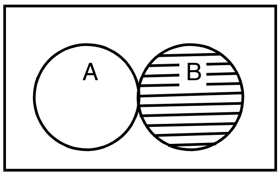
$B - A =$ Shaded portion.

Fig. 1.24

(ii)



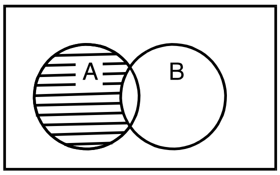
$A - B = A$



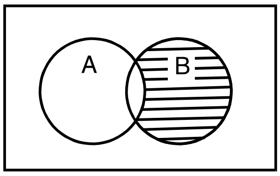
$B - A = B$

Fig. 1.25

(iii)



$A - B =$ Shaded Portion



$B - A =$ Shaded Portion

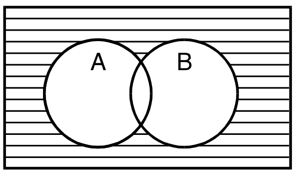
Fig. 1.26

TERMINAL EXERCISE

1. (i) False (ii) True (iii) False (iv) False

2. (i) $\{1, 2, 3, 4, 6, 7, 8, 9\}$ (ii) $\{8, 10, 12\}$

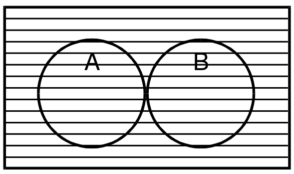
3. (i)



$(A \cup B)' =$ Shaded Portion

Fig. 1.27

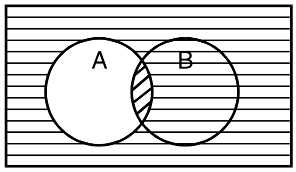
(ii)



$(A \cap B)' =$ Shaded Portion

Fig. 1.28

(iii)



$(A-B)' =$ Shaded Portion

Fig. 1.29

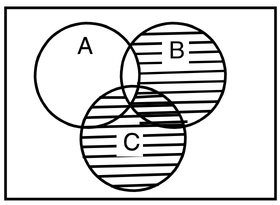
5. (i) $\{4, 6, 8, 10\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   (iii) ϕ (iv) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

6. (i) $(A \cap B) \cap (B \cap C)$ (ii) $(A \cap B) \cap C$

   (iii) $[(A \cup B) \cup C]'$ (iv) $A \cup (B \cap C)$.

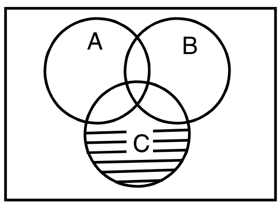
7. (i)



$A' \cap (B \cup C) =$ Shaded Portion

Fig.1.30

(ii)



$A' \cap (C - B) =$ Shaded Portion

Fig. 1.31

9. (*a*) True (*b*) False

   (*c*) True (*d*) True

10. (*a*) ϕ, $\{a\}$ (*b*) ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

    (c) ϕ

11. (*a*) $(-4, 6]$ (*b*) $(-12, -10)$

    (*c*) $[0, 7)$ (*d*) $[3, 4]$

2

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RELATIONS AND FUNCTIONS-I

In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student etc. In mathematics also, we come across many relations such as number m is greater than number n, line ℓ is perpendicular to line m etc. the concept of relation has been established in mathematical form. The word "function" was introduced by leibnitz in 1694. Function is defiend as a special type of relation. In the present chapter we shall discuss cartesion product of sets, relation between two sets , conditions for a relation to be a function, different types of functions and their properties.

OBJECTIVES

After studying this lesson, you will be able to :

- define cartesion product of two sets.
- define. relation, function and cite examples there of
- find domain and range of a function
- draw graph of functions.
- define and cite examples of even and odd turnotions.
- determine whether a function is odd or even or neither
- define and cite examples of functions like $|x|, [x]$ the greatest integer functions, polynomial functions, logarithmic and exponential functions
- to Find sum. difference, product and quotient of real functions.

EXPECTED BACKGROUND KNOWLEDGE

- concept of ordered pairs.

2.1 CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

A={1, 2}, B= {3, 4, 5}.

Set of all ordered pairs of elements of A and B is $\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

This set is denoted by $A \times B$ and is called the cartesian product of sets A and B.

i.e. $A\times B = \{(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)\}$

Cartesian product of sets B and A is denoted by $B\times A$.

In the present example, it is given by

$B\times A = \{(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)\}$, Clearly $A\times B \neq B\times A$.

In the set builder form :

$$A\times B = \{(a,b) : a \in A \text{ and } b \in B\} \text{ and } B\times A = \{(b,a) : b \in B \text{ and } a \in A\}$$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A \times B = B \times A = \phi$.

Example 2.1

(1) Let $A=\{a,b,c\}$, $B=\{d,e\}$, $C=\{a,d\}$.

Find (i) $A\times B$ (ii) $B\times A$ (iii) $A\times(B \cup C)$ (iv) $(A \cap C) \times B$

(v) $(A \cap B) \times C$ (vi) $A \times (B - C)$.

Solution : (i) $A\times B = \{(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)\}$.

(ii) $B\times A = \{(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)\}$.

(iii) $A = \{a, b, c\}$, $B \cup C = \{a,d,e\}$.

$\therefore$ $A \times (B \cup C) = \{(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e)$.

(iv) $A \cap C = \{a\}$, $B=\{d, e\}$. $\therefore$ $(A \cap C)\times B=\{(a, d), (a, e)\}$

(v) $A \cap B = \phi$, $C=\{a,d\}$, $\therefore$ $A \cap B \times C = \phi$

(vi) $A = \{a,b,c\}$, $B - C = \{e\}$. $\therefore$ $A \times (B - C) = \{(a,e),(b,e),(c,e)\}$.

2.1.1 Number of elements in the Cartesian product of two finite sets

Let A and B be two non-empty sets. We know that $A \times B = \{(a, b); a \in A \text{ and } b \in B\}$

Then number of elements in Cartesian product of two finite sets A and B

i.e. $n(A \times B) = n(A).\, n(B)$

Example 2.2 Suppose $A = \{1,2,3\}$ and $B=\{x, y\}$, show that $n(A\times B) = n(A)\times n(B)$

Solution : Here $n(A) = 3$, $n(B) = 2$

$\therefore$ $A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$

$n(A \times B) = n(A) \times n(B), = 3 \times 2 = 6$

Example 2.3 If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$

Solution : We know that $n(A \times B) = n(A) \times n(B)$

$$n(A \times B) = 5 \times 4 = 20$$

2.1.2 Cartesian product of the set of real numbers R with itself upto R × R × R

Ordered triplet $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$

Here (a, b, c) is called an ordered triplet.

The Cartesian product $R \times R$ represents the set $R \times R = \{(x, y) : x, y \in R\}$ which represents the coordinates of all the points in two dimensional plane and the Cartesian product $R \times R \times R$ represent the set $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ which represents the coordinates of all the points in three dimensional space.

Example 2.4 If $A = \{1, 2\}$, form the set $A \times A \times A$.

Solution : $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)$

$(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

2.2 RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim} and B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly $R \subseteq A \times B$, i.e. $R = \{(a, b) : a \in A, b \in B \text{ and } aRb\}$

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) $R = \phi$, R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{(a, a) \ \forall \ a \in A\}$, is called the identity relation.

2.2.1 Domain and Range of a Relation

If R is a relation between two sets then the set of first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation. In the previous example. Domain = {Mohan, Sohan, David, Karim}, Range = {Rita, Marry, Fatima}

Example 2.5 Given that $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$.

R is a relation from A to B defined by

$$R = \{(a, b) : a \in A,\ b \in B \text{ and a is divisible by b}\}$$

find (i) R in the roster form (ii) Domain of R (iii) Range of R (iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6} (iii) Range of R = {2, 3}

(iv)

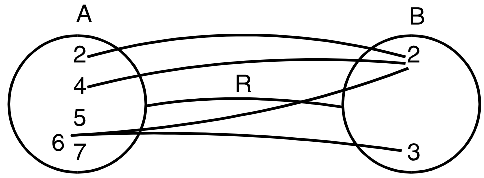


Fig. 2.1

Example 2.6 If R is a relation 'is greater than' from A to B, where

A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(2,1), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {2, 3, 4, 5} (iii) Range of R = {1, 2}

2.2.2 Co-domain of a Relation

If R is a relation from A to B, then B is called codomain of R.

For example, let A = {1, 3, 4, 5, 7} and B = {2, 4, 6, 8} and R be the relation 'is one less than' from A to B, then R = {(1, 2), (3, 4), (5, 6), (7, 8)}

so codomain of R = {2, 4, 6, 8}

Example 2.7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$ and write down the domain, range and codomain of R.

Solution : R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

Domain of A = {1, 2, 3, 4, 5}

Range of R = {2, 3, 4, 5, 6} and Codomain of R = {1, 2, 3, 4, 5, 6}

CHECK YOUR PROGRESS 2.1

1. Given that A = {4, 5, 6, 7}, B = {8, 9}, C = {10}

Verify that $A \times (B - C) = (A \times B) - (A \times C)$.

2. If U is a universal set and A, B are its subsets. Where U= {1, 2, 3, 4, 5}.
A = {1,3,5}, B = {x : x is a prime number}, find A' × B'
3. If A = {4, 6, 8, 10}, B = {2, 3, 4, 5}, R is a relation defined from A to B where
R= {(a, b) : $a \in A$, $b \in B$ and a is a multiple of b}
find (i)R in the Roster form (ii) Domain of R (iii) Range of R .
4. If R be a relation from N to N defined by R= {(x,y) : $4x + y = 12$, $x, y \in N$ }
find (i) R in the Roster form (ii) Domain of R (iii) Range of R.
5. If R be a relation on N defined by R={ (x, x^2) : x is a prime number less than 15}
Find (i) R in the Roster form (ii) Domain of R (iii) Range of R
6. If R be a relation on set of real numbers defined by R={(x,y) : $x^2 + y^2 = 0$ }, Find
(i) R in the Roster form (ii) Domain of R (iii) Range of R.
7. If $(x + 1, y - 2) = (3, 1)$, find the values of x and y.
8. If A = {–1, 1} find A × A × A.
9. If A × B = {(a, x), (a, y), (b, x), (b, y)}. Find A and B.
10. If n(A) = 6 and n(B) = 3, then find n(A × B).

2.3 DEFINITION OF A FUNCTION

Consider the relation f : {(a,1), (b,2), (c,3), (d,5)}from set A = {a,b,c,d}to set B = {1,2,3,4}.

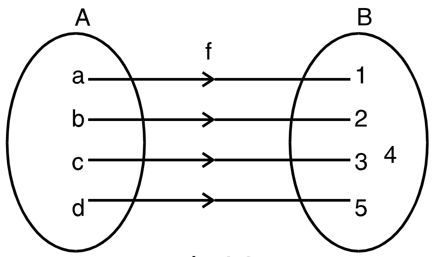


Fig. 2.2

In this relation we see that each element of A has a unique image in *B*. This relation *f* from set *A* to *B* where every element of *A* has a unique image in *B* is defined as a function from *A* to *B*. So we observe that ***in a function no two ordered pairs have the same first element.***

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and (ii) the set {1, 2, 3, 5} is called the range. From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

2.3.1 Real valued function of a real variable

A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, then it is called a real function.

Let R be the set of all real numbers and X, Y be two non-empty subsets of R, then a rule 'f' which associates to each $x \in$ X, a unique element y of Y is called a real valued function of the real variable or simply a real function and we write it as $f : X \rightarrow Y$

A real function 'f' is a rule which associates to each possible real number x, a unique real number $f(x)$.

Example 2.8 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A = \{1,2,4\}$, $B = \{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A = \{a,b,c,d,e\}$ $B = \{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A = \{2,3,4,5,6\}$, $B = \{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A = \{0,1,2,3,4,5\}$, $B = \{-1,-2,-3,-4,-5\}$

(f) $\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\}$,

$A = \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}$ $B = \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$

(g) $\{(a,b),(a,2),(b,3),(b,4)\}$, $A = \{a,b\}$, $B = \{b,2,3,4\}$.

Solution :

(a) It is a function. Domain $= \{1,3,4,8\}$, Range $= \{-2,7,-6,1\}$

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function. Because Domain $= \{a,b,c,d\} \neq A$, Range $= \{b,c\}$

(d) It is a function. Domain $= \{2,3,4,5,6\}$, Range $= \{4,9,16,25,36\}$

(e) It is not a function .

Because Domain $= \{1,2,3,4,5\} \neq A$, Range $= \{-1,-2,-3,-4,-5\}$

(f) It is a function .

Domain $= \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}$, Range $= \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3}\right\}$

(g) It is not a function. Because first two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 2.9 State whether each of the following relations represent a function or not.

(a)

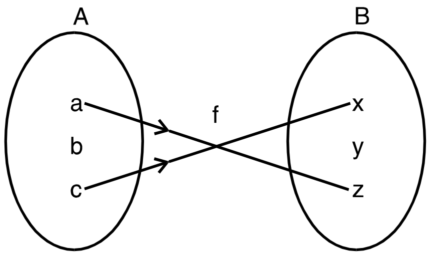


Fig. 2.3

(b)

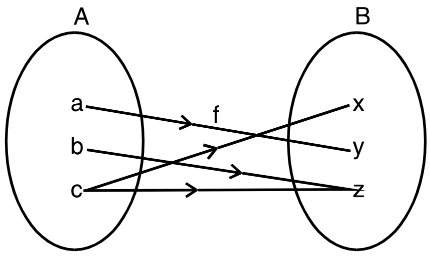


Fig. 2.4

(c)

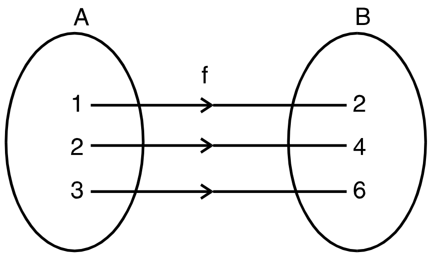


Fig. 2.5

(d)

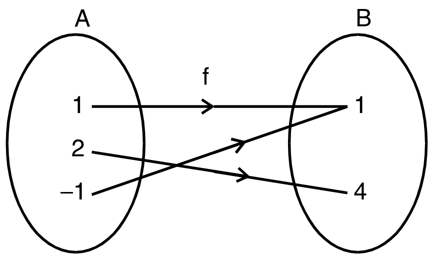


Fig. 2.6

Solution :

(a) *f* is not a function because the element *b* of *A* does not have an image in *B*.

(b) *f* is not a function because the element *c* of *A* does not have a unique image in *B*.

(c) *f* is a function because every element of *A* has a unique image in *B*.

(d) *f* is a function because every element in *A* has a unique image in *B*.

Example 2.10 Which of the following relations from $R \rightarrow$. R are functions?

(a) $y = 3x + 2$ (b) $y < x + 3$ (c) $y = 2x^2 + 1$

Solution : (a) $y = 3x + 2$ Here corresponding to every element $x \in R, \exists$ a unique element $y \in R.$ $\therefore$ It is a function.

(b) $y < x + 3.$

For any real value of x we get more than one real value of y. $\therefore$ It is not a function.

(c) $y = 2x^2 + 1$

For any real value of x, we will get a unique real value of y. $\therefore$ It is a function.

Example 2.11 Let N be the set of natural numbers. Define a real valued function $f : N \to N$ by $f(x) = 2x + 1$. Using the definition find $f(1)$, $f(2)$, $f(3)$, $f(4)$

Solution : $f(x) = 2x + 1$

$$f(1) = 2 \times 1 + 1 = 2 + 1 = 3, \quad f(2) = 2 \times 2 + 1 = 4 + 1 = 5$$

$$f(3) = 2 \times 3 + 1 = 6 + 1 = 7, \quad f(4) = 2 \times 4 + 1 = 6 + 1 = 9$$

CHECK YOUR PROGRESS 2.2

1. Which of the following relations are functions from A to B ?

(a) $\{(1,-2),(3,7),(4,-6),(8,11)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,11\}$

(b) $\{(1,0),(1,-1),(2,3),(4,10)\}$, $A = \{1,2,4\}$, $B = \{1,0,-1,3,10\}$

(c) $\{(a,2),(b,3),(c,2),(d,3)\}$, $A = \{a,b,c,d\}$, $B = \{2,3\}$

(d) $\{(1,1),(1,2),(2,3),(-3,4)\}$, $A = \{1,2,-3\}$, $B = \{1,2,3,4\}$

(e) $\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),\ldots,\left(10,\frac{1}{10}\right)\right\}$, $A = \{1,2,3,4\}$, $B = \left\{\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{11}\right\}$

(f) $\{(1,1),(-1,1),(2,4),(-2,4)\}$, $A = \{0,1,-1,2,-2\}$, $B = \{1,4\}$

2. Which of the following relations represent a function ?

(a)

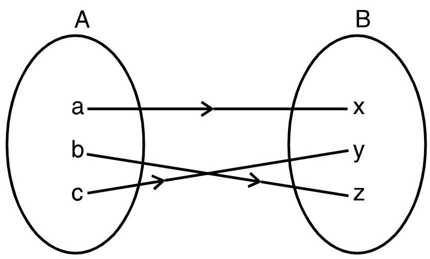


Fig. 2.7

(b)

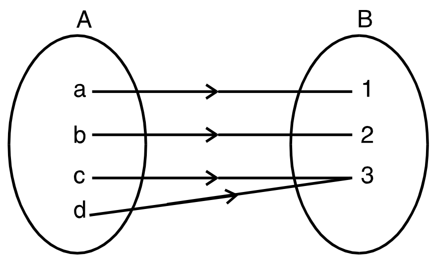


Fig. 2.8

(c)

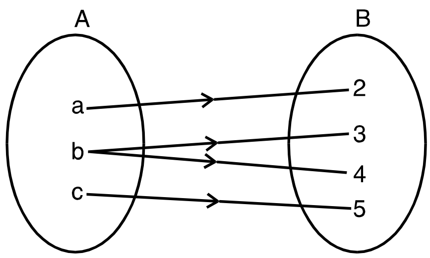


Fig. 2.9

(d)

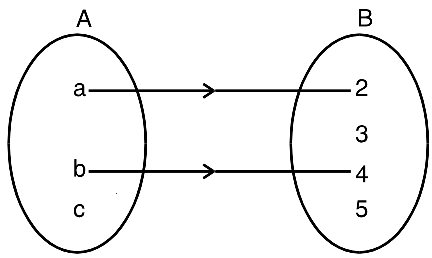


Fig. 2.10

3. Which of the following relations defined from $R \to R$ are functions ?

   (a) $y = 2x + 1$ (b) $y > x + 3$ (c) $y < 3x + 1$ (d) $y = x^2 + 1$

4. Write domain and range for each of the following functions :

   (a) $\left\{\left(\sqrt{2}, 2\right), \left(\sqrt{5}, -1\right), \left(\sqrt{3}, 5\right)\right\}$, (b) $\left\{\left(-3, \frac{1}{2}\right), \left(-2, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\right\}$

   (c) $\{(1, 1), (0, 0), (2, 2), (-1, -1)\}$

   (d) $\{(\text{Deepak}, 16), (\text{Sandeep}, 28), (\text{Rajan}, 24)\}$

5. Write domain and range for each of the following mappings :

(a)

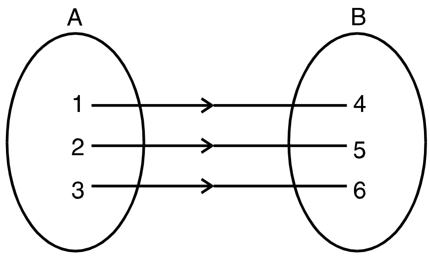


Fig. 2.11

(b)

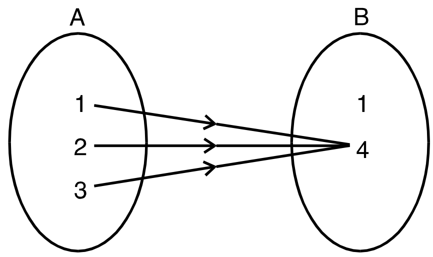


Fig. 2.12

(c)

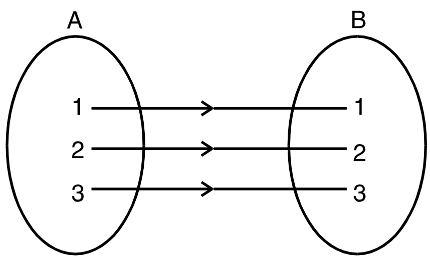


Fig. 2.13

(d)

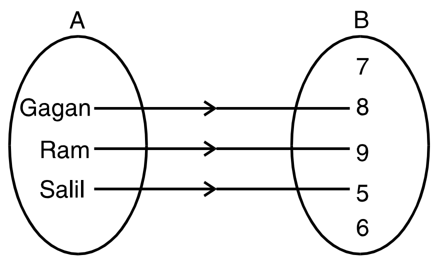


Fig. 2.14

(e)

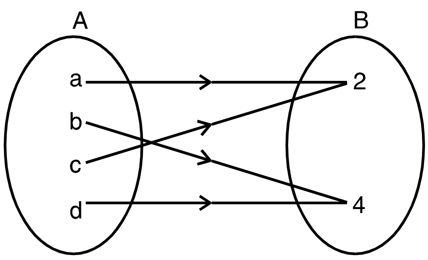


Fig. 2.15

2.3.2 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 2.12 Find the domain of each of the following functions :

$$\text{(a) } y = \frac{1}{x} \qquad \text{(b) } y = \frac{1}{x-2} \qquad \text{(c) } y = \frac{1}{(x+2)(x-3)}$$

Solution : The function $y = \frac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{ \ldots\ldots., \left(-2, -\frac{1}{2}\right), (-1, -1), (1, 1)\left(2, \frac{1}{2}\right), \ldots. \right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e., $\frac{1}{0}$ is not defined. $\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

Note : Here range $= R - \{ 0 \}$

(b) x can take all real values except 2 because the corresponding image, i.e., $\frac{1}{(2-2)}$ does not exist. $\therefore$ Domain $= R - \{2\}$

(c) Value of y does not exist for $x = -2$ and $x = 3$ $\therefore$ Domain $= R - \{-2, 3\}$

Example 2.13 Find domain of each of the following functions :

$$\text{(a) } y = +\sqrt{x-2} \qquad \text{(b) } y = +\sqrt{(2-x)(4+x)}$$

Solution :(a) Consider the function $y = +\sqrt{x-2}$

In order to have real values of y, we must have $(x-2) \geq 0$ i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2.

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2-x)(4+x) \geq 0$

We can achieve this in the following two cases :

Case I : $(2-x) \geq 0$ and $(4+x) \geq 0 \Rightarrow x \leq 2$ and $x \geq -4$

$\therefore$ Domain consists of all real values of x such that $-4 \leq x \leq 2$

Case II : $2 - x \leq 0$ and $4 + x \leq 0 \Rightarrow 2 \leq x$ and $x \leq -4$.

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4.

$\therefore$ From both the cases, we have Domain $= -4 \le x \le 2 \ \forall \ x \in R$

Example 2.14 For the function

$$f(x) = y = 2x + 1, \text{ find the range when domain } = \{-3, -2, -1, 0, 1, 2, 3\}.$$

Solution : For the given values of *x*, we have

$$f(-3) = 2(-3) + 1 = -5, \ f(-2) = 2(-2) + 1 = -3$$

$$f(-1) = 2(-1) + 1 = -1, \ f(0) = 2(0) + 1 = 1$$

$$f(1) = 2(1) + 1 = 3, \ f(2) = 2(2) + 1 = 5, \ f(3) = 2(3) + 1 = 7$$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3, -5), (-2, -3), (-1, -1), (0, 1)(1, 3), (2, 5)(3, 7)\}$

$\therefore$ Range $= \{-5, -3, -1, 1, 3, 5, 7\}$

Example 2.15 If $f(x) = x + 3, \ 0 \le x \le 4$, find its range.

Solution : Here $0 \le x \le 4$ or $0 + 3 \le x + 3 \le 4 + 3$ or $3 \le f(x) \le 7$

$\therefore$ Range $= \{f(x) : 3 \le f(x) \le 7\}$

Example 2. 16 If $f(x) = x^2, \ -3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$ or $0 \le x^2 \le 9$ or $0 \le f(x) \le 9$

$\therefore$ Range $= \{f(x) : 0 \le f(x) \le 9\}$

CHECK YOUR PROGRESS 2.3

1. Find the domain of each of the following functions $x \in R$:

(a) (i) $y = 2x$, (ii) $y = 9x + 3$, (iii) $y = x^2 + 5$

(b) (i) $y = \dfrac{1}{3x - 1}$, (ii) $y = \dfrac{1}{(4x + 1)(x - 5)}$

(iii) $y = \dfrac{1}{(x - 3)(x - 5)}$, (iv) $y = \dfrac{1}{(3 - x)(x - 5)}$

(c) (i) $y = \sqrt{6 - x}$, (ii) $y = \sqrt{7 + x}$, (iii) $y = \sqrt{3x + 5}$

(d) (i) $y = \sqrt{(3 - x)(x - 5)}$ (ii) $y = \sqrt{(x - 3)(x + 5)}$

(iii) $y = \dfrac{1}{\sqrt{(3+x)(7+x)}}$ (iv) $y = \dfrac{1}{\sqrt{(x-3)(7+x)}}$

2. Find the range of the function, given its domain in each of the following cases.

(a) (i) $f(x) = 3x + 10,\ x \in \{1, 5, 7, -1, -2\}$,

(ii) $f(x) = 2x^2 + 1,\ x \in \{-3, 2, 4, 0\}$

(iii) $f(x) = x^2 - x + 2,\ x \in \{1, 2, 3, 4, 5\}$

(b) (i) $f(x) = x - 2,\ 0 \le x \le 4$ (ii) $f(x) = 3x + 4,\ -1 \le x \le 2$

(c) (i) $f(x) = x^2,\ -5 \le x \le 5$ (ii) $f(x) = 2x,\ -3 \le x \le 3$

(iii) $f(x) = x^2 + 1,\ -2 \le x \le 2$ (iv) $f(x) = \sqrt{x},\ 0 \le x \le 25$

(d) (i) $f(x) = x + 5,\ x \in R$ (ii) $f(x) = 2x - 3,\ x \in R$

(iii) $f(x) = x^3,\ x \in R$ (iv) $f(x) = \dfrac{1}{x},\ \{x : x < 0\}$

(v) $f(x) = \dfrac{1}{x-2},\ \{x : x \le 1\}$ (vi) $f(x) = \dfrac{1}{3x-2},\ \{x : x \le 0\}$

(vii) $f(x) = \dfrac{2}{x},\ \{x : x > 0\}$ (viii) $f(x) = \dfrac{x}{x+5},\ \{x : x \ne -5\}$

2.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

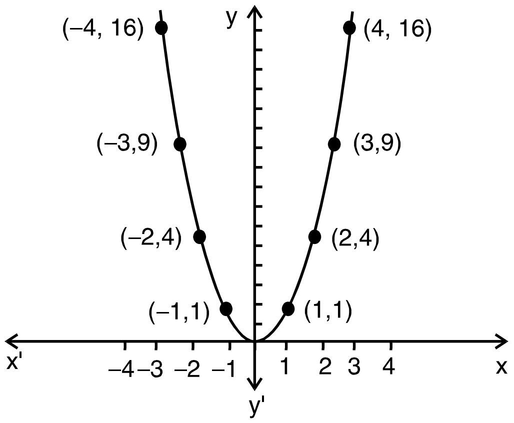


Fig. 2.16

Does this represent a function?

Yes, this represent a function because corresponding to each value of x $\exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

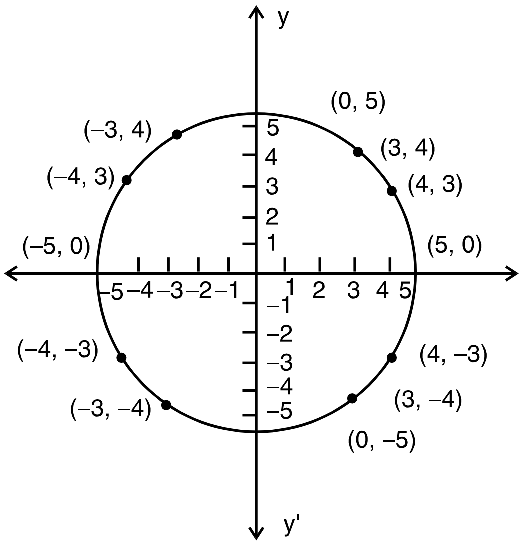


Fig. 2.17

This graph represents a circle. Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 2.4

1. (i) Does the graph represent a function?

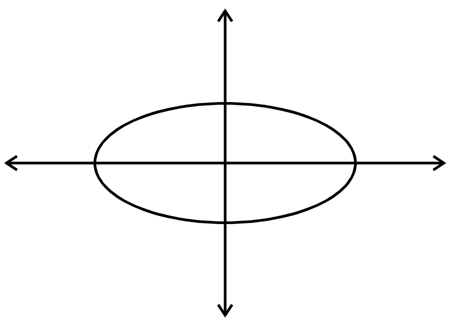


Fig. 2.18

(ii) Does the graph represent a function ?

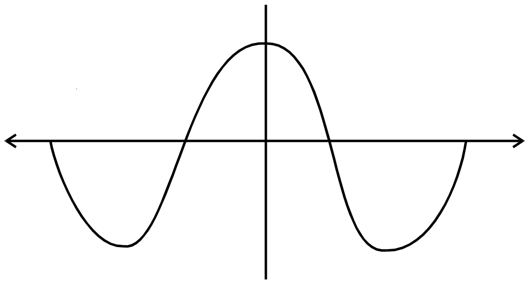


Fig. 2.19

2. Draw the graph of each of the following functions :

(a) $y = 3x^2$ (b) $y = -x^2$ (c) $y = x^2 - 2$

(d) $y = 5 - x^2$ (e) $y = 2x^2 + 1$ (f) $y = 1 - 2x^2$

3. Which of the following graphs represents a function ?

(a)

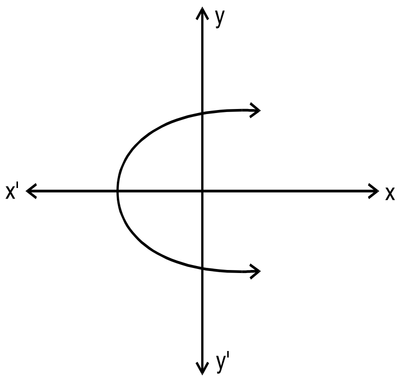


Fig. 2.20

(b)

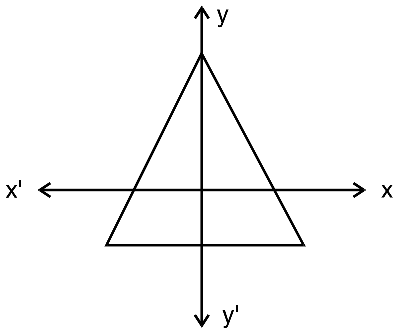


Fig. 2.21

(c)

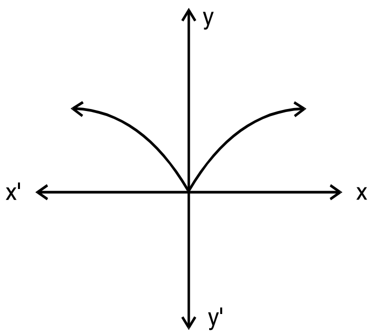


Fig. 2.22

(d)

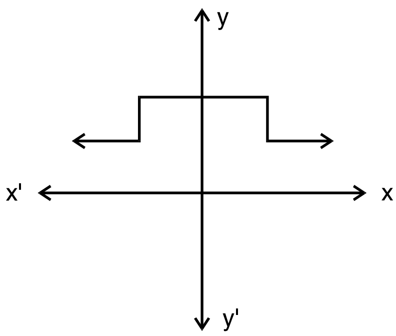


Fig. 2.23

(e)

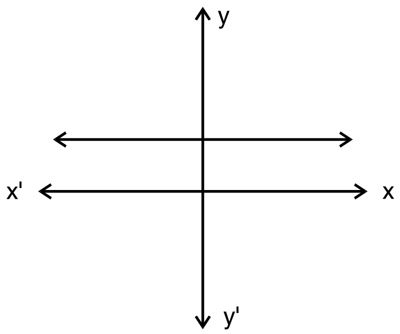


Fig . 2.24

Hint : If any line || to y-axis cuts the graph in more than one point, graph does not represent a function.

2.5 SOME SPECIAL FUNCTIONS

2.5.1 Monotonic Function

Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a,b) if it is either increasing or decreasing on that interval.

For function to be increasing on an interval (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

and for function to be decreasing on (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Consider the function $F : R \to R$ defined by $f(x) = x^2$.

Now $\quad \forall \quad x_1, x_2 \in [0, \infty], \; x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $[0, \infty]$.

($\because$ It is only increasing function on this interval)

But $\quad \forall \; x_1, x_2 \in (-\infty, 0), \; x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $(-\infty, 0)$

($\because$ It is only a decreasing function on this interval)

Therefore if we talk of the whole domain given function is not monotonic on R but it is monotonic on $(-\infty, 0)$ and $(0, \infty)$. Again consider the function $F : R \to R$ defined by $f(x) = x^3$.

Clearly $\forall \; x_1 \; x_2 \in$ domain $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\therefore$ Given function is ***monotonic*** on R i.e. on the whole domain.

2.5.2 Even Function

A function is said to be an even function if for each x of domain $F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) If $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) If $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) If $F(x) = |x|$ then $F(-x) = |-x| = |x| = F(x)$

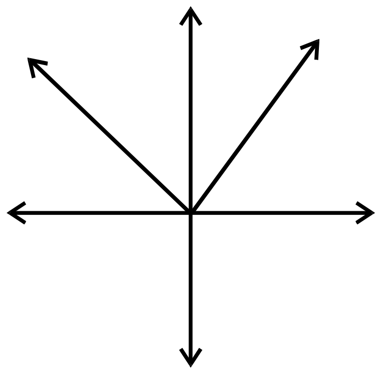


Fig. 2.25

The graph of this even function (modulus function) is shown in the figure above.

Observation Graph is symmetrical about y-axis.

2.5.3 Odd Function

A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

For example,

(i) If $f(x) = x^3$

then $f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

Graph of the odd function y = x is given in Fig.2.26

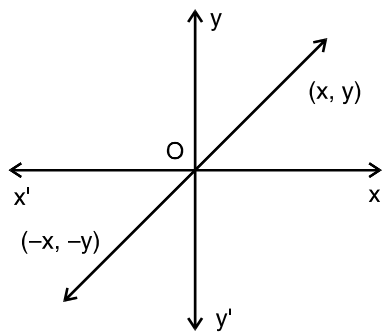


Fig. 2.26

Observation Graph is symmetrical about origin.

2.5.4 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 2.27.

Let us draw the graph of $y = [x], x \in R$

$[x] = 1, \quad 1 \le x < 2$

$[x] = 2, \quad 2 \le x < 3$

$[x] = 3, \quad 3 \le x < 4$

$[x] = 0, \quad 0 \le x < 1$

$[x] = -1, \quad -1 \le x < -0$

$[x] = -2, \quad -2 \le x < -1$

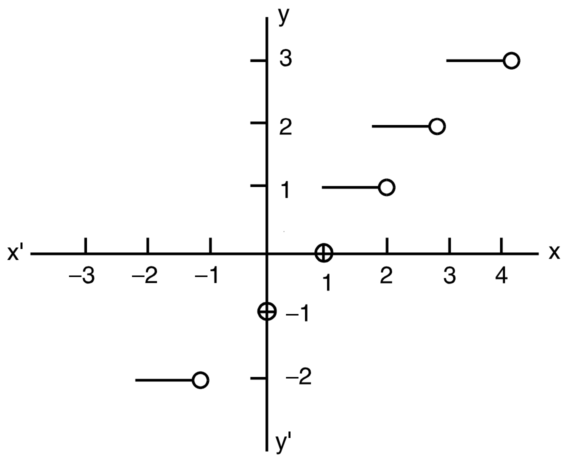


Fig. 2.27

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

2.5.5 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example, (i) $f(x) = 3x^2 - 4x - 2$, (ii) $f(x) = x^3 - 5x^2 - x + 5$, (iii) $f(x) = 3$ are all polynomial functions.

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.

2.5.6 Rational Function

Function of the type $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \ne 0$ and $g(x)$ and $h(x)$ are polynomial functions are called rational functions.

For example, $\quad f(x) = \dfrac{x^2 - 4}{x + 1}, x \ne -1$ is a rational function.

2.5.7 Reciprocal Function: Functions of the type $y = \dfrac{1}{x}, x \ne 0$ is called a reciprocal function.

2.5.8 Exponential Function A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots.. + \frac{1}{\underline{|n}} + \ldots.. \qquad \ldots..(1)$$

It is well known that the sum of this infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 an 3.

Now consider the infinite series $1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+.....+\frac{x^n}{\underline{|n}}+.....$

It can be shown that the sum of this infinite series also tends to a finite limit, which we denote by

e^x . Thus, $e^x=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+.....+\frac{x^n}{\underline{|n}}+.....$ ......(2)

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x)=e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function $y=e^x$ is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0=1$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 2.28).

x	–3	–2	–1	0	1	2	3
$y=e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

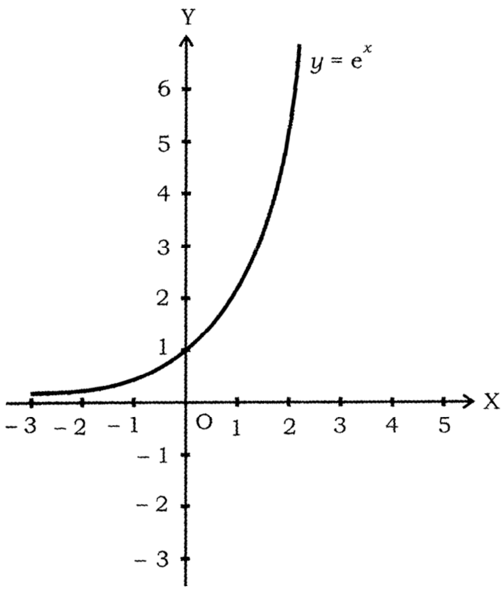


Fig. 2.28

If we take the base different from e, say a, we would get exponential function

$$f(x) = a^x \text{ ,provbided } a > 0, \quad a \neq 1.$$

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$y = 2^x$ (See Fig. 2.29)

and $y = 3^x$ (See Fig. 2.30)

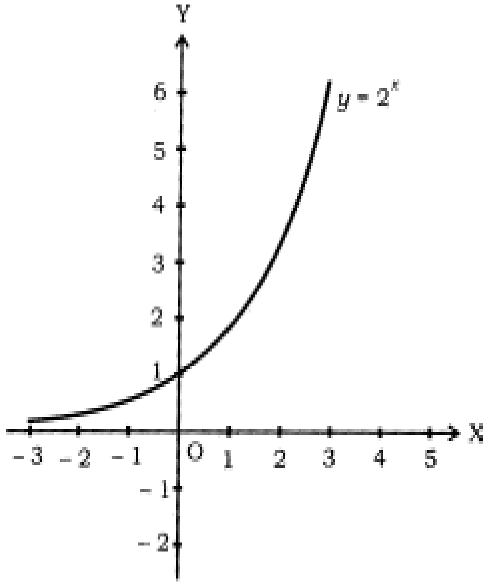


Fig. 2.29

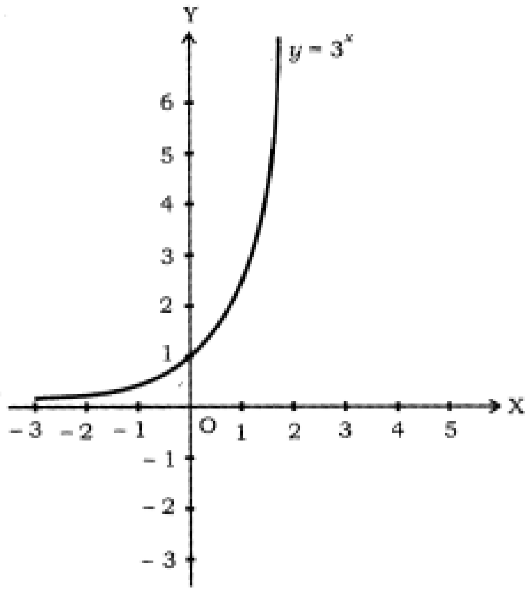


Fig. 2.30

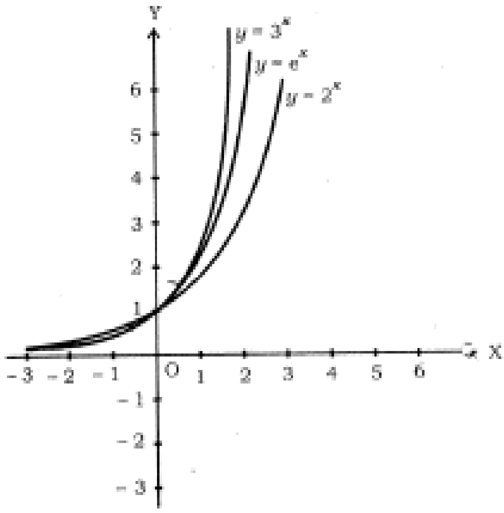


Fig. 2.31

2.5.9 Logarithmic Functions

Now Consider the function $y = e^x$ .....(3)

We write it equivalently as $x = \log_e y$ Thus, $y = \log_e x$ .....(4)

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line $y = x$. The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line y = x.

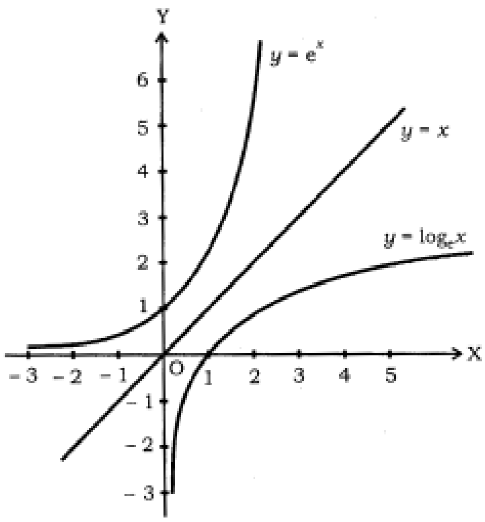


Fig. 2.32

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics : If a > 0, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n},\ a^m \div a^n = a^{m-n},\ (a^m)^n = a^{mn},\ a^0 = 1$$

(ii) The corresponding laws of logarithms are

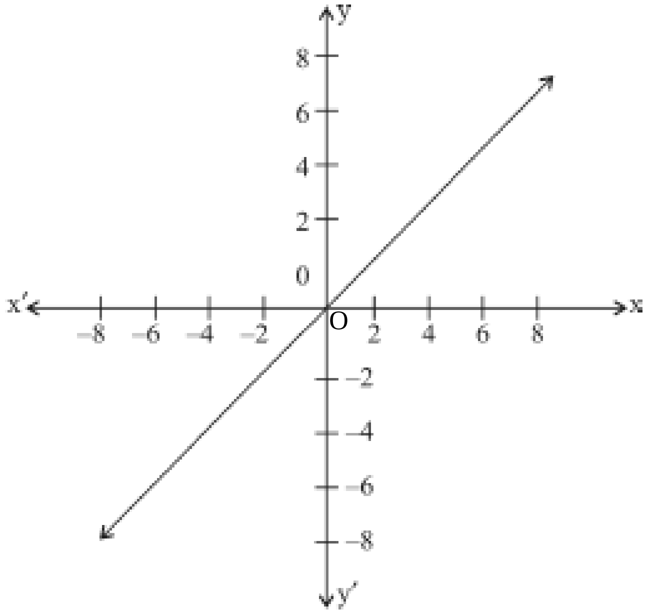
$$\log_a(mn) = \log_a m + \log_a n,\ \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m^n) = n\log_a m,\ \log_b m = \frac{\log_a m}{\log_a b} \text{ or } \log_b m = \log_a m \log_b a$$

Here a, b > 0, $a \neq 1$, $b \neq 1$.

2.5.10 Identity Function

Let R be the set of real numbers. Define the real valued function f:R→R by $y = f(x) = x$ for each $x \in$R. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line. It passes through the origin.

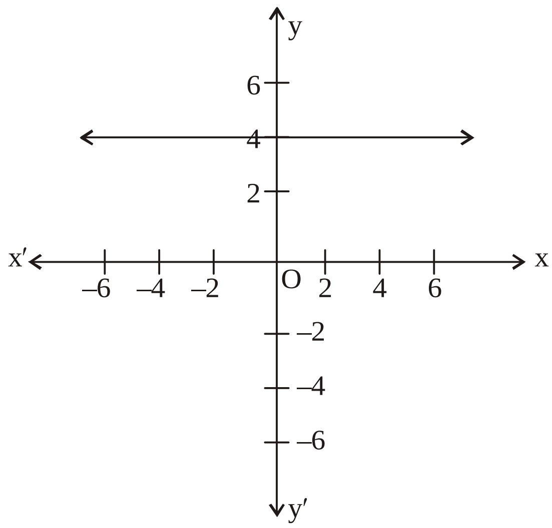


$f(x) = x$ **Fig. 2.33**

2.5.11 Constant Function

Define the function f : R → R by $y = f(x) = c$, $x \in$R where c is a constant and each $x \in$R. Here domain of f is R and its range is $\{c\}$.

The graph is a line parallel to x-axis. For example, $f(x) = 4$ for each $x \in$R, then its graph will be shown as



$f(x) = 4$ **Fig. 2.34**

MODULE - I
Sets, Relations and Functions

Notes

2.5.12 Signum Function

The function $f : R \to R$ defined by $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$ is called a signum function.

The domain of the signum function is R and the range is the set {–1, 0, 1}.

The graph of the signum function is given as under :

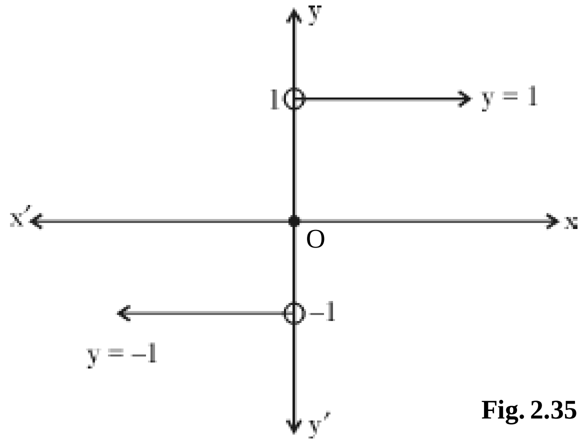


Fig. 2.35

CHECK YOUR PROGRESS 2.5

1. Which of the following statements are true or false.

(i) function $f(x) = 2x^4 + 7x^2 + 9x$ is an even function.

(ii) Odd function is symmetrical about y-axis.

(iii) $f(x) = x^{1/2} - x^3 + x^5$ is a polynomial function.

(iv) $f(x) = \frac{x-3}{3+x}$ is a rational function for all $x \in R$.

(v) $f(x) = \frac{\sqrt{5}}{3}$ is a constant function.

(vi) Domain of the function defined by $f(x) = \frac{1}{x}$ is the set of real numbers except 0.

(vii) Greatest integer function is neither even nor odd.

2. Which of the following functions are even or odd functions ?

(a) $f(x) = \frac{x^2 - 1}{x + 1}$ (b) $f(x) = \frac{x^2}{5 + x^2}$ (c) $f(x) = \frac{1}{x^2 + 5}$

(d) $f(x) = \dfrac{2}{x^3}$ (e) $f(x) = \dfrac{x}{x^2+1}$ (f) $f(x) = \dfrac{5}{x-5}$

(g) $f(x) = \dfrac{x-3}{3+x}$ (h) $f(x) = x - x^3$

3. Draw the graph of the function $y = [x] - 2$.

4. Specify the following functions as polynomial function, rational function, reciprocal function or constant function.

(a) $y = 3x^8 - 5x^7 + 8x^5$ (b) $y = \dfrac{x^2+2x}{x^3-2x+3}, \; x^3 - 2x + 3 \neq 0$

(c) $y = \dfrac{3}{x^2}, \; x \neq 0$ (d) $y = 3 + \dfrac{2x+1}{x}, \; x \neq 0$

(e) $y = 1 - \dfrac{1}{x}, \; x \neq 0$ (f) $y = \dfrac{x^2-5x+6}{x-2}, \; x \neq 2$

(g) $y = \dfrac{1}{9}$.

2.6 Sum, difference, product and quotient of functions

(i) Addition of two real functions :

Let $f : X \to R$ and $g : X \to R$ be any two functions, where $X \subset R$. Then, we define $(f + g) : X \to R$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$

(ii) **Subtraction of a real function from another :**

Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g) : X \to R$ by $(f - g)x = f(x) - g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

then $(f - g)(x) = f(x) - g(x) = x^2 - (2x + 1) = x^2 - 2x - 1$

(iii) **Multiplication of two real functions :**

The product of two real functions $f : X \to R$ and $g : X \to R$ is a function $f\,g : X \to R$ defined by $(f\,g)(x) = f(x) \,.\, g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $f\,g(x) = f(x) \,.\, g(x) = x^2 . (2x + 1) = 2x^3 + x^2$

(iv) **Quotient of two real functions :**

Let f and g be two real functions defined from $X \to R$ where $X \subset R$. The real quotient of f by g denoted by $\frac{f}{g}$ is a function defined by

$$\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0,\ x \in X$$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq \frac{-1}{2}$

Example 2. 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)(x)$, $(f - g)(x)$, $(f\,g)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution : We have $f(x) = \sqrt{x}, g(x) = x$

Then $(f + g)(x) = f(x) + g(x) = \sqrt{x} + x$

$(f - g)(x) = f(x) - g(x) = \sqrt{x} - x$

$(f\,g)(x) = f(x) \,.\, g(x) = \sqrt{x} \,.\, x = x^{3/2}$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = x^{\frac{-1}{2}}, x \neq 0$$

CHECK YOUR PROGRESS 2.6

1. A function f is defined as $f(x) = 3x + 4$. Write down the values of

   (i) $f(0)$ (ii) $f(7)$ (iii) $f(-3)$

2. Let $f, g : R \to R$ be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $(f + g)$, $(f - g)$ $(f\,g)$ and $\left(\frac{f}{g}\right)$.

LET US SUM UP

- Cartesian product of two sets A and B is the set of all ordered pairs of the elements of A and B. It is denoted by $A \times B$ i.e $A \times B = \{(a,b)\!: a \in A \text{ and } b \in B\}$

- Relation is a sub set of A×B where *A* and *B* are sets.

  i.e. $R \subseteq A \times B = \{(a, b) : a \in A \text{ and } b \in B \text{ and } aRb\}$
- Function is a special type of relation.
- Functrions $f : A \to B$ is a rule of correspondence from *A* to *B* such that to every element of *A* $\exists$ a unique element in *B*.
- Functions can be described as a set of ordered pairs.
- Let *f* be a function from *A* to *B*.

  Domain : Set of all first elements of ordered pairs belonging to *f*.

  Range : Set of all second elements of ordered pairs belonging to *f*.
- Functions can be written in the form of equations such as $y = f(x)$

  where x is independent variable, y is dependent variable.

  Domain : Set of independent variable.

  Range : Set of dependent variable.

  Every equation does not represent a function.
- **Vertical line test :** To check whether a graph is a function or not, we draw a line parallel to y-axis. If this line cuts the graph in more than one point, we say that graph does not represent a function.
- A function is said to be monotonic on an interval if it is either increasing or decreasing on that interval.
- A function is called even function if $f(x) = f(-x)$, and odd function if $f(-x) = -f(x)$, $x, -x \in D_f$
- $f, g : X \to R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$

  $(f \, . \, g)x = f(x) \, . \, g(x)$, $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided $g(x) \neq 0$.
- A real function has the set of real number or one of its subsets both as its domain and as its range.

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/ index.shtml

http://en.wikipedia.org/wiki/functions

http://en.wikipedia.org/wiki/relations

TERMINAL EXERCISE

1. Given $A = \{a, b, c,\}, \quad B = \{2, 3\}$. Find the number of relations from A to B.

2. Given that $A = \{7, 8, 9\}$, $B = \{9, 10, 11\}$, $C = \{11, 12\}$ verify that

   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

3. Which of the following equations represent functions? In each of the case $x \in R$

   (a) $y = \dfrac{2x+3}{4-5x}, x \neq \dfrac{4}{5}$ (b) $y = \dfrac{3}{x}, x \neq 0$ (c) $y = \dfrac{3}{x^2-16}, x \neq 4, -4$

   (d) $y = \sqrt{x-1}, \; x \geq 1$ (e) $y = \dfrac{1}{x^2+1}$ (f) $x^2 + y^2 = 25$

4. Write domain and range of the following functions :

   $f_1 : \{(0,1), (2,3), (4,5)(6,7), ...(100, 101)\}$

   $f_2 : \{(-2, 4), (-4, 16), (-6, 36)........\}$

   $f_3 : \left\{(1,1), \left(\dfrac{1}{2}, -1\right), \left(\dfrac{1}{3}, 1\right)\left(\dfrac{1}{4}, -1\right), ...\right\}$

   $f_4 : \{........(3, 0), (-1, 2), (4, -1)\}$

   $f_5 : \{........(-3, 3), (-2, 2), (-1, 1)(0, 0)(1, 1), (2, 2), ...\}$

5. Write domain of the following functions :

   (a) $f(x) = x^3$ (b) $f(x) = \dfrac{1}{x^2-1}$ (c) $f(x) = \sqrt{3x+1}$

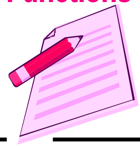
   (d) $f(x) = \dfrac{1}{\sqrt{(x+1)(x+6)}}$ (e) $f(x) = \dfrac{1}{\sqrt{(x-1)(2x-5)}}$

6. Write range of each of the following functions :

   (a) $y = 3x + 2, \; x \in R$ (b) $y = \dfrac{1}{x-2,}, \; x \in R - \{2\}$

   (c) $y = \dfrac{x-1}{x+1}, x \in \{0, 2, 3, 5, 7, 9\}$ (d) $y = \dfrac{2}{\sqrt{x}}, \quad x \in R^+$

   (All non-negative real values)



7. Draw the graph of each of the following functions :

(a) $y = x^2 + 3,\ x \in R$ (b) $y = \dfrac{1}{x-2},\ x \in R - \{2\}$

(c) $y = \dfrac{x-1}{x+1},\ x \in \{0, 2, 3, 5, 7, 9\}$ (d) $y = \dfrac{1}{\sqrt{x}},\ x \in R^+$.

8. Which of the following graphs represent a function ?

(a)

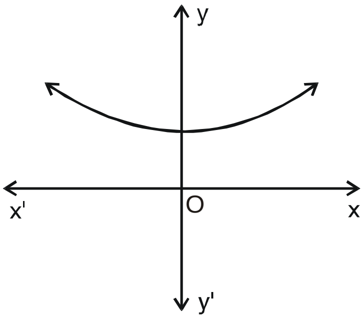


Fig. 2.36

(b)

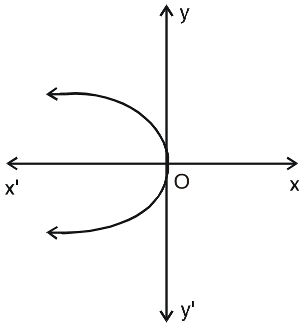


Fig. 2.37

(c)

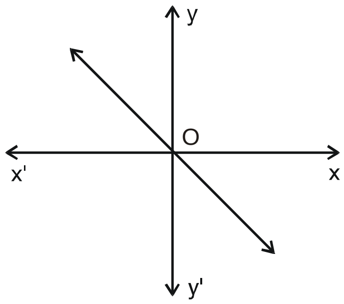


Fig. 2.38

(d)

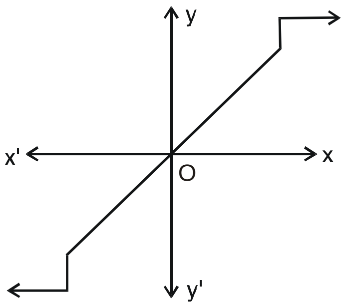


Fig. 2.39

(e)

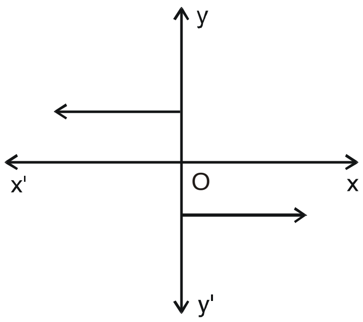


Fig. 2.40

(f)

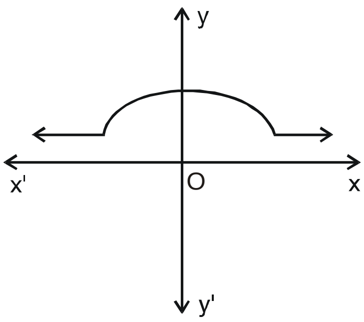


Fig. 2.41

(g)

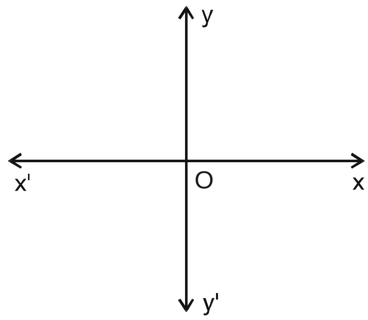


Fig. 2.42

(h)

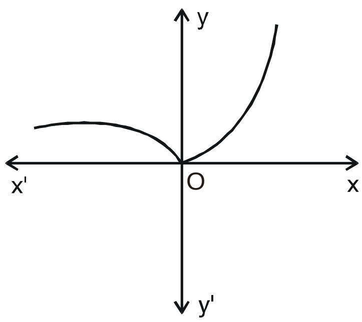


Fig. 2.43

9. Which of the following functions are rational functions ?

(a) $f(x)=\dfrac{2x-3}{x+2}, x\in R-\{-2\}$ (b) $f(x)=\dfrac{x}{\sqrt{x}}, x\in R^+$

(c) $f(x)=\dfrac{x+2}{x^2+4x+4}, x\in R-\{-2\}$ (d) $y=x, x\in R$

10. Which of the following functions are polynomial functions ?

(a) $f(x)=x^2+\sqrt{3}\,x+2$ (b) $f(x)=(x+2)^2$

(c) $f(x)=3-x+2x^3-x^4$ (d) $f(x)=\sqrt{x}+x-5,\ \ x\geq 0$

(e) $f(x)=\sqrt{x^2-4}, x\notin(-2,2)$

11. Which of the following functions are even or odd functions ?

(a) $f(x)=\sqrt{9-x^2}\ \ x\in[-3,3]$ (b) $f(x)=\dfrac{x^2-1}{x^2+1}$

(c) $f(x)=|x|$ (d) $f(x)=x-x^5$

(e)

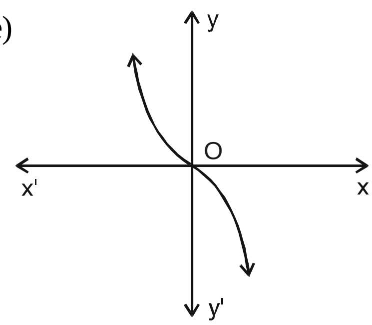


Fig. 2.44

(f)

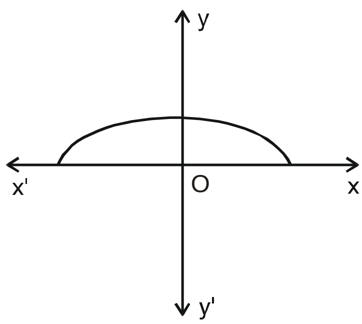


Fig. 2.45

(g)

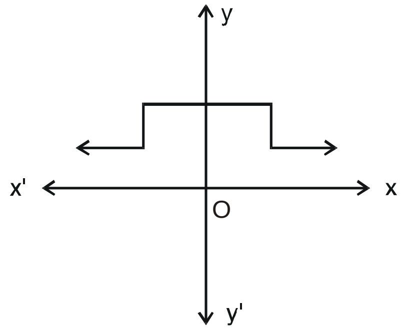


Fig. 2.46

12. Let 'f' be a function defined by $f(x) = 5x^2 + 2$, $x \in R$.
    (i) Find the image of 3 under f.
    (ii) Find $f(3) \times f(2)$
    (iii) Find x such that $f(x) = 22$

13. Let $f(x) = x + 2$ and $g(x) = 2x - 3$ be two real functions. Find the following functions
    (i) $f + g$ (ii) $f - g$
    (iii) $f \cdot g$ (iv) $\dfrac{f}{g}$

14. If $f(x) = (2x + 5)$, $g(x) = x^2 - 1$ are two real valued functions, find the following functions
    (i) $f + g$ (ii) $f - g$ (iii) $f\,g$
    (iv) $\dfrac{f}{g}$ (v) $\dfrac{g}{f}$

ANSWERS

CHECK YOUR PROGRESS 2.1

2. $\{(2,1),(4,1),(2,4),(4,4)\}$.

3. (i) $R = \{(4,2),(4,4),(6,2),(6,3),(8,2),(8,4),(10,2),(10,5)\}$.

   (ii) Domain of $R = \{4,6,8,10\}$. (iii) Range of $R = \{2,3,4,5\}$.

4. (i) $R = \{(1,8),(2,4)\}$. (ii) Domain of $R = \{1,2\}$.

   (iii) Range of $R = \{1,2\}\{8,4\}$

5. (i) $R = \{(2,4),(3,9),(5,25),(7,49),(11,121),(13,169)\}$

   Domain of $R = \{2,3,5,7,11,13\}$, Range of $R = \{4,9,25,49,121,169\}$.

6. (i) Domain of $R = \phi$ (ii) Domain of $R = \phi$ (iii) Range of $R = \phi$

7. x = 2, y = 3

8. {(-1,-1,-1), (-1,-1,1), (-1,1,-1), (-1,1,1), (1,-1,-1), (1,-1,1), (1,1,-1), (1,1,1)

9. A={a,b}, B={x,y}.

10. 18

CHECK YOUR PROGRESS 2.2

1. (a), (c), (f) 2. (a), (b) 3. (a), (d)

4. (a) Domain $= \{\sqrt{2}, \sqrt{5}, \sqrt{3}\}$, Range $= \{2, -1, 5\}$

   (b) Domain $= \{-3, -2, -1\}$, Range $= \left\{\frac{1}{2}\right\}$

   (c) Domain $= \{1, 0, 2, -1\}$, Range $= \{1, 0, 2, -1\}$

   (d) Domain $= \{$Deepak, Sandeep, Rajan$\}$, Range $= \{16, 28, 24\}$.

5. (a) Domain $= \{1, 2, 3\}$, Range $= \{4, 5, 6\}$

   (b) Domain $= \{1, 2, 3\}$, Range $= \{4\}$

   (c) Domain $= \{1, 2, 3\}$, Range $= \{1, 2, 3\}$

(d) Domain $= \{$Gagan, Ram, Salil$\}$, Range $= \{8, 9, 5\}$

(e) Domain $= \{a.b, c, d\}$, Range $= \{2, 4\}$

CHECK YOUR PROGRESS 2.3

1. (a) (i) Domain = Set of real numbers. (ii) Domain = Set of real numbers.

   (iii) Domain = Set of a real numbers.

   (b) (i) Domain $= R - \left\{\frac{1}{3}\right\}$ (ii) Domain $= R - \left\{-\frac{1}{4}, 5\right\}$

   (iii) Domain $= R - \{3, 5\}$ (iv) Domain $= R - \{3, 5\}$

   (c) (i) Domain $= \{x \in R : x \le 6\}$ (ii) Domain $= \{x \in R : x \ge -7\}$

   (iii) Domain $= \left\{x : x \in R, x \ge -\frac{5}{3}\right\}$

   (d) (i) Domain $= \{x : x \in R \text{ and } 3 \le x \le 5\}$

   (ii) Domain $= \{x : x \in R \;\; x \ge 3, x \le -5\}$

   (iii) Domain $= \{x : x \in R \;\; x \ge -3, x \le -7\}$

   (iv) Domain $= \{x : x \in R \;\; x \ge 3, x \le -7\}$

2. (a) (i) Range $= \{13, 25, 31, 7, 4\}$ (ii) Range $= \{19, 9, 33, 1\}$

   (iii) Range $= \{2, 4, 8, 14, 22\}$

   (b) (i) Range $= \{f(x) : -2 \le f(x) \le 2\}$ (ii) Range $= \{f(x) : 1 \le f(x) \le 10\}$

   (c) (i) Range $= \{f(x) : 1 \le f(x) \le 25\}$ (ii) Range $= \{f(x) : -6 \le f(x) \le 6\}$

   (iii) Range $= \{f(x) : 1 \le f(x) \le 5\}$ (iv) Range $= \{f(x) : 0 \le f(x) \le 5\}$

   (d) (i) Range = R (ii) Range = R (iii) Range = R

   (iv) Range $= \{f(x) : f(x) < 0\}$

   (v) Range $= \{f(x) : -1 \le f(x) < 0\}$

   (vi) Range $= \{f(x) : 0.5 \le f(x) < 0\}$ (vii) Range $= \{f(x) : f(x) > 0\}$

   (viii) Range : All values of $f(x)$ except values at $x = -5$.

CHECK YOUR PROGRESS 2.4

1. (i) No. (ii) Yes

2. (a)

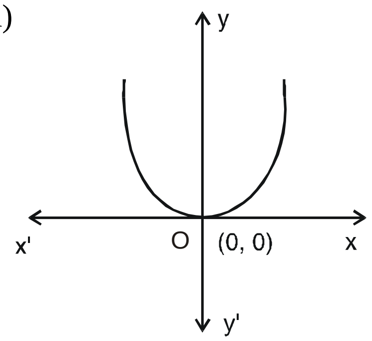


Fig. 2.47

(b)

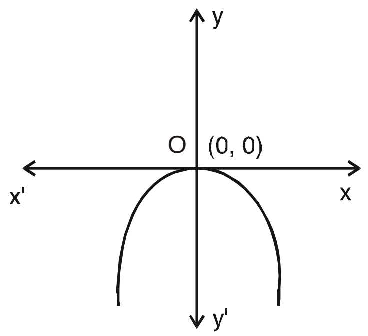


Fig. 2.48

(c)

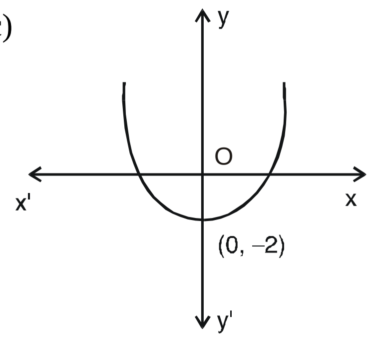


Fig. 2.49

(d)

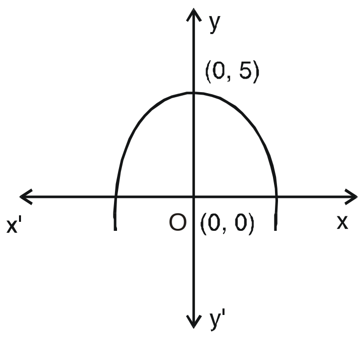


Fig. 2.50

(e)

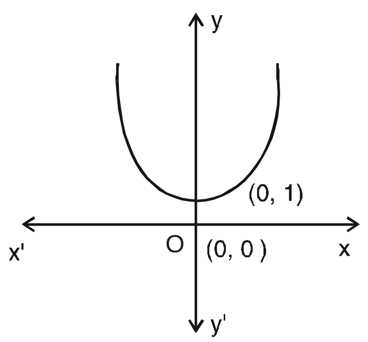


Fig. 2.51

(f)

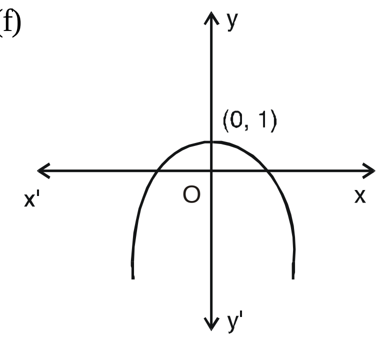


Fig. 2.52

3. (c), (d) and (e).

CHECK YOUR PROGRESS 2.5

1. v, vi, vii are true statements.
   (i), (ii), (iii), (iv) are false statement.

2. (b) (c) are even functions. (d) (e) (h) are odd functions.

3.

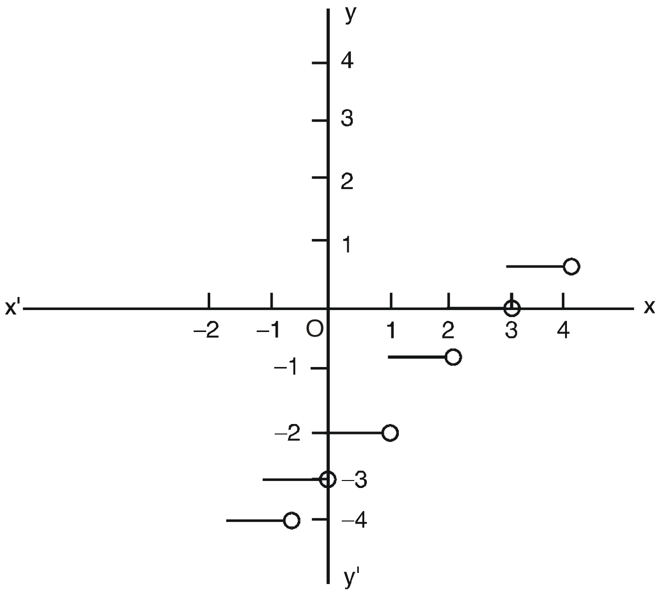


Fig. 2.53

4. (a) Polynomial function (b) Rational function. (c) Rational function.
(d) Rational function. (e) Rational function. (f) Rational function.
(g) Constant function.

CHECK YOUR PROGRESS 2.6

1. (i) 4 (ii) 25 (iii) -5

2. (f+g) x=3x-2, (f-g) x=4-x , (f.g) x= $2x^2 - x - 3$, $\left(\frac{f}{g}\right)x = \frac{x-1}{2x-3}$, $x \neq \frac{3}{2}$

TERMINAL EXERCISE

2. 2^6 i.e., 64.

3. (a), (b), (c), (d), (e) are functions.

4. f_1 – Domain $= \{0, 2, 4, 6, ...100\}$ Range $= \{1, 3, 5, 7, ...101\}$.

f_2 – Domain $= \{-2, -4, -6, ...\}$. Range $= \{4, 16, 36, ...\}$.

f_3 – Domain $= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$. Range $= \{1, -1\}$.

f_4 – Domain $= \{3, -1, 4\}$. Range $= \{0, 2, -1\}$.

f_5 – Domain $= \{...... -3, -2, -1, 0, 1, 2, 3, ...\}$. Range $= \{0, 1, 2, 3, ...\}$.

5. (a) Domain $=$ R.

(b) Domain $=$ R $- \{-1, 1\}$.

(c) Domain $= x \geq -\dfrac{1}{3} \quad \forall\, x \in$ R.

(d) Domain $= x \geq -1, x \leq -6$.

(e) Domain $= x \geq \dfrac{5}{2}, x \leq 1$.

6. (a) Range $=$ R

(b) Range $=$ All values of y except at $x = 2$.

(c) Range $= \left\{-1, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}\right\}$

(d) Range $=$ All values of y for $x > 0$

8. (a), (c), (e), (f), (h). Use hint given in check your progress 15.7, Q. No. 7 for the solution.

9. (a), (c)

10. (a), (b), (c)

11. Even functions : (a), (b), (c), (f), (g)

Odd functions : (d), (e)

12. (i) $f(3) = 47$ (ii) $f(3) \times f(2) = 1034$ (iii) $x = 2, -2$

13. (i) $f + g = 3x - 1$ (ii) $f - g = -x + 5$ (iii) $fg = 2x^2 + x - 6$

(iv) $f/g = \dfrac{x+2}{2x-3}, x \neq \dfrac{3}{2}$

14. (i) $f + g = x^2 + 2x + 4$ (ii) $f - g = -x^2 + 2x + 6$

(iii) $f \,.\, g = 2x^3 + 5x^2 - 2x - 5$ (iv) $f/g = \dfrac{2x+5}{x^2-1}, x \neq \pm 1$

(v) $g/f = \dfrac{x^2-1}{2x+5}, x \neq \dfrac{-5}{2}$

3

TRIGONOMETRIC FUNCTIONS-I

311en03

We have read about trigonometric ratios in our earlier classes.

Recall that we defined the ratios of the sides of a right triangle as follows :

$\sin\,\theta = \frac{c}{b}$, $\cos\,\theta = \frac{a}{b}$, $\tan\,\theta = \frac{c}{a}$

and $\text{cosec}\,\theta = \frac{b}{c}$, $\sec\,\theta = \frac{b}{a}$, $\cot\,\theta = \frac{a}{c}$

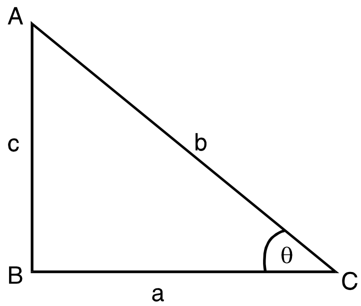


Fig.3.1

We also developed relationships between these trigonometric ratios as $\sin^2\theta + \cos^2\theta = 1$,

$\sec^2\theta = 1 + \tan^2\theta$, $\text{cosec}^2\theta = 1 + \cot^2\theta$

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type

y = sin x, y = cos x, y = tan x, y = cot x, y = sec x, y = cosec x, y = a sin x, y = b cos x, etc., where *x*, *y* are real numbers. We shall draw the graphs of functions of the type

y = sin x, y = cos x, y = tan x, y = cotx, y = secx, and y = cosecx y = a sin x, y = a cos x.

OBJECTIVES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;
- state the formula $\ell = r\,\theta$ where r and θ have their usual meanings;
- solve problems using the relation $\ell = r\,\theta$;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions; and
- interpret the graphs of trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Definition of an angle.
- Concepts of a straight angle, right angle and complete angle.
- Circle and its allied concepts.
- Special products : $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$
- Knowledge of Pythagoras Theorem and Pythagorean numbers.

3.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clock-wise or anticlock-wise.

3.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

3.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

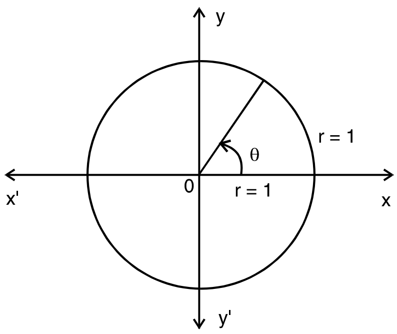


Fig. 3.2

Note : A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

3.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π $(\because r = 1)$ subtend an angle of 2π radians.

Hence 2π radians $= 360°$, $\Rightarrow$ π radians $= 180°$, $\Rightarrow$ $\frac{\pi}{2}$ radians $= 90°$

$$\Rightarrow \quad \frac{\pi}{4} \text{ radians } = 45° \quad \Rightarrow \quad 1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ} = \left(\frac{180}{\pi}\right)^{\circ}$$

$$\text{or } 1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} \text{ radians}$$

Example 3.1 Convert

(i) 90° into radians (ii) 15° into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution :

(i) $1° = \frac{2\pi}{360}$ radians

$$\Rightarrow \quad 90° = \frac{2\pi}{360} \times 90 \text{ radians} \quad \text{or} \quad 90° = \frac{\pi}{2} \text{ radians}$$

(ii) $15° = \frac{2\pi}{360} \times 15$ radians or $15° = \frac{\pi}{12}$ radians

(iii) $1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ}$, $\frac{\pi}{6}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{6}\right)^{\circ}$

$\frac{\pi}{6}$ radians $= 30°$

(iv) $\frac{\pi}{10}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{10}\right)^{\circ}$, $\frac{\pi}{10}$ radians $= 18°$

CHECK YOUR PROGRESS 3.1

1. Convert the following angles (in degrees) into radians :

   (i) 60° (ii) 15° (iii) 75° (iv) 105° (v) 270°

2. Convert the following angles into degrees:

   (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{\pi}{20}$ (iv) $\frac{\pi}{60}$ (v) $\frac{2\pi}{3}$

3. The angles of a triangle are 45°, 65° and 70°. Express these angles in radians

4. The three angles of a quadrilateral are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$. Find the fourth angle in radians.

5. Find the angle complementary to $\frac{\pi}{6}$.

3.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of 2½ radians willbe subtended if arc is 2½ times the radius.

All this can be read from the following table :

Length of the arc (*l*)	Angle subtended at the centre of the circle θ (in radians)
r	1
2r	2
(2½)r	2½
4r	4

Therefore, $\theta = \frac{\ell}{r}$ or $\ell = r\,\theta$, where r = radius of the circle,

θ = angle substended at the centre in radians, and ℓ = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{\ell}{r}$ is valid only when the angle is measured in radians.

Example 3.2 Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : $\ell = 10cm$ and $r = 35$ cm.

$$\theta = \frac{\ell}{r} \text{ radians} \quad \text{or} \quad \theta = \frac{10}{35} \text{ radians , or} \quad \theta = \frac{2}{7} \text{ radians}$$

Example 3.3 A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solution : Angle θ is given in degrees. To apply the formula $\ell = r\,\theta, \theta$ must be changed to radians.

$$\theta = 45° = 45 \times \frac{\pi}{180} \text{ radians} \quad \text{....(1)} \quad = \frac{\pi}{4} \text{ radians}$$

$$\ell = 500 \text{ m} \quad \text{....(2)}$$

$$\ell = r\theta \text{ gives } r = \frac{\ell}{\theta} \qquad \therefore \qquad r = \frac{500}{\frac{\pi}{4}} \text{ m} \qquad \text{[using (1) and (2)]}$$

$$= 500 \times \frac{4}{\pi} \text{ m}, \; = 2000 \times 0.32 \text{ m} \left(\frac{1}{\pi} = 0.32\right), \; = 640 \text{ m}$$

Example 3.4 A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\frac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60} \text{ km} = \frac{1}{4} \text{ km}$$

$$\therefore \qquad \text{We have } \ell = \frac{1}{4} \text{ km and } r = \frac{5}{6} \text{ km}$$

$$\therefore \qquad \theta = \frac{\ell}{r} = \frac{\frac{1}{4}}{\frac{5}{6}} \text{ radians} = \frac{3}{10} \text{ radians}$$

CHECK YOUR PROGRESS 3.2

1. Express the following angles in radians :

   (a) $30°$ (b) $60°$ (c) $150°$

2. Express the following angles in degrees :

   (a) $\frac{\pi}{5}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$

3. Find the angle in radians and in degrees subtended by an arc of length 2.5 cm at the centre of a circle of radius 15 cm.

4. A train is travelling at the rate of 20 km per hour on a circular track. Through what angle will it turn in 3 seconds if the radius of the track is $\frac{1}{12}$ of a km?.

5. A railroad curve is to be laid out on a circle. What should be the radius of the circular track if the railroad is to turn through an angle of 60° in a distance of 100 m?

6. Complete the following table for l, r, θ having their usual meanings.

	l	r	θ
(a)	1.25m	....	135°
(b)	30 cm	....	$\frac{\pi}{4}$
(c)	0.5 cm	2.5 m	....
(d)	....	6 m	120°
(e)	....	150 cm	$\frac{\pi}{15}$
(f)	150 cm	40 m	....
(g)	....	12 m	$\frac{\pi}{6}$
(h)	1.5 m	0.75 m	....
(i)	25 m	....	75°

3.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between *0* and 2π, there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the coordinates of the point *P*.

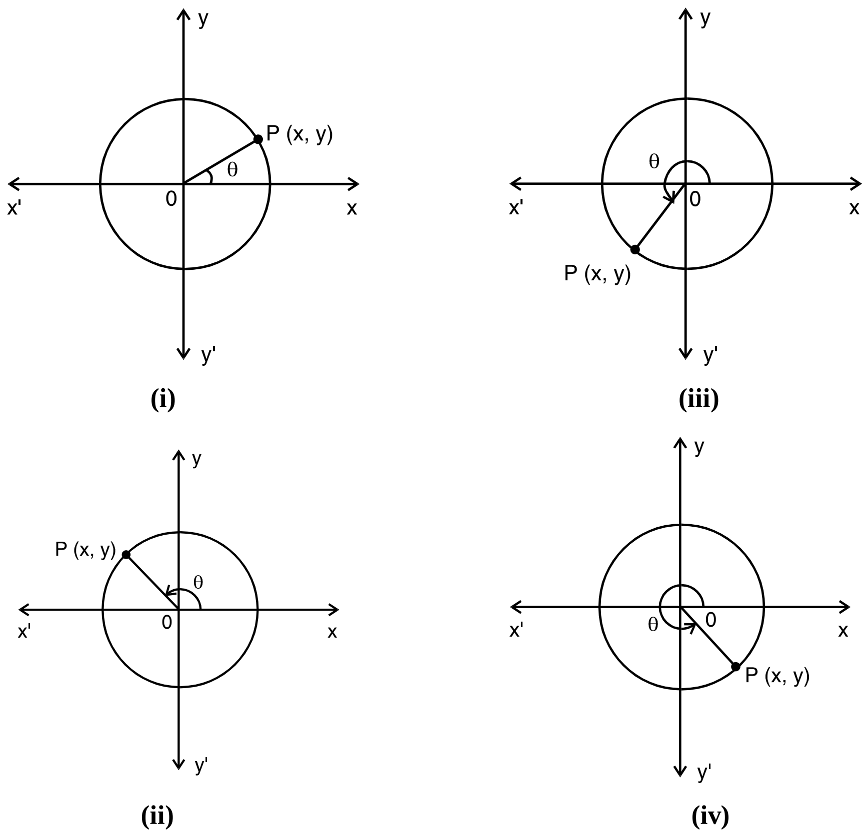


Fig. 3.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \frac{\pi}{2}$, then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (*x*, *y*). The values of *x* and *y* will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point *P* (on the unit circle) and the corresponding coordinates (*x*, *y*), we define trigonometric functions as :

$$\sin\theta = y, \ \cos\theta = x$$

$$\tan\theta = \frac{y}{x} \ (\text{for } x \neq 0), \ \cot\theta = \frac{x}{y} \ (\text{for } y \neq 0)$$

$$\sec\theta = \frac{1}{x} \ (\text{for } x \neq 0), \ \text{cosec}\,\theta = \frac{1}{y} \ (\text{for } y \neq 0)$$

Now let the point *P* moves from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I	quadrant, both *x* and *y* are positve.
II	quadrant, *x* will be negative and *y* will be positive.
III	quadrant, *x* as well as *y* will be negative.
IV	quadrant, *x* will be positive and *y* will be negative.

or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and θ, the real number generated by the position of the point, then $\sin\theta$ = y and $\cos\theta$ = x. This means the coordinates of the point P can also be written as ($\cos\theta$, $\sin\theta$)

From Fig. 3.4, you can easily see that the values of x will be between –1 and + 1 as *P* moves on the unit circle. Same will be true for *y* also.

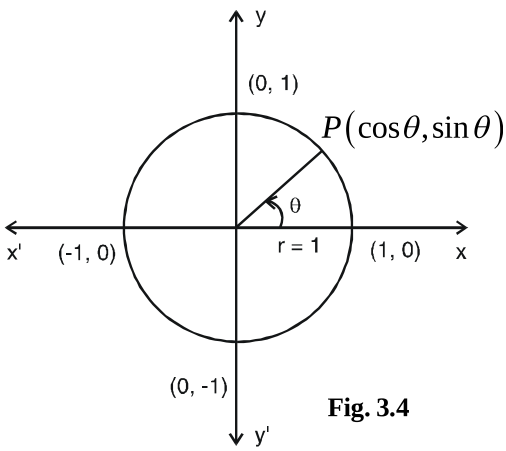


Fig. 3.4

Thus, for all P on the unit circle

$$-1 \le x \le 1 \qquad \text{and } -1 \le y \le 1$$

Thereby, we conclude that for all real numbers θ

$$-1 \le \cos\theta \le 1 \text{ and } \qquad -1 \le \sin\theta \le 1$$

In other words, $\sin\theta$ and $\cos\theta$ can not be numerically greater than 1

Example 3.5 What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution :

(i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 3.6 Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution : (i) From Fig. 3.5, we can see that the coordinates of the point A are (0,1)

$\therefore\ \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

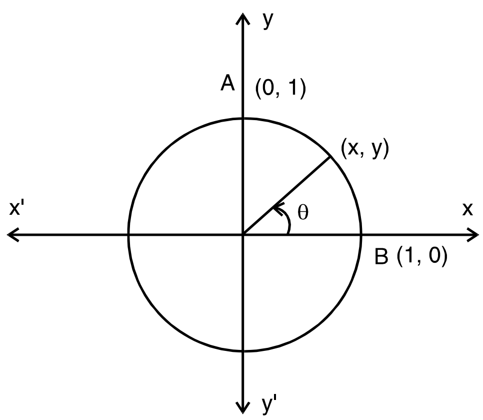


Fig. 3.5

(ii) Coordinates of the point B are (1, 0) $\qquad \therefore\ \cos 0 = 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined , Thus $\tan\frac{\pi}{2}$ is not defined.

Example 3.7 Write the minimum and maximum values of $\cos\theta$.

Solution : We know that $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is -1.

CHECK YOUR PROGRESS 3.3

1. What will be the sign of the following ?

(i) $\cos\frac{2\pi}{3}$ (ii) $\tan\frac{5\pi}{6}$ (iii) $\sec\frac{2\pi}{3}$

(iv) $\sec\frac{35\pi}{18}$ (v) $\tan\frac{25\pi}{18}$ (vi) $\cot\frac{3\pi}{4}$

(vii) $\operatorname{cosec}\frac{8\pi}{3}$ (viii) $\cot\frac{7\pi}{8}$

2. Write the value of each of the following :

(i) $\cos\frac{\pi}{2}$ (ii) $\sin 0$ (iii) $\cos\frac{2\pi}{3}$ (iv) $\tan\frac{3\pi}{4}$

(v) $\sec 0$ (vi) $\tan\frac{\pi}{2}$ (vii) $\tan\frac{3\pi}{2}$ (viii) $\cos 2\pi$

3.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$, $y = \sin\theta$

As $\tan\theta = \frac{y}{x}, \ (x \ne 0)$, $= \frac{\sin\theta}{\cos\theta}, \ \theta \ne \frac{n\pi}{2}$

and $\cot\theta = \frac{x}{y}, \ (y \ne 0)$,

i.e., $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \ (\theta \ne n\pi)$

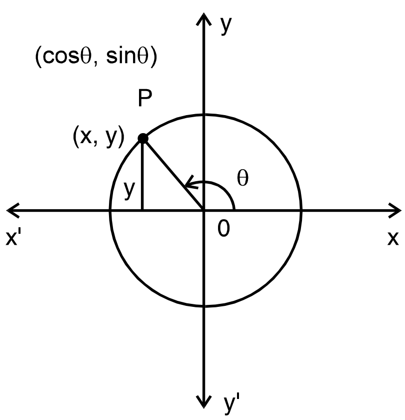


Fig. 3.6

Similarly, $\sec\theta = \frac{1}{\cos\theta}$ $\left(\theta \ne \frac{n\pi}{2}\right)$

and $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$ $\quad (\theta \neq n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$, i.e., $(\cos \theta)^2 + (\sin \theta)^2 = 1$

or, $\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$

Note : $(\cos \theta)^2$ is written as $\cos^2 \theta$ and $(\sin \theta)^2$ as $\sin^2 \theta$

Again $x^2 + y^2 = 1$ or $1 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{1}{x}\right)^2$, for $x \neq 0$

or, $1 + (\tan \theta)^2 = (\sec \theta)^2$, i.e. $\mathbf{\sec^2 \theta = 1 + \tan^2 \theta}$

Similarly, $\mathbf{\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta}$

Example 3.8 Prove that $\sin^4 \theta + \cos^4 \theta = 1 - 2\sin^2 \theta \cos^2 \theta$

Solution : L.H.S. $= \sin^4 \theta + \cos^4 \theta$

$$= \sin^4 \theta + \cos^4 \theta + 2\sin^2 \theta \cos^2 \theta - 2\sin^2 \theta \cos^2 \theta$$

$$= \left(\sin^2 \theta + \cos^2 \theta\right)^2 - 2\sin^2 \theta \cos^2 \theta$$

$$= 1 - 2\sin^2 \theta \cos^2 \theta \quad \left(\because \sin^2 \theta + \cos^2 \theta = 1\right), = \text{R.H.S.}$$

Example 3.9 Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution : L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}} = \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} = \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{R.H.S.}$$

Example 3.10 If $\sin\theta = \dfrac{21}{29}$, prove that $\sec\theta + \tan\theta = -2\dfrac{1}{2}$, given that θ lies in the second quadrant.

Solution : $\sin\theta = \dfrac{21}{29}$ Also, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \cos^2\theta = 1-\sin^2\theta = 1-\frac{441}{841} = \frac{400}{841} = \left(\frac{20}{29}\right)^2$$

$$\Rightarrow \quad \cos\theta = \frac{-20}{29} \quad (\cos\theta \text{ is negative as } \theta \text{ lies in the second quardrant})$$

$$\therefore \quad \tan\theta = \frac{-21}{20} \quad (\tan\theta \text{ is negative as } \theta \text{ lies in the second qudrant})$$

$$\therefore \quad \sec\theta + \tan\theta = \frac{-29}{20} + \frac{-21}{20} = \frac{-29-21}{20}, \; = \frac{-5}{2} = -2\frac{1}{2} = \text{R.H.S.}$$

CHECK YOUR PROGRESS 3.4

1. Prove that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

2. If $\tan\theta = \frac{1}{2}$, find the other five trigonometric functions. where θ lies in the first quardrant)

3. If $\operatorname{cosec}\theta = \frac{b}{a}$, find the other five trigonometric functions, if θ lies in the first quardrant.

4. Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

5. If $\cot\theta + \operatorname{cosec}\theta = 1.5$, show that $\cos\theta = \frac{5}{13}$

6. If $\tan\theta + \sec\theta = m$, find the value of $\cos\theta$

7. Prove that $(\tan A + 2)(2\tan A + 1) = 5\tan A + 2\sec^2 A$

8. Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

9. Prove that $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$

10. Prove that $\dfrac{\tan\theta}{1+\cos\theta} + \dfrac{\sin\theta}{1-\cos\theta} = \cot\theta + \operatorname{cosec}\theta \cdot \sec\theta$

11. If $\sec x = \frac{13}{5}$ and x lies in the fourth quadrant, Find other five trigonometric ratios.

3.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$ are summarised below in the form of a table :

MODULE - I
Sets, Relations and Functions

Notes

Function ↓ \ Real Numbers → (θ)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function : , $\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Example 3.11 Find the value of the following :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3}$ (b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}$

Solution :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}, = 4(1)^2 - (2)^2 - \left(\frac{1}{2}\right)^2, = 4 - 4 - \frac{1}{4} = -\frac{1}{4}$

Example 3.12 If $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$, verify that $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Solution : L.H.S. $= \cos(A+B) = \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos\frac{\pi}{2} = 0$

R.H.S. $= \cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

$\therefore$ L.H.S. = 0 = R.H.S.

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

CHECK YOUR PROGRESS 3.5

1. Find the value of

(i) $\sin^2\frac{\pi}{6}+\tan^2\frac{\pi}{4}+\tan^2\frac{\pi}{3}$ (ii) $\sin^2\frac{\pi}{3}+\operatorname{cosec}^2\frac{\pi}{6}+\sec^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{3}-\sin\frac{2\pi}{3}\sin\frac{\pi}{3}$ (iv) $4\cot^2\frac{\pi}{3}+\operatorname{cosec}^2\frac{\pi}{4}+\sec^2\frac{\pi}{3}\tan^2\frac{\pi}{4}$

(v) $\left(\sin\frac{\pi}{6}+\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\right)+\frac{1}{4}$

2. Show that

$$\left(1+\tan\frac{\pi}{6}\tan\frac{\pi}{3}\right)+\left(\tan\frac{\pi}{6}-\tan\frac{\pi}{3}\right)=\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{3}$$

3. Taking $A=\frac{\pi}{3}, B=\frac{\pi}{6}$, verify that

(i) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

4. If $\theta=\frac{\pi}{4}$, verify: (i) $\sin 2\theta=2\sin\theta\cos\theta$

(ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta \quad =2\cos^2\theta-1 \quad =1-2\sin^2\theta$

5. If $A=\frac{\pi}{6}$, verify that, (i) $\cos 2A=2\cos^2 A-1$

(ii) $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$ (iii) $\sin 2A=2\sin A\cos A$

3.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

3.4.1 Variations of sinθ as θ Varies Continuously From 0 to 2π.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O and radius OP = unity, draw a circle. Let OP starting from OX and moving in anticlockwise direction make an angle θ with the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP=1$.

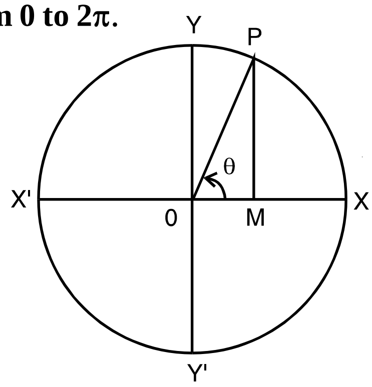


Fig. 3.7

$\therefore$ The variations of sin θ are the same as those of MP.

I Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$

PM is positive and increases from 0 to 1.

$\therefore$ sin θ is positive.

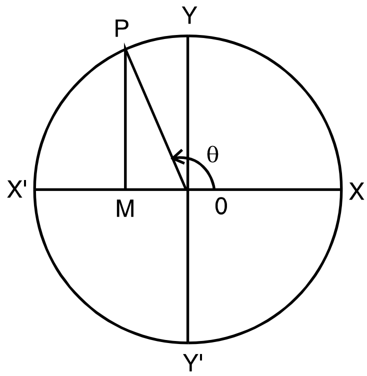


Fig. 3.8

II Quadrant $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point P is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus sin θ is positive.

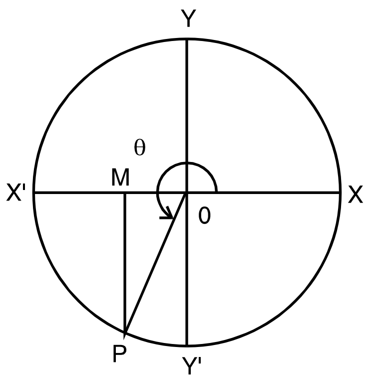


Fig. 3.9

III Quadrant $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point P can move in the third quadrant only. Hence $PM = y$ is negative and decreases from 0 to –1 as θ varies from π to $\frac{3\pi}{2}$. In this interval sin θ decreases from 0 to –1. In this interval sin θ is negative.

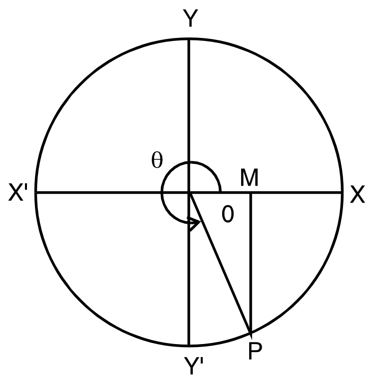


Fig. 3.10

IV Quadrant $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from -1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus sin θ is negative in this interval.

3.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let *X′OX* and *Y′OY* be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of sin θ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41, \frac{1}{\sqrt{2}} = .707, \frac{\sqrt{3}}{2} = .87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin θ	0	.5	.87	1	.87	.5	0	−.5	−.87	−1	−.87	−.5	0

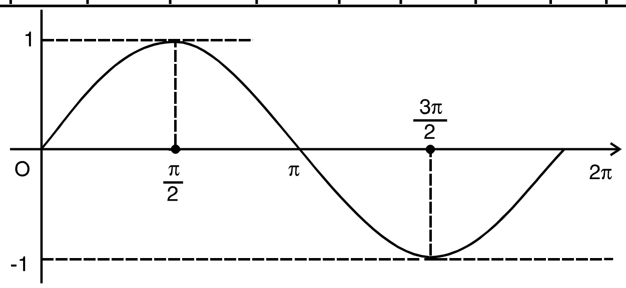


Fig. 3.11

Some Observations

(i) Maximum value of sin θ is 1. (ii) Minimum value of sin θ is −1.

(iii) It is continuous everywhere. (iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π.

It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. With the help of the graph drawn in Fig. 6.11 we can always draw another graph y = sin θ in the interval of [2π, 4π] (see Fig. 3.12)

What do you observe ?

The graph of y = sin θ in the interval [2π, 4π] is the same as that in 0 to 2π. Therefore, this graph can be drawn by using the property sin $(2\pi + \theta) = \sin\theta$. Thus, sin θ repeats itself when θ is increased by 2π. This is known as the periodicity of sin θ.

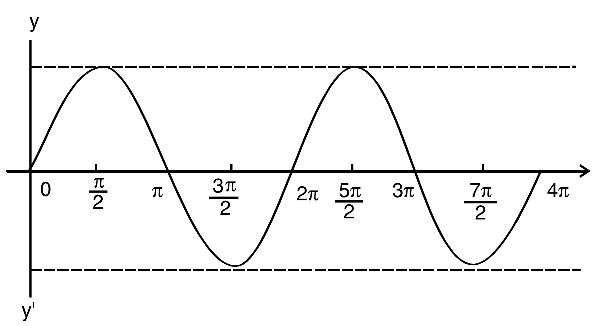


Fig. 3.12

We shall discuss in details the periodicity later in this lesson.

Example 3.13 Draw the graph of $y = \sin 2\theta$ in the interval 0 to π.

Solution :

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ :	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2\theta$:	0	.87	1	.87	0	–.87	–1	–.87	0

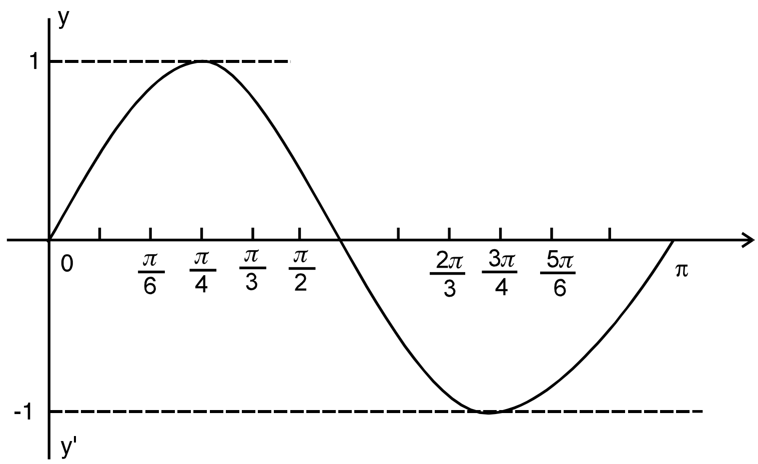


Fig. 3.13

The graph is similar to that of $y = \sin\theta$

Some Observations

1. The other graphs of $\sin\theta$, like $a \sin\theta$, $3 \sin 2\theta$ can be drawn applying the same method.
2. Graph of $\sin\theta$, in other intervals namely $[4\pi, 6\pi]$, $[-2\pi, 0]$, $[-4\pi, -2\pi]$, can also be drawn easily. This can be done with the help of properties of allied angles: $\sin(\theta + 2\pi) = \sin\theta$, $\sin(\theta - 2\pi) = \sin\theta$. i.e., θ repeats itself when increased or decreased by 2π.

CHECK YOUR PROGRESS 3.6

1. What are the maximum and minimum values of $\sin\theta$ in $[0, 2\pi]$?
2. Explain the symmetry in the graph of $\sin\theta$ in $[0, 2\pi]$
3. Sketch the graph of $y = 2\sin\theta$, in the interval $[0, \pi]$

4. For what values of θ in $[\pi, 2\pi]$, $\sin\theta$ becomes, (a) $\frac{-1}{2}$ (b) $\frac{-\sqrt{3}}{2}$

5. Sketch the graph of $y = \sin x$ in the interval of $[-\pi, \pi]$

3.4.3 Graph of cosθ as θ Varies From 0 to 2π

As in the case of $\sin\theta$, we shall also discuss the changes in the values of $\cos\theta$ when θ assumes values in the intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

I Quadrant : In the interval $\left[0, \frac{\pi}{2}\right]$, point *P* lies in the first quadrant, therefore, $OM = x$ is positive but decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{2}$.

Thus in this interval $\cos\theta$ decreases from 1 to 0.

$\therefore$ $\cos\theta$ is positive in this quadrant.

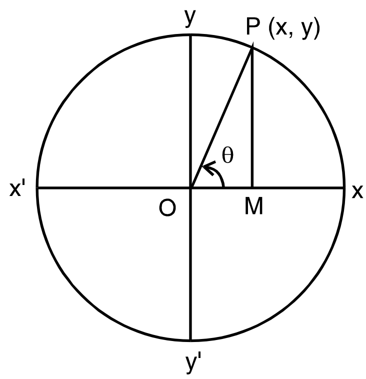


Fig. 3.14

II Quadrant : In the interval $\left[\frac{\pi}{2}, \pi\right]$, point *P* lies in the second quadrant and therefore point M lies on the negative side of x-axis. So in this case $OM = x$ is negative and decreases from 0 to -1 as θ increases from $\frac{\pi}{2}$ to π. Hence in this inverval $\cos\theta$ decreases from 0 to -1.

$\therefore$ $\cos\theta$ is negative.

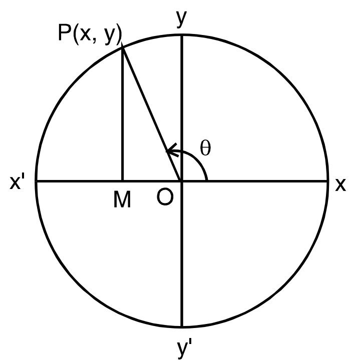


Fig. 3.15

III Quadrant : In the interval $\left[\pi, \frac{3\pi}{2}\right]$, point *P* lies in the third quadrant and therefore, $OM = x$ remains negative as it is on the negative side of x-axis. Therefore $OM = x$ is negative but increases from–1 to 0 as θ increases from π to $\frac{3\pi}{2}$. Hence in this interval $\cos\theta$ increases from -1 to 0.

$\therefore$ $\cos\theta$ is negative.

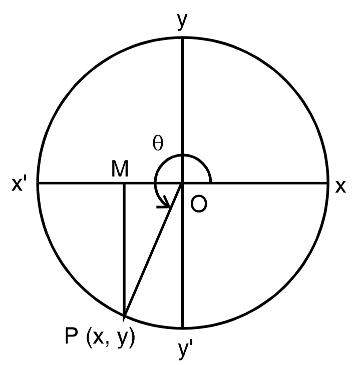


Fig. 3.16

IV Quadrant : In the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, point P lies in the fourth quadrant and M moves on the positive side of x-axis. Therefore $OM = x$ is positive. Also it increases from 0 to 1 as θ increases from $\frac{3\pi}{2}$ to 2π.

Thus in this interval $\cos\theta$ increases from 0 to 1.

$\therefore\quad \cos\theta$ is positive.

Let us tabulate the values of cosines of some suitable values of θ.

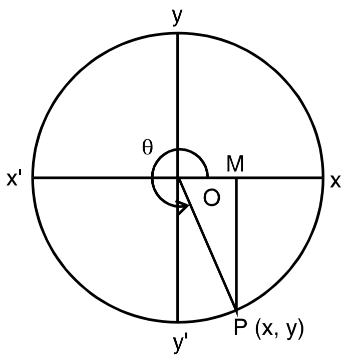


Fig. 3.17

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	.87	.5	0	0.5	−.87	−1	−.87	−.5	0	0.5	.87	1

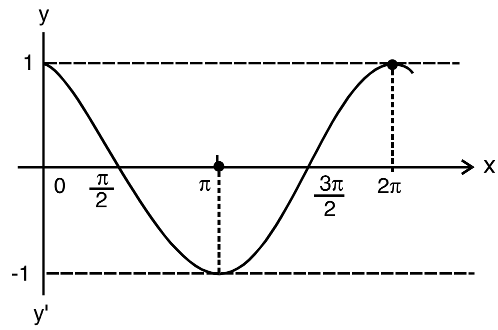


Fig. 3.18

Let $X'OX$ and $Y'OY$ be the axes. Values of θ are measured along x-axis and those of $\cos\theta$ along y-axis.

Some observations

(i) Maximum value of $\cos\theta = 1$. (ii) Minimum value of $\cos\theta = -1$.

(iii) It is continuous everywhere.

(iv) $\cos(\theta + 2\pi) = \cos\theta$. Also $\cos(\theta - 2\pi) = \cos\theta$. Cos θ repeats itself when θ is increased or decreased by 2π. It is called periodicity of $\cos\theta$. We shall discuss in details about this in the later part of this lesson.

(v) Graph of $\cos\theta$ in the intervals $[2\pi, 4\pi]$ $[4\pi, 6\pi]$ $[-2\pi, 0]$, will be the same as in $[0, 2\pi]$.

Example 3.14 Draw the graph of $\cos\theta$ as θ varies from $-\pi$ to π. From the graph read the values of θ when $\cos\theta = \pm 0.5$.

Solution :

θ :	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$:	–1.0	–0.87	–0.5	0	.50	–.87	1.0	0.87	0.5	0	–0.5	–0.87	–1

$$\cos\theta = 0.5$$

when $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

$$\cos\theta = -0.5$$

when $\theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

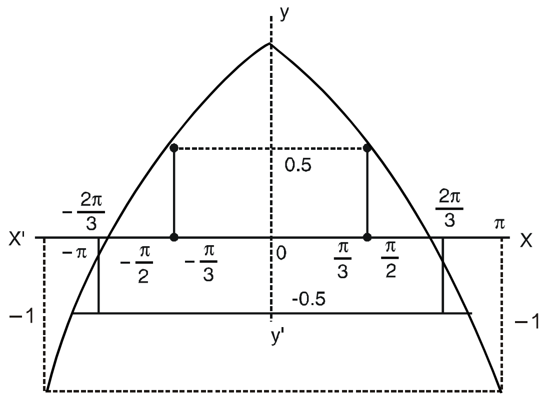


Fig. 3.19

Example 3.15 Draw the graph of $\cos 2\theta$ in the interval 0 to π.

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2\theta$	1	0.5	0	–0.5	–1	–0.5	0	0.5	1

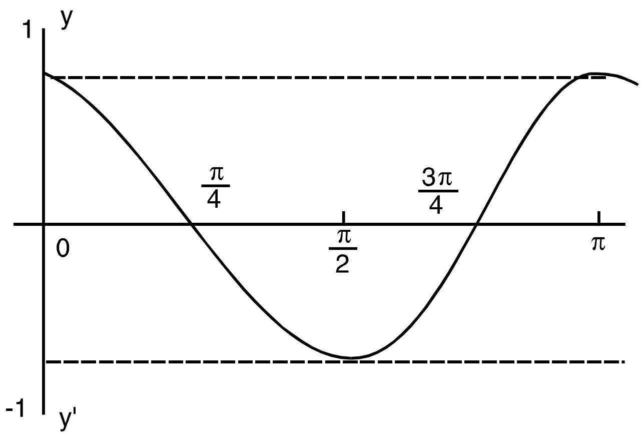


Fig. 3.20

CHECK YOUR PROGRESS 3.7

1. (a) Sketch the graph of $y = \cos\theta$ as θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$.

(b) Draw the graph of $y = 3\cos\theta$ as θ varies from 0 to 2π.

(c) Draw the graph of $y = \cos 3\theta$ from $-\pi$ to π and read the values of θ when $\cos\theta = 0.87$ and $\cos\theta = -0.87$.

(d) Does the graph of $y = \cos\theta$ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ lie above x-axis or below x-axis?

(e) Draw the graph of $y = \cos\theta$ in $[2\pi, 4\pi]$

3.4.4 Graph of tan θ as θ Varies from 0 to 2π

In I Quadrant : $\tan\theta$ can be written as $\dfrac{\sin\theta}{\cos\theta}$

Behaviour of $\tan\theta$ depends upon the behaviour of $\sin\theta$ and $\dfrac{1}{\cos\theta}$

In I quadrant, $\sin\theta$ increases from 0 to 1, $\cos\theta$ decreases from 1 to 0

But $\dfrac{1}{\cos\theta}$ increases from 1 indefintely (and write it as increasses from 1 to ∞) $\tan\theta > 0$

$\therefore$ $\tan\theta$ increases from 0 to ∞. (See the table and graph of $\tan\theta$).

In II Quadrant : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 1 to 0.

$\cos\theta$ decreases from 0 to -1.

$\tan\theta$ is negative and increases from $-\infty$ to 0

In III Quadrant : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 0 to -1

$\cos\theta$ increases from -1 to 0

$\therefore$ $\tan\theta$ is positive and increases from 0 to ∞

In IV Quadrant : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sin\theta$ increases from -1 to 0

$\cos\theta$ increases from 0 to 1

$\tan\theta$ is negative and increases form $-\infty$ to 0

Graph of tan θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0°$	$\frac{\pi}{2}+0°$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0°$	$\frac{3\pi}{2}+0°$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$	0	.58	1.73	$+\infty$	-1.73	-.58	0	.58	1.73	$+\infty$	$-\infty$	-1.73	-.58	0	0

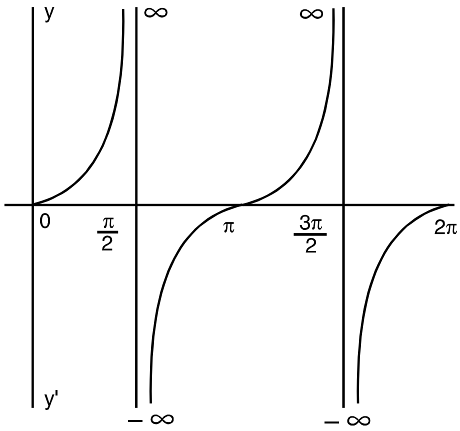


Fig. 3.21

Observations

(i) $\tan(180° + \theta) = \tan\theta$. Therefore, the complete graph of $\tan\theta$ consists of infinitely many repetitions of the same to the left as well as to the right.

(ii) Since $\tan(-\theta) = -\tan\theta$, therefore, if $(\theta, \tan\theta)$ is any point on the graph then $(-\theta, -\tan\theta)$ will also be a point on the graph.

(iii) By above results, it can be said that the graph of $y = \tan\theta$ is symmetrical in opposite quadrants.

(iv) $\tan\theta$ may have any numerical value, positve or negative.

(v) The graph of $\tan\theta$ is discontinuous (has a break) at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

(vi) As θ passes through these values, $\tan\theta$ suddenly changes from $+\infty$ to $-\infty$.

3.4.5 Graph of cot θ as θ Varies From 0 to 2π

The behaviour of $\cot\theta$ depends upon the behaviour of $\cos\theta$ and $\frac{1}{\sin\theta}$ as $\cot\theta = \cos\theta\frac{1}{\sin\theta}$

We discuss it in each quadrant.

I Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 1 to 0

$\sin\theta$ increases from 0 to 1

$\therefore$ $\cot\theta$ also decreases from $+\infty$ to 0 but $\cot\theta > 0$.

II Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 0 to -1

$\sin\theta$ decreases from 1 to 0

$\Rightarrow \quad \cot\theta < 0$ or $\cot\theta$ decreases from 0 to $-\infty$

III Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from -1 to 0

$\sin\theta$ decreases from 0 to -1

$\therefore$ $\cot\theta$ decreases from $+\infty$ to 0.

IV Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from 0 to 1

$\sin\theta$ increases from -1 to 0

$\therefore$ $\cot\theta < 0$

$\cot\theta$ decreases from 0 to $-\infty$

Graph of $\cot\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	∞	1.73	.58	0	-.58	-1.73	$-\infty$	$+\infty$	1.73	.58	0	-.58	-1.73	$-\infty$

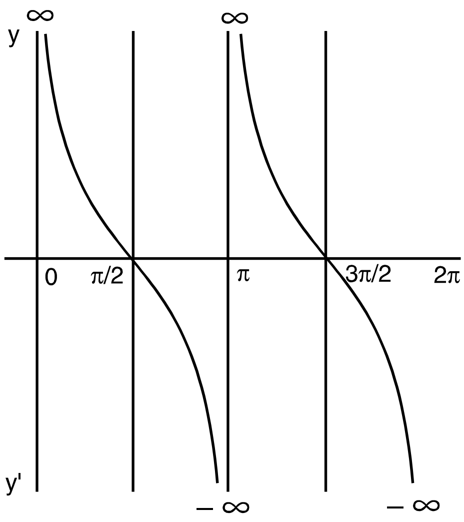


Fig. 3.22

Observations

(i) Since $\cot(\pi+\theta) = \cot\theta$, the complete graph of $\cot\theta$ consists of the portion from $\theta = 0$ to $\theta = \pi$ or $\theta = \dfrac{\pi}{2}$ to $\theta = \dfrac{3\pi}{2}$.

(ii) $\cot\,\theta$ can have any numerical value - positive or negative.

(iii) The graph of $\cot\,\theta$ is discontinuous, i.e. it breaks at 0, π, 2π,.

(iv) As θ takes values 0, π, 2π, $\cot\,\theta$ suddently changes from $-\infty$ to $+\infty$

CHECK YOUR PROGRESS 3.8

1. (a) What is the maximum value of $\tan\,\theta$?

   (b) What changes do you observe in $\tan\,\theta$ at $\frac{\pi}{2}, \frac{3\pi}{2}$?

   (c) Draw the graph of $y = \tan\,\theta$ from $-\pi$ to π. Find from the graph the value of θ for which $\tan\theta = 1.7$.

2. (a) What is the maximum value of $\cot\,\theta$?

   (b) Find the value of θ when $\cot\,\theta = -1$, from the graph.

3.4.6 To Find the Variations And Draw The Graph of sec θ As θ Varies From 0 to 2π.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O, draw a circle of unit radius.

Let P be any point on the circle. Join OP and draw $PM \perp X'OX$.

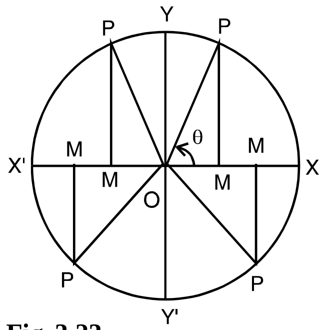


Fig. 3.23

$$\sec\theta = \frac{OP}{OM} = \frac{1}{OM}$$

$\therefore$ Variations will depend upon OM.

I Quadrant : sec θ is positive as OM is positive.

Also $\sec 0 = 1$ and $\sec\frac{\pi}{2} = \infty$ when we approach $\frac{\pi}{2}$ from the right.

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, $\sec\theta$ increases from 1 to ∞.

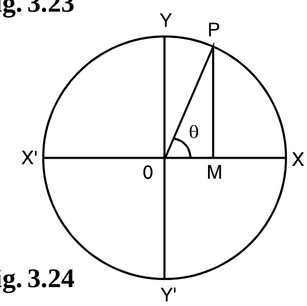


Fig. 3.24

II Quadrant : $\sec\theta$ is negative as OM is negative.

$\sec\frac{\pi}{2} = -\infty$ when we approach $\frac{\pi}{2}$ from the left. Also $\sec\pi = -1$.

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, $\sec\theta$ changes from $-\infty$ to -1.

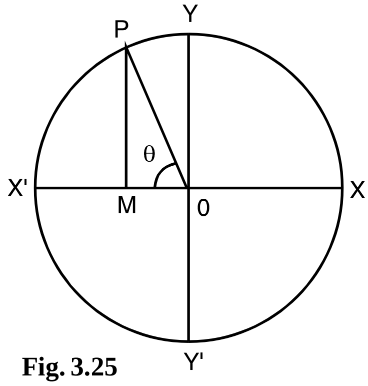


Fig. 3.25

It is observed that as θ passes through $\frac{\pi}{2}$, sec θ changes from $+\infty$ to $-\infty$.

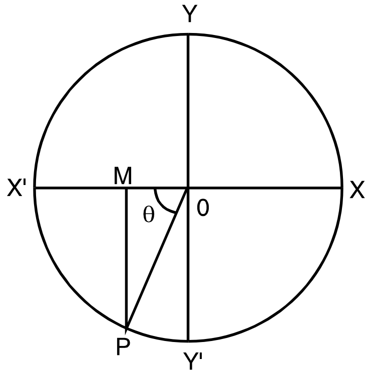


Fig. 3.26

III Quadrant : sec θ is negative as *OM* is negative. $\sec\pi = -1$ and $\sec\frac{3\pi}{2} = -\infty$ when the angle approaches $\frac{3\pi}{2}$ in the counter clockwise direction. As θ varies from π to $\frac{3\pi}{2}$, sec θ decreases from -1 to $-\infty$.

IV Quadrant : sec θ is positive as OM is positive. when θ is slightly greater than $\frac{3\pi}{2}$, sec θ is positive and very large. Also $\sec 2\pi = 1$. Hence sec θ decreases from ∞ to 1 as θ varies from $\frac{3\pi}{2}$ to 2π.

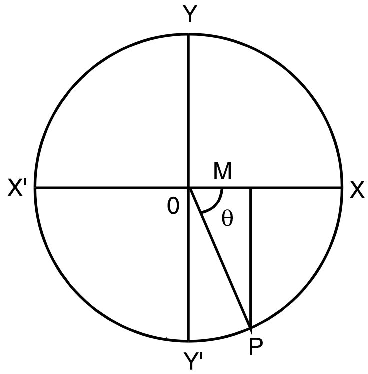


Fig. 3.27

It may be observed that as θ passes through $\frac{3\pi}{2}$; sec θ changes from $-\infty$ to $+\infty$.

Graph of sec θ as θ varies from 0 to 2π

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cot θ	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$	$+\infty$	2	1.15	

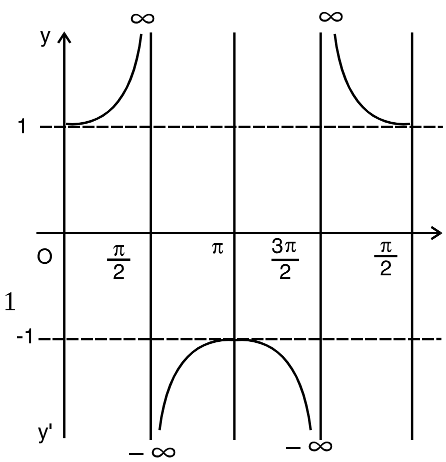


Fig. 3.28

Observations

(a) $\sec\theta$ cannot be numerically less than 1.

(b) Graph of $\sec\theta$ is discontinuous, discontinuities (breaks) occuring at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(c) As θ passes through $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, see θ changes abruptly from $+\infty$ to $-\infty$ and then from $-\infty$ to $+\infty$ respectively.

3.4.7 Graph of cosec θ as θ Varies From 0 to 2π

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O* draw a circle of unit radius. Let *P* be any point on the circle. Join *OP* and draw *PM* perpendicular to *X'OX*.

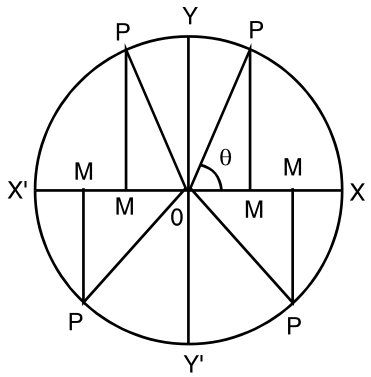


Fig. 3.29

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{MP}$$

$\therefore$ The variation of $\operatorname{cosec}\theta$ will depend upon *MP*.

I Quadrant : cosec θ is positive as *MP* is positive.

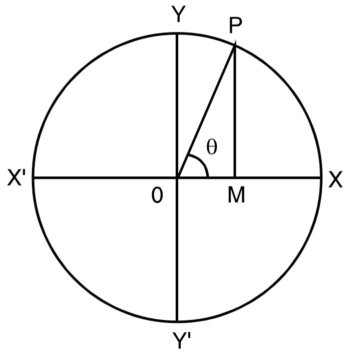


Fig. 3.30

cosec $\frac{\pi}{2} = 1$ when θ is very small, *MP* is also small and therefore, the value of cosec θ is very large.

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, cosec θ decreases from ∞ to 1.

II Quadrant : PM is positive. Therefore, cosec θ is positive. cosec $\frac{\pi}{2} = 1$ and cosec $\pi = \infty$ when the revolving line approaches π in the counter clockwise direction.

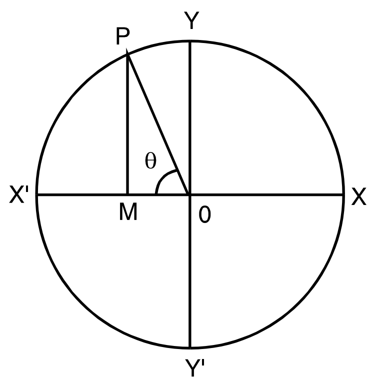


Fig. 3.31

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, cosec θ increases from 1 to ∞.

III Quadrant : *PM* is negative

$\therefore$ cosec θ is negative. When θ is slightly greater than π, cosec θ is very large and negative.

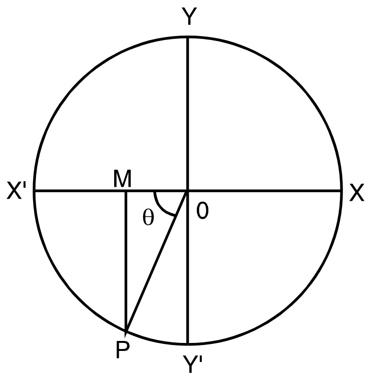


Fig. 3.32

Also cosec $\frac{3\pi}{2} = -1$.

$\therefore$ As θ varies from π to $\frac{3\pi}{2}$, cosec θ changes from $-\infty$ to -1.

It may be observed that as θ passes through π, cosec θ changes from $+\infty$ to $-\infty$.

IV Quadrant :

PM is negative.

Therefore, cosec $\theta = -\infty$ as θ approaches 2π.

$\therefore$ as θ varies from $\frac{3\pi}{2}$ to 2π, $\operatorname{cosec}\theta$ varies from -1 to $-\infty$.

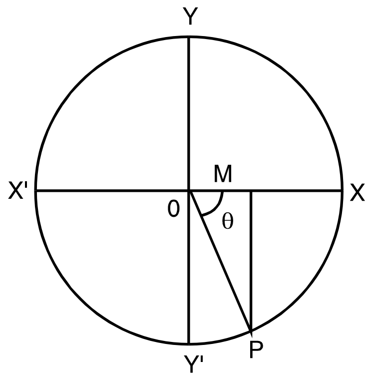


Fig. 3.33

Graph of cosec θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cosec θ	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$

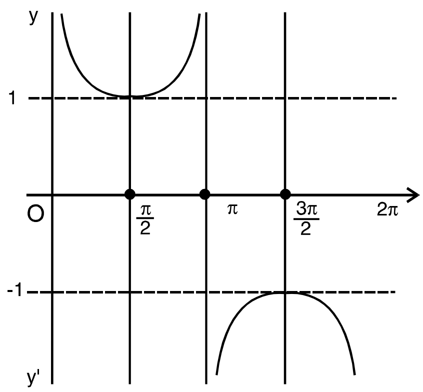


Fig. 3.34

Observations

(a) cosec θ cannot be numerically less than 1.

(b) Graph of cosec θ is discountinous and it has breaks at $\theta = 0, \pi, 2\pi$.

(c) As θ passes through π, cosec θ changes from $+\infty$ to $-\infty$. The values at 0 and 2π are $+\infty$ and $-\infty$ respectively.

Example 3.16 Trace the changes in the values of sec θ as θ lies in $-\pi$ to π.

Soluton :

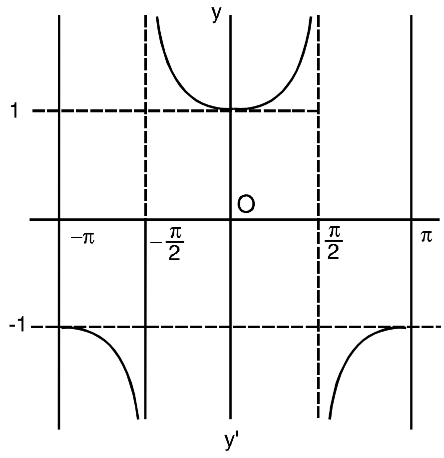


Fig. 3.35

CHECK YOUR PROGRESS 3.9

1. (a) Trace the changes in the values of sec θ when θ lies between -2π and 2π and draw the graph between these limits.

   (b) Trace the graph of cosec θ ,when θ lies between -2π and 2π.

3.5 PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS

From your daily experience you must have observed things repeating themselves after regular intervals of time. For example, days of a week are repeated regularly after 7 days and months of a year are repeated regularly after 12 months. Position of a particle on a moving wheel is another example of the type. The property of repeated occurence of things over regular intervals is known as ***periodicity.***

Definition : A function $f(x)$ is said to be periodic if its value is unchanged when the value of the variable is increased by a constant, that is if $f(x + p) = f(x)$ for all x.

If p is smallest positive constant of this type, then p is called the period of the function $f(x)$.

If $f(x)$ is a periodic function with period p, then $\frac{1}{f(x)}$ is also a periodic function with period p.

3.5.1 Periods of Trigonometric Functions

(i) $\sin x = \sin(x + 2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

(ii) $\cos x = \cos(x + 2n\pi)$; $n = 0, \pm 1, \pm 2 ,.....$

Also there is no p, lying in 0 to 2π, for which

$$\sin x = \sin(x + p)$$

$$\cos x = \cos(x + p), \text{ for all } x$$

$\therefore$ 2π is the smallest positive value for which

$\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$

$\Rightarrow$ sin x and cos x each have the period 2π.

(iii) The period of cosec x is also 2π because $\operatorname{cosec} x = \dfrac{1}{\sin x}$.

(iv) The period of sec x is also 2π as $\sec x = \dfrac{1}{\cos x}$.

(v) Also $\tan(x+\pi) = \tan x$. Suppose $p\ (0 < p < \pi)$ is the period of tan x, then

$\tan(x+p) = \tan x$, for all x. Put x = 0, then tan p = 0, i.e., p = 0 or π.

$\Rightarrow$ the period of tan x is π.

$\therefore$ p can not have values between 0 and π for which $\tan x = \tan(x+p)$

$\therefore$ The period of $\tan x$ is π

(vi) Since $\cot x = \dfrac{1}{\tan x}$, therefore, the period of cot x is also π.

Example 3.17 Find the period of each the following functions :

(a) $y = 3\sin 2x$ (b) $y = \cos\dfrac{x}{2}$ (c) $y = \tan\dfrac{x}{4}$

Solution :

(a) Period is $\dfrac{2\pi}{2}$, i.e., π.

(b) $y = \cos\dfrac{1}{2}x$, therefore period $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

(c) Period of $y = \tan\dfrac{x}{4} = \dfrac{\pi}{\frac{1}{4}} = 4\pi$

CHECK YOUR PROGRESS 3.10

1. Find the period of each of the following functions :

(a) $y = 2\sin 3x$ (b) $y = 3\cos 2x$

(c) $y = \tan 3x$ (d) $y = \sin^2 2x$

LET US SUM UP

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length *l* subtends an angle of θ radians at the centre of the circle with radius *r*, we have $l = r\theta$.
- If the coordinates of a point *P* of a unit circle are (*x*, *y*) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and

  $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (*x*, *y*) of a point *P* can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point *P* makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

Real numbers θ → / Functions	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continuous every where
  - — Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - — Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - — Period of these functions is 2π.

- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.

— The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0, 2\pi)$.

— Its period is π.

— The graph of $\cot\theta$ has discontinuities (breaks) at 0, π, 2π. Its period is π.

- $\sec\theta$ cannot have any value numerically less than 1.

(i) It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. It repeats itself after 2π.

(ii) $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

It has discontinuities (breaks) at 0, π, 2π. It repeats itself after 2π.

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Trigonometric_functions

http://mathworld.wolfram.com/ Trigonometric_functions.html

TERMINAL EXERCISE

1. A train is moving at the rate of 75 km/hour along a circular path of radius 2500 m. Through how many radians does it turn in one minute ?

2. Find the number of degrees subtended at the centre of the circle by an arc whose length is 0.357 times the radius.

3. The minute hand of a clock is 30 cm long. Find the distance covered by the tip of the hand in 15 minutes.

4. Prove that

(a) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (b) $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta$

(c) $\dfrac{\tan\theta}{1+\tan^2\theta} - \dfrac{\cot\theta}{1+\cot^2\theta} = 2\sin\theta\cos\theta$ (d) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\tan\theta + \sec\theta)^2$

(e) $\sin^8\theta - \cos^8\theta = \left(\sin^2\theta - \cos^2\theta\right)\left(1 - 2\sin^2\theta\cos^2\theta\right)$

(f) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

5. If $\theta = \frac{\pi}{4}$, verify that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

6. Evaluate :

(a) $\sin \frac{25\pi}{6}$ (b) $\sin \frac{21\pi}{4}$

(c) $\tan\left(\frac{3\pi}{4}\right)$ (d) $\sin \frac{17}{4}\pi$

(e) $\cos \frac{19}{3}\pi$

7. Draw the graph of cos x from $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$.

8. Define a periodic function of x and show graphically that the period of tan x is π, i.e. the position of the graph from x = π to 2π is repetition of the portion from x = 0 to π.

ANSWERS

CHECK YOUR PROGRESS 3.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{5\pi}{12}$ (iv) $\frac{7\pi}{12}$ (v) $\frac{3\pi}{2}$

2. (i) 45° (ii) 15° (iii) 9° (iv) 3° (v) 120°

3. $\frac{\pi}{4}, \frac{13\pi}{36}, \frac{14\pi}{36}$ 4. $\frac{5\pi}{6}$ 5. $\frac{\pi}{3}$

CHECK YOUR PROGRESS 3.2

1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$

2. (a) 36° (b) 30° (c) 20°

3. $\frac{1}{6}$ radian; 9.55° 4. $\frac{1}{5}$ radian 5. 95.54 m

6. (a) 0.53 m (b) 38.22 cm (c) 0.002 radian
(d) 12.56 m (e) 31.4 cm (f) 3.75 radian
(g) 6.28 m (h) 2 radian (i) 19.11 m.

CHECK YOUR PROGRESS 3.3

1. (i) – ve (ii) – ve (iii) – ve (iv) + ve
(v) + ve (vi) – ve (vii) + ve (viii) – ve

2. (i) zero (ii) zero (iii) $-\frac{1}{2}$ (iv) – 1
(v) 1 (vi) Not defined (vii) Not defined (viii) 1

CHECK YOUR PROGRESS 3.4

2. $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\cot\theta = 2$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$

3. $\sin\theta = \frac{a}{b}$, $\cos\theta = \frac{\sqrt{b^2 - a^2}}{b}$, $\sec\theta = \frac{b}{\sqrt{b^2 - a^2}}$,

$\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$, $\cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$ 6. $\frac{2m}{1+m^2}$

11. $\cos x = \frac{5}{13}, \sin x = \frac{-12}{13}, \operatorname{cosec} x = \frac{-13}{12}, \tan x = \frac{-12}{5}, \cot x = \frac{-5}{12}$

CHECK YOUR PROGRESS 3.5

1. (i) $4\frac{1}{4}$ (ii) $6\frac{1}{2}$ (iii) -1 (iv) $\frac{22}{3}$ (v) Zero

CHECK YOUR PROGRESS 3.6

1. 1, -1 3. Graph of $y = 2\sin\theta$, $[0, \pi]$

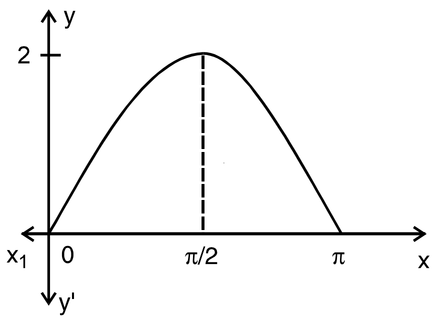


Fig. 3.36

4. (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{4\pi}{3}, \frac{5\pi}{3}$ 5. $y = \sin x$ from $-\pi$ to π

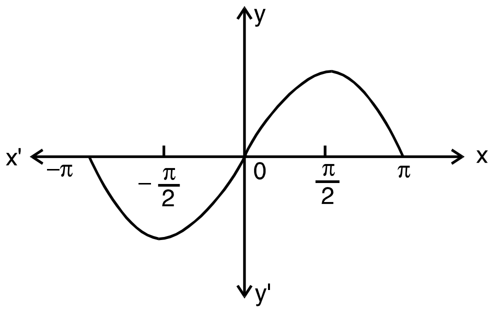


Fig. 3.37

CHECK YOUR PROGRESS 3.7

1. (a) $y = \cos\theta$, $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

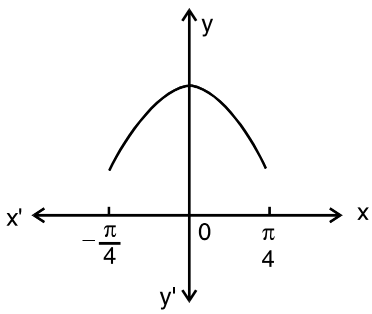


Fig. 3.38

MODULE - I
Sets, Relations and Functions

Notes

(b) $y = 3\cos\theta;\ 0 \text{ to } 2\pi$

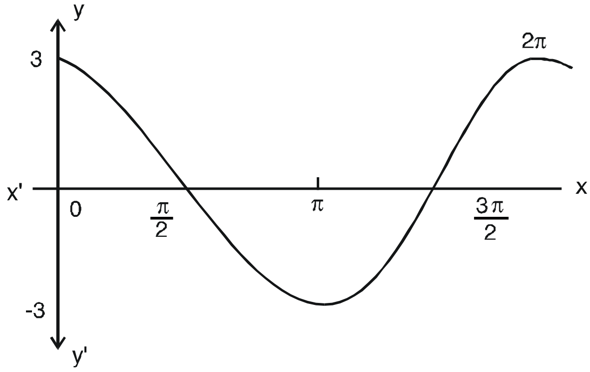


Fig. 3.39

(c) $y = \cos 3\theta, -\pi \text{ to } \pi$

$$\cos\theta = 0.87$$

$$\theta = \frac{\pi}{6}, -\frac{\pi}{6}$$

$$\cos\theta = -0.87$$

$$\theta = \frac{5\pi}{6}, -\frac{5\pi}{6}$$

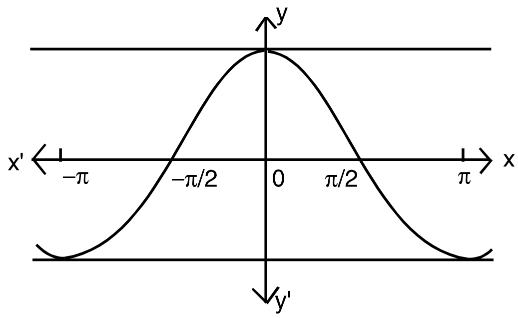


Fig. 3.40

(d) Graph of $y = \cos\theta$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ lies below the x-axis.

(e) $y = \cos\theta$

θ lies in 2π to 4π

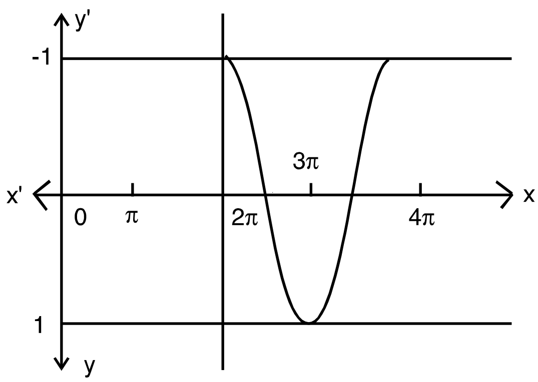


Fig. 3.41

CHECK YOUR PROGRESS 3.8

1. (a) Infinite (b) At $\frac{\pi}{2}, \frac{3\pi}{2}$ there are breaks in graphs.

(c) $\quad y = \tan 2\theta, \quad -\pi$ to π

At $\theta = \frac{\pi}{3}, \tan\theta = 1.7$

2. (a) Infinite $\qquad$ (b) $\cot\theta = -1$ at $\theta = \frac{3\pi}{4}$

CHECK YOUR PROGRESS 3.9

1. (a) $\qquad y = \sec\theta$

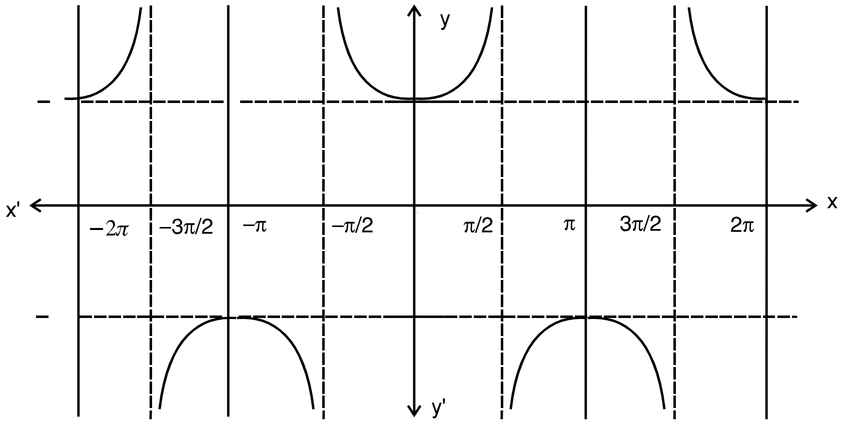


Fig. 3.42

Points of discontinuity of $\sec 2\theta$ are at $\frac{\pi}{4}, \frac{3\pi}{4}$ in the interval $[0, 2\pi]$.

(b) In tracing the graph from 0 to -2π, use $\text{cosec}(-\theta) = -\text{cosec}\,\theta$.

CHECK YOUR PROGRESS 3.10

1. (a) Period is $\frac{2\pi}{3}$ $\qquad$ (b) Period is $\frac{2\pi}{2} = \pi$ $\qquad$ (c) Period of y is $\frac{\pi}{3}$

(d) $y = \sin^2 2x = \frac{1-\cos 4x}{2} = \frac{1}{2} - \frac{1}{2}\cos 4x$; Period of y is $\frac{2\pi}{4}$ i.e $\frac{\pi}{2}$

(e) $y = 3\cot\left(\frac{x+1}{3}\right)$, Period of y is $\frac{\pi}{\frac{1}{3}} = 3\pi$

TERMINAL EXERCISE

1. $\frac{1}{2}$ radian $\qquad$ 2. $20.45°$ $\qquad$ 3. 15π cm

6. (a) $\frac{1}{2}$ $\qquad$ (b) $-\frac{1}{\sqrt{2}}$ $\qquad$ (c) -1 $\qquad$ (d) $\frac{1}{\sqrt{2}}$ $\qquad$ (e) $\frac{1}{2}$

MODULE - I
Sets, Relations and Functions

Notes

7.

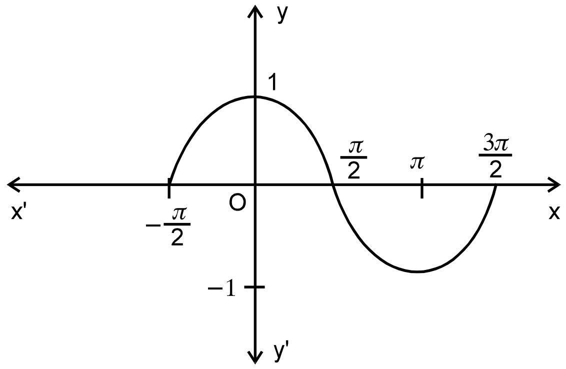
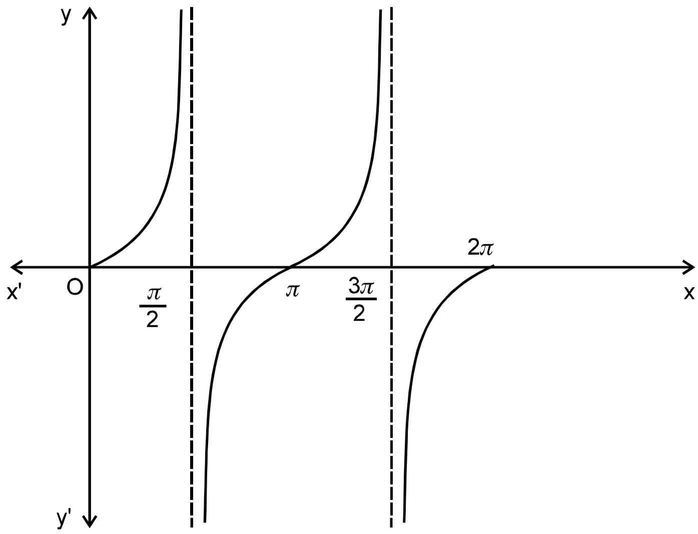


Fig. 3.43

8.



$y = \sec\theta$

Fig. 3.44

4

31len04

TRIGONOMETRIC FUNCTIONS-II

In the previous lesson, you have learnt trigonometric functions of real numbers, drawn and interpretd the graphs of trigonometric functions. In this lesson we will establish addition and subtraction formulae for $\cos(A \pm B)$, $\sin(A \pm B)$ and $\tan(A \pm B)$. We will also state the formulae for the multiple and sub multiples of angles and solve examples thereof. The general solutions of simple trigonometric functions will also be discussed in the lesson.

OBJECTIVES

After studying this lesson, you will be able to :

- write trigonometric functions of $-x, \frac{x}{2}, x \pm y, \frac{\pi}{2} \pm x, \pi \pm x$ where x, y are real nunbers;
- establish the addition and subtraction formulae for :

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,

  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- solve problems using the addition and subtraction formulae;
- state the formulae for the multiples and sub-multiples of angles such as cos2A, sin 2A, tan 2A, cos 3A, sin 3A, tan 3A, $\sin\frac{A}{2}, \cos\frac{A}{2}$ and $\tan\frac{A}{2}$; and
- solve simple trigonometric equations of the type :

  $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

EXPECTED BACKGROUND KNOWLEDGE

- Definition of trigonometric functions.
- Trigonometric functions of complementary and supplementary angles.
- Trigonometric identities.

4.1 ADDITION AND MULTIPLICATION OF TRIGONOMETRIC FUNCTIONS

In earlier sections we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers *A* and *B*, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ etc.

$$\frac{\pi}{12} \text{ can be expressed as } \frac{\pi}{4}-\frac{\pi}{6} \text{ and } \frac{5\pi}{12} \text{ can be expressed as } \frac{\pi}{4}+\frac{\pi}{6}$$

How can we express $\frac{7\pi}{12}$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

4.1.1 Addition Formulae

For any two numbers A and B,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

In given figure trace out

$\angle SOP = A$

$\angle POQ = B$

$\angle SOR = -B$

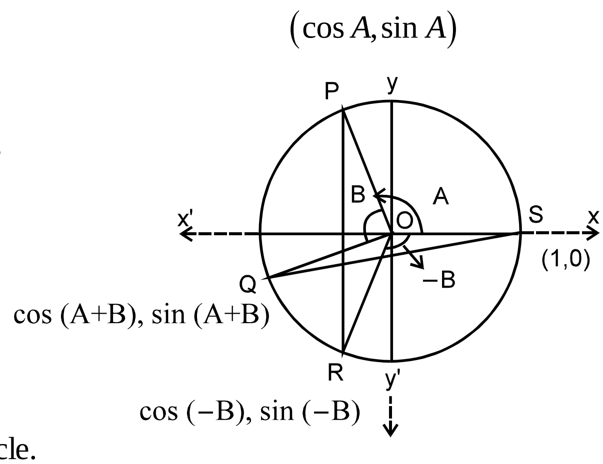


Fig. 4.1

where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be (cos A, sin A),

[cos (A + B), sin (A + B)],

$[\cos(-B), \sin(-B)]$, and (1, 0).

From the given figure, we have

side OP = side OQ, $\angle POR = \angle QOS$ (each angle = $\angle B + \angle$ QOR), side OR = side OS

$\Delta POR \cong \Delta QOS$ (by SAS) $\therefore$ PR = QS

$$PR = \sqrt{(\cos A - \cos(-B))^2 + (\sin A - \sin(-B))^2}$$

$$QS = \sqrt{(\cos(A+B) - 1)^2 + (\sin(A+B) - 0)^2}$$

Since $PR^2 = QS^2$ $\therefore$ $\cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B + 2\sin A \sin B$

$$= \cos^2(A+B) + 1 - 2\cos(A+B) + \sin^2(A+B)$$

$\Rightarrow$ $$1 + 1 - 2(\cos A \cos B - \sin A \sin B) = 1 + 1 - 2\cos(A+B)$$

$\Rightarrow$ $$\cos A \cos B - \sin A \sin B = \cos(A+B) \quad \text{(I)}$$

Corollary 1

For any two numbers A and B, cos (A – B) = cos A cos B + sin A sin B

Proof : Replace B by $-B$ in (I)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

[$\because$ $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$]

Corollary 2

For any two numbers A and B, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof : We know that $\cos\left(\frac{\pi}{2} - A\right) = \sin A$ and $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$\therefore$ $$\sin(A+B) = \cos\left[\left(\frac{\pi}{2} - (A+B)\right)\right] = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

or $$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \text{.....(II)}$$

Corollary 3

For any two numbers A and B, $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Proof : Replacing B by $-B$ in (2), we have

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Example 4.1

(a) Find the value of each of the following :

(i) $\sin\frac{5\pi}{12}$ (ii) $\cos\frac{\pi}{12}$ (iii) $\cos\frac{7\pi}{12}$

(b) If $\sin A = \frac{1}{\sqrt{10}}$, $\sin B = \frac{1}{\sqrt{5}}$ show that $A + B = \frac{\pi}{4}$

Solution :

(a) (i) $$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Observe that $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$

(iii) $$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(b) $$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos A = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and } \cos B = \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$$

Substituting all these values in the above formula, we get

$$\sin(A+B) = \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} + \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \text{ or } A + B = \frac{\pi}{4}$$

CHECK YOUR PROGRESS 4.1

1. (a) Find the values of each of the following :

(i) $\sin\frac{\pi}{12}$ (ii) $\sin\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}+\cos\frac{\pi}{9}\cdot\sin\frac{2\pi}{9}$

(b) Prove the following :

(i) $\sin\left(\frac{\pi}{6}+A\right)=\frac{1}{2}\left(\cos A+\sqrt{3}\sin A\right)$ (ii) $\sin\left(\frac{\pi}{4}-A\right)=\frac{1}{\sqrt{2}}\left(\cos A-\sin A\right)$

(c) If $\sin A=\frac{8}{17}$ and $\sin B=\frac{5}{13}$, find $\sin(A-B)$

2. (a) Find the value of $\cos\frac{5\pi}{12}$.

(b) Prove that:

(i) $\cos\theta+\sin\theta=\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$ (ii) $\sqrt{3}\sin\theta-\cos\theta=2\sin\left(\theta-\frac{\pi}{6}\right)$

(iii) $\cos(n+1)A\cos(n-1)A+\sin(n+1)A\sin(n-1)A=\cos 2A$

(iv) $\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}-B\right)+\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-B\right)=\cos(A+B)$

Corollary 4 : $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

Proof : $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

Dividing by cos A cos B, we have

$$\tan(A+B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

or $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \qquad ......(III)$$

Corollary 5 : $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

Proof : Replacing B by $-B$ in (III), we get the required result.

Corollary 6 : $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

Proof : $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

Dividing by $\sin A \sin B$, we have ......(IV)

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

Corollary 7 : $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \tan A}{1 - \tan A}$

Proof : $$\tan\left(\frac{\pi}{4} + A\right) = \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4} \cdot \tan A} = \frac{1 + \tan A}{1 - \tan A} \text{ as } \tan\frac{\pi}{4} = 1$$

Similarly, it can be proved that $\tan\left(\dfrac{\pi}{4} - A\right) = \dfrac{1 - \tan A}{1 + \tan A}$

Example 4.2 Find $\tan\dfrac{\pi}{12}$

Solution : $$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4} \cdot \tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$$\therefore \quad \tan\frac{\pi}{12} = 2 - \sqrt{3}$$

Example 4.3 Prove the following :

(a) $$\frac{\cos\frac{7\pi}{36} + \sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36} - \sin\frac{7\pi}{36}} = \tan\frac{4\pi}{9}$$

(b) $$\tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \cdot \tan 3A$$

Solution : (a) Dividing numerator and denominator by $\cos\frac{7\pi}{36}$, we get

$$\text{L.H.S.} = \frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \frac{1+\tan\frac{7\pi}{36}}{1-\tan\frac{7\pi}{36}} = \frac{\tan\frac{\pi}{4}+\tan\frac{7\pi}{36}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{7\pi}{36}}$$

$$= \tan\left(\frac{\pi}{4}+\frac{7\pi}{36}\right) = \tan\frac{16\pi}{36} = \tan\frac{4\pi}{9} = \text{R.H.S.}$$

(b) $\tan 7A = \tan(4A+3A) = \dfrac{\tan 4A + \tan 3A}{1-\tan 4A\ \tan 3A}$

or $\quad \tan 7A - \tan 7A\ \tan 4A\ \tan 3A = \tan 4A + \tan 3A$

or $\quad \tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A$

CHECK YOUR PROGRESS 4.2

1. Fill in the blanks :

(i) $\quad \sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-A\right) = \text{.........}$

(ii) $\quad \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \text{.........}$

2. (a) Prove that :

(i) $\quad \tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}-\theta\right) = 1.$

(ii) $\quad \cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(iii) $\quad \tan\frac{\pi}{12}+\tan\frac{\pi}{6}+\tan\frac{\pi}{12}\cdot\tan\frac{\pi}{6} = 1$

(b) If $\tan A = \frac{a}{b}$; $\tan B = \frac{c}{d}$, Prove that $\tan(A+B) = \dfrac{ad+bc}{bd-ac}$.

(c) Find the value of $\cos\frac{11\pi}{12}$.

3. (a) Prove that : (i) $\tan\left(\frac{\pi}{4}+A\right)\tan\left(\frac{3\pi}{4}+A\right)=-1$

(ii) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan\left(\frac{\pi}{4}+\theta\right)$ (iii) $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$

4.2 TRANSFORMATION OF PRODUCTS INTO SUMS AND VICE VERSA

4.2.1 Transformation of Products into Sums or Differences

We know that $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

By adding and subtracting the first two formulae, we get respectively

$$2\sin A\cos B=\sin(A+B)+\sin(A-B) \quad .....(1)$$

and $$2\cos A\sin B=\sin(A+B)-\sin(A-B) \quad .....(2)$$

Similarly, by adding and subtracting the other two formulae, we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B) \quad ....(3)$$

and $$2\sin A\sin B=\cos(A-B)-\cos(A+B) \quad ....(4)$$

We can also quote these as

$$2\sin A\cos B=\sin(\text{sum})+\sin(\text{difference})$$

$$2\cos A\sin B=\sin(\text{sum})-\sin(\text{difference})$$

$$2\cos A\cos B=\cos(\text{sum})+\cos(\text{difference})$$

$$2\sin A\sin B=\cos(\text{difference})-\cos(\text{sum})$$

4.2.2 Transformation of Sums or Differences into Products

In the above results put

$$A+B=C \text{ and } A-B=D$$

Then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$ and (1), (2), (3) and (4) become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

4.2.3 Further Applications of Addition and Subtraction Formulae

We shall prove that (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$ or $\cos^2 B - \sin^2 A$

Proof : (i) $\sin(A+B)\sin(A-B)$

$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$

$$= \sin^2 A\left(1-\sin^2 B\right) - \left(1-\sin^2 A\right)\sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

(ii) $\cos(A+B)\cos(A-B)$

$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$$

$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$

$$= \cos^2 A\left(1-\sin^2 B\right) - \left(1-\cos^2 A\right)\sin^2 B = \cos^2 A - \sin^2 B$$

$$= \left(1-\sin^2 A\right) - \left(1-\cos^2 B\right) = \cos^2 B - \sin^2 A$$

Example 4.4 Express the following products as a sum or difference

(i) $2\sin 3\theta\cos 2\theta$ (ii) $\cos 6\theta\cos\theta$ (iii) $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

Solution :

(i) $2\sin 3\theta\cos 2\theta = \sin(3\theta+2\theta) + \sin(3\theta-2\theta) = \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta\cos\theta = \frac{1}{2}(2\cos 6\theta\cos\theta) = \frac{1}{2}[\cos(6\theta+\theta) + \cos(6\theta-\theta)]$

$$= \frac{1}{2}(\cos 7\theta + \cos 5\theta)$$

(iii) $$\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{5\pi-\pi}{12}\right) - \cos\left(\frac{5\pi+\pi}{12}\right)\right] = \frac{1}{2}\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right]$$

Example 4.5 Express the following sums as products.

(i) $\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9}$ (ii) $\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36}$

Solution :

(i) $$\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 2\cos\frac{5\pi+7\pi}{9\times 2}\cos\frac{5\pi-7\pi}{9\times 2}$$

$$= 2\cos\frac{2\pi}{3}\cos\frac{\pi}{9} \quad \left[\because \cos\left(-\frac{\pi}{9}\right) = \cos\frac{\pi}{9}\right]$$

$$= 2\cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{9} = -2\cos\frac{\pi}{3}\cos\frac{\pi}{9}$$

$$= -\cos\frac{\pi}{9} \quad \left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

(ii) $$\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36} = \sin\left(\frac{\pi}{2} - \frac{13\pi}{36}\right) + \cos\frac{7\pi}{36}$$

$$= \cos\frac{13\pi}{36} + \cos\frac{7\pi}{36}$$

$$= 2\cos\frac{13\pi+7\pi}{36\times 2}\cos\frac{13\pi-7\pi}{36\times 2} = 2\cos\frac{5\pi}{18}\cos\frac{\pi}{12}$$

Example 4.6 Prove that $\dfrac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$

Solution :

$$\text{L.H.S.} = \frac{2\sin\dfrac{7A+9A}{2}\sin\dfrac{9A-7A}{2}}{2\cos\dfrac{9A+7A}{2}\sin\dfrac{9A-7A}{2}}$$

$$= \frac{\sin 8A\ \sin A}{\cos 8A\ \sin A} = \frac{\sin 8A}{\cos 8A} = \tan 8A = \text{R.H.S.}$$

Example 4.7 Prove the following :

(i) $$\cos^2\left(\frac{\pi}{4}-A\right)-\sin^2\left(\frac{\pi}{4}-B\right)=\sin(A+B)\cos(A-B)$$

(ii) $$\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$$

Solution :

(i) Applying the formula

$\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$, we have

$$\text{L.H.S.} = \cos\left[\frac{\pi}{4}-A+\frac{\pi}{4}-B\right]\cos\left[\frac{\pi}{4}-A-\frac{\pi}{4}+B\right]$$

$$=\cos\left[\frac{\pi}{2}-(A+B)\right]\cos\left[-(A-B)\right]=\sin(A+B)\cos(A-B)=\text{R.H.S.}$$

(ii) Applying the formula

$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$, we have

$$\text{L.H.S.} = \sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{\pi}{8}-\frac{A}{2}\right)\sin\left(\frac{\pi}{8}+\frac{A}{2}-\frac{\pi}{8}+\frac{A}{2}\right)$$

$$=\sin\frac{\pi}{4}\sin A = \frac{1}{\sqrt{2}}\sin A = \text{R.H.S.}$$

Example 4.8 Prove that

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{3}\cos\frac{4\pi}{9}=\frac{1}{16}$$

Solution : $$\text{L.H.S. } \cos\frac{\pi}{3}\left[\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$$

$$=\frac{1}{2}\cdot\frac{1}{2}\left[2\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9} \qquad \left[\because \cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=\frac{1}{4}\left[\cos\frac{\pi}{3}+\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[2\cos\frac{4\pi}{9}\cos\frac{\pi}{9}\right]$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[\cos\frac{5\pi}{9}+\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\cos\frac{5\pi}{9} + \frac{1}{16} \qquad .....(1)$$

Now $$\cos\frac{5\pi}{9} = \cos\left[\pi - \frac{4\pi}{9}\right] = -\cos\frac{4\pi}{9} \qquad .....(2)$$

From (1) and (2), we get L.H.S. $= \frac{1}{16} =$ R.H.S.

CHECK YOUR PROGRESS 4.3

1. Express each of the following as sums or differences :

   (a) $2\cos 3\theta \sin 2\theta$ (b) $2\sin 4\theta \sin 2\theta$

   (c) $2\cos\frac{\pi}{4}\cos\frac{\pi}{12}$ (d) $2\sin\frac{\pi}{3}\cos\frac{\pi}{6}$

2. Express each of the following as a product :

   (a) $\sin 6\theta + \sin 4\theta$ (b) $\sin 7\theta - \sin 3\theta$

   (c) $\cos 2\theta - \cos 4\theta$ (d) $\cos 7\theta + \cos 5\theta$

3. Prove that :

   (a) $\sin\frac{5\pi}{18} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ (b) $\dfrac{\cos\frac{\pi}{9} - \cos\frac{7\pi}{18}}{\sin\frac{7\pi}{18} - \sin\frac{\pi}{9}} = 1$

   (c) $\sin\frac{5\pi}{18} - \sin\frac{7\pi}{18} + \sin\frac{\pi}{18} = 0$ (d) $\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 0$

4. Prove that :

   (a) $\sin^2(n+1)\theta - \sin^2 n\theta = \sin(2n+1)\theta \cdot \sin\theta$

   (b) $\cos\beta\cos(2\alpha-\beta) = \cos^2\alpha - \sin^2(\alpha-\beta)$

   (c) $\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{12} = \frac{\sqrt{3}}{4}$

5. Show that $\cos^2\left(\frac{\pi}{4}+\theta\right) - \sin^2\left(\frac{\pi}{4}-\theta\right)$ is independent of θ.

6. Prove that :

(a) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

(b) $\sin\dfrac{\pi}{18}\sin\dfrac{5\pi}{6}\sin\dfrac{5\pi}{18}\sin\dfrac{7\pi}{18}=\dfrac{1}{16}$

(c) $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$

4.3 TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that $\sin(A+B)=\sin A\cos B+\cos A\sin B$

By putting B = A, we get $\sin 2A=\sin A\cos A+\cos A\sin A=2\sin A\cos A$

$\therefore$ sin 2A can also be written as

$$\sin 2A=\frac{2\sin A\cos A}{\cos^2 A+\sin^2 A}\qquad(\because 1=\cos^2 A+\sin^2 A)$$

Dividing numerator and denomunator by $\cos^2 A$, we get

$$\sin 2A=\frac{2\left(\dfrac{\sin A\cos A}{\cos^2 A}\right)}{\dfrac{\cos^2 A}{\cos^2 A}+\dfrac{\sin^2 A}{\cos^2 A}}=\frac{2\tan A}{1+\tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that $\cos(A+B)=\cos A\cos B-\sin A\sin B$

Putting B = A, we have $\cos 2A=\cos A\cos A-\sin A\sin A$

or $\qquad \cos 2A=\cos^2 A-\sin^2 A$

Also $\cos 2A=\cos^2 A-(1-\cos^2 A)=\cos^2 A-1+\cos^2 A$

i.e, $\qquad \cos 2A=2\cos^2 A-1 \qquad\Rightarrow\qquad \cos^2 A=\dfrac{1+\cos 2A}{2}$

Also $\cos 2A=\cos^2 A-\sin^2 A=1-\sin^2 A-\sin^2 A$

i.e., $\qquad \cos 2A=1-2\sin^2 A \qquad\Rightarrow\qquad \sin^2 A=\dfrac{1-\cos 2A}{2}$

$$\therefore\qquad \cos 2A=\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$$

Dividing the numerator and denominator of R.H.S. by $\cos^2 A$, we have

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Thus we have derived the following formulae :

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}, \quad \cos^2 A = \frac{1 + \cos 2A}{2}, \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

Example 4.9 Prove that $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$

Solution : $\dfrac{\sin 2A}{1 + \cos 2A} = \dfrac{2 \sin A \cos A}{2 \cos^2 A} = \dfrac{\sin A}{\cos A} = \tan A$

Example 4.10 Prove that $\cot A - \tan A = 2 \cot 2A$.

Solution : $\cot A - \tan A = \dfrac{1}{\tan A} - \tan A = \dfrac{1 - \tan^2 A}{\tan A}$

$$= \frac{2(1 - \tan^2 A)}{2 \tan A}$$

$$= \frac{2}{\left(\dfrac{2 \tan A}{1 - \tan^2 A}\right)}$$

$$= \frac{2}{\tan 2A} = 2 \cot 2A.$$

Example 4.11 Evaluate $\cos^2 \dfrac{\pi}{8} + \cos^2 \dfrac{3\pi}{8}$.

Solution : $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}=\frac{1+\cos\frac{\pi}{4}}{2}+\frac{1+\cos\frac{3\pi}{4}}{2}$

$$=\frac{1+\frac{1}{\sqrt{2}}}{2}+\frac{1-\frac{1}{\sqrt{2}}}{2}=\frac{\left(\sqrt{2}+1\right)+\left(\sqrt{2}-1\right)}{2\sqrt{2}}=1$$

Example 4.12 Prove that $\frac{\cos A}{1-\sin A}=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$.

Solution : R.H.S. $=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\tan\frac{A}{2}}$

$$=\frac{1+\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1-\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$$

$$=\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}$$

[Multiplying Numerator and Denominator by $\left(\frac{\cos A}{2}-\frac{\sin A}{2}\right)$]

$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\cos\frac{A}{2}\sin\frac{A}{2}}=\frac{\cos A}{1-\sin A}=\text{L.H.S}.$$

CHECK YOUR PROGRESS 4.4

1. If $A=\frac{\pi}{3}$, verify that

   (a) $\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

2. Find the value of sin 2A when (assuming $0 < A < \frac{\pi}{2}$)

(a) $\cos A = \dfrac{3}{5}$ (b) $\sin A = \dfrac{12}{13}$ (c) $\tan A = \dfrac{16}{63}$.

3. Find the value of cos 2A when

(a) $\cos A = \dfrac{15}{17}$ (b) $\sin A = \dfrac{4}{5}$ (c) $\tan A = \dfrac{5}{12}$

4. Find the value of tan 2A when

(a) $\tan A = \dfrac{3}{4}$ (b) $\tan A = \dfrac{a}{b}$

5. Evaluate $\sin^2 \dfrac{\pi}{8} + \sin^2 \dfrac{3\pi}{8}$.

6. Prove the following :

(a) $\dfrac{1+\sin 2A}{1-\sin 2A} = \tan^2\left(\dfrac{\pi}{4} + A\right)$ (b) $\dfrac{\cot^2 A + 1}{\cos^2 A - 1} = \sec 2A$

7. (a) Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cos A$ (b) Prove that $\tan A + \cot A = 2\,\text{cosec}\, 2A$.

8. (a) Prove that $\dfrac{\cos A}{1+\sin A} = \tan\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right)$

(b) Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2 \dfrac{\alpha-\beta}{2}$

4.3.1 Trigonometric Functions of 3A in Terms of A

(a) sin 3A in terms of sin A

Substituting 2A for B in the formula

$\sin(A+B) = \sin A \cos B + \cos A \sin B$, we get

$\sin(A+2A) = \sin A \cos 2A + \cos A \sin 2A$

$= \sin A\left(1 - 2\sin^2 A\right) + (\cos A \times 2\sin A \cos A)$

$= \sin A - 2\sin^3 A + 2\sin A\left(1-\sin^2 A\right)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$\therefore \quad \sin 3A = 3\sin A - 4\sin^3 A \quad$ ....(1)

(b) **cos 3A in terms of cos A**

Substituting 2A for B in the formula

$$\cos(A+B)=\cos A\cos B-\sin A\sin B\text{, we get}$$

$$\cos(A+2A)=\cos A\cos 2A-\sin A\sin 2A$$

$$=\cos A\left(2\cos^2 A-1\right)-(\sin A)\times 2\sin A\cos A$$

$$=2\cos^3 A-\cos A-2\cos A\left(1-\cos^2 A\right)$$

$$=2\cos^3 A-\cos A-2\cos A+2\cos^3 A$$

$\therefore$ $\cos 3A=4\cos^3 A-3\cos A$ ....(2)

(c) **tan 3A in terms of tan A**

Putting $B=2A$ in the formula $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$, we get

$$\tan(A+2A)=\frac{\tan A+\tan 2A}{1-\tan A\tan 2A}=\frac{\tan A+\dfrac{2\tan A}{1-\tan^2 A}}{1-\tan A\times\dfrac{2\tan A}{1-\tan^2 A}}$$

$$=\frac{\dfrac{\tan A-\tan^3 A+2\tan A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A-2\tan^2 A}{1-\tan^2 A}}=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}\qquad ....(3)$$

(d) **Formulae for** $\sin^3 A$ **and** $\cos^3 A$

$\because$ $\sin 3A=3\sin A-4\sin^3 A$

$\therefore$ $4\sin^3 A=3\sin A-\sin 3A$ or $\sin^3 A=\dfrac{3\sin A-\sin 3A}{4}$

Similarly, $\cos 3A=4\cos^3 A-3\cos A$

$\therefore$ $3\cos A+\cos 3A=4\cos^3 A$ or $\cos^3 A=\dfrac{3\cos A+\cos 3A}{4}$

Example 4.13 Prove that

$$\sin\alpha\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right)=\frac{1}{4}\sin 3\alpha$$

Solution : $\sin\alpha\sin\left(\dfrac{\pi}{3}+\alpha\right)\sin\left(\dfrac{\pi}{3}-\alpha\right)$

$$= \frac{1}{2}\sin\alpha\left[\cos 2\alpha - \cos\frac{2\pi}{3}\right] = \frac{1}{2}\sin\alpha\left[1-2\sin^2\alpha-\left(1-2\sin^2\frac{\pi}{3}\right)\right]$$

$$= 2\frac{1}{2}\sin\alpha\left[\sin^2\frac{\pi}{3} - \sin^2\alpha\right]$$

$$= \sin\alpha\left[\frac{3}{4} - \sin^2\alpha\right] = \frac{3\sin\alpha - 4\sin^3\alpha}{4} = \frac{1}{4}\sin 3\alpha$$

Example 4.14 Prove that $\cos^3 A \sin 3A + \sin^3 A \cos 3A = \frac{3}{4}\sin 4A$

Solution : $\cos^3 A \sin 3A + \sin^3 A \cos 3A$

$$= \cos^3 A\left(3\sin A - 4\sin^3 A\right) + \sin^3 A\left(4\cos^3 A - 3\cos A\right)$$

$$= 3\sin A\cos^3 A - 4\sin^3 A\cos^3 A + 4\sin^3 A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos A\left(\cos^2 A - \sin^2 A\right) = \left(3\sin A\cos A\right)\cos 2A$$

$$= \frac{3\sin 2A}{2}\times\cos 2A = \frac{3}{2}\frac{\sin 4A}{2} = \frac{3}{4}\sin 4A.$$

Example 4.15 Prove that $\cos^3\frac{\pi}{9} + \sin^3\frac{\pi}{18} = \frac{3}{4}\left(\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right)$

Solution : L.H.S. $= \frac{1}{4}\left[3\cos\frac{\pi}{9} + \cos\frac{\pi}{3}\right] + \frac{1}{4}\left(3\sin\frac{\pi}{18} - \sin\frac{\pi}{6}\right)$

$$= \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] = \text{R.H.S.}$$

CHECK YOUR PROGRESS 4.5

1. If $A = \frac{\pi}{3}$, verify that (a) $\sin 3A = 3\sin A - 4\sin^3 A$

   (b) $\cos 3A = 4\cos^3 A - 3\cos A$ (c) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

2. Find the value of sin 3A when (a) $\sin A = \frac{2}{3}$ (b) $\sin A = \frac{p}{q}$.

3. Find the value of cos 3A when (a) $\cos A = -\frac{1}{3}$ (b) $\cos A = \frac{c}{d}$.

4. Prove that $\cos\alpha \cos\left(\frac{\pi}{3}-\alpha\right)\cos\left(\frac{\pi}{3}+\alpha\right) = \frac{1}{4}\cos 3\alpha$.

5. (a) Prove that $\sin^3\frac{2\pi}{9} - \sin^3\frac{\pi}{9} = \frac{3}{4}\left(\sin\frac{2\pi}{9} - \sin\frac{\pi}{9}\right)$

   (b) Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$ is constant.

6. (a) Prove that $\cot 3A = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

   (b) Prove that

$$\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A = 8\cos A\cos^3 3A$$

4.4 TRIGONOMETRIC FUNCTIONS OF SUBMULTIPLES OF ANGLES

$\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A = \frac{1-\cos 2A}{2}, \quad \cos^2 A = \frac{1+\cos 2A}{2}, \quad \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

Replacing A by $\frac{A}{2}$, we easily get the following formulae for the sub-multiple $\frac{A}{2}$:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \quad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \text{and} \quad \tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\frac{A}{2}$. This will be clear from the following examples

Example 4.16 Find the values of $\sin\left(-\frac{\pi}{8}\right)$ and $\cos\left(-\frac{\pi}{8}\right)$.

MODULE - I
Sets, Relations and Functions

Notes

Solution : We use the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

and take the lower sign, i.e., negative sign, because $\sin\left(-\frac{\pi}{8}\right)$ is negative.

$$\sin\left(-\frac{\pi}{8}\right) = -\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$= -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

Similarly,
$$\cos\left(-\frac{\pi}{8}\right) = +\sqrt{\frac{1+\cos\left(-\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 4.17 If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$, find the values of

(i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

Solution : $\because$ A lies in the 4th-quardrant, $\frac{3\pi}{2} < A < 2\pi$

$$\Rightarrow \quad 3\frac{\pi}{4} < \frac{A}{2} < \pi$$

$$\therefore \quad \sin\frac{A}{2} > 0,\ \cos\frac{A}{2} < 0,\ \tan\frac{A}{2} < 0.$$

$$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{7}{25}}{2}} = -\sqrt{\frac{32}{50}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

and
$$\tan\frac{A}{2} = -\sqrt{\frac{1-\cos A}{1+\cos A}} = -\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}} = -\sqrt{\frac{18}{32}} = -\sqrt{\frac{9}{16}} = -\frac{3}{4}$$

CHECK YOUR PROGRESS 4.6

1. If $A = \frac{\pi}{3}$, verify that (a) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$

   (b) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$ (c) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

2. Find the values of $\sin\frac{\pi}{12}$ and $\sin\frac{\pi}{24}$.

3. Determine the values of

   (a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$.

4.5 TRIGONOMETRIC EQUATIONS

You are familiar with the equations like simple linear equations, quadratic equations in algebra. You have also learnt how to solve the same.

Thus, (i) $x - 3 = 0$ gives one value of *x* as a solution.

(ii) $x^2 - 9 = 0$ gives two values of *x*.

You must have noticed, the number of values depends upon the degree of the equation.

Now we need to consider as to what will happen in case *x*'s and *y*'s are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, will give

$$\sin\theta = 1 \text{ and } \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots.$$

Clearly, the solution of simple equations with only finite number of values does not necessarily hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

4.5.1 To find the general solution of the equation sin θ = sin α

It is given that $\sin\theta = \sin\alpha, \Rightarrow \quad \sin\theta - \sin\alpha = 0$

or $$2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$\therefore$ Either $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p,\ q \in Z$$

$$\Rightarrow \quad \theta = (2p+1)\pi - \alpha \quad \text{or} \quad \theta = 2q\pi + \alpha \quad ....(1)$$

From (1), we get

$\theta = n\pi + (-1)^n \alpha,\ n \in Z$ as the geeneral solution of the equation $\sin\theta = \sin\alpha$

4.5.2 To find the general solution of the equation cos θ = cos α

It is given that, $\cos\theta = \cos\alpha$, $\Rightarrow \quad \cos\theta - \cos\alpha = 0$

$$\Rightarrow \quad -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\therefore \quad \text{Either, } \sin\frac{\theta+\alpha}{2} = 0 \quad \text{or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = p\pi \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p, q \in Z$$

$$\Rightarrow \quad \theta = 2p\pi - \alpha \quad \text{or} \quad \theta = 2z\pi + \alpha \qquad ....(1)$$

From (1), we have

$\theta = 2n\pi \pm \alpha, n \in Z$ as the general solution of the equation $\cos\theta = \cos\alpha$

4.5.3 To find the general solution of the equation tan θ = tan α

It is given that, $\tan\theta = \tan\alpha$, $\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\alpha}{\cos\alpha} = 0$

$$\Rightarrow \quad \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0, \quad \Rightarrow \quad \sin(\theta-\alpha) = 0$$

$$\Rightarrow \quad \theta - \alpha = n\pi,\ n \in Z, \quad \Rightarrow \quad \theta = n\pi + \alpha,\ n \in Z$$

Similarly, for $\quad \operatorname{cosec}\theta = \operatorname{cosec}\alpha$, the general solution is $\theta = n\pi + (-1)^n \alpha$

and, for $\quad \sec\theta = \sec\alpha$, the general solution is $\theta = 2n\pi \pm \alpha$

and for $\quad \cot\theta = \cot\alpha$, $\theta = n\pi + \alpha$ is its general solution

Example 4.18 Find the general solution of the following equations :

(a) (i) $\sin\theta = \dfrac{1}{2}$ (ii) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (b) (i) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\dfrac{1}{2}$

(c) $\cot\theta = -\sqrt{3}$ (d) $4\sin^2\theta = 1$

Solution : (a) (i) $\sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$

$$\therefore \quad \theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in Z$$

(ii) $\sin\theta = \frac{-\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

$$\therefore \quad \theta = n\pi + (-1)^n \frac{4\pi}{3}, \quad n \in Z$$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \ \therefore \quad \theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in Z$

(ii) $\cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$$\therefore \quad \theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in Z$$

(c) $\cot\theta = -\sqrt{3}, \ \tan\theta = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$

$$\therefore \quad \theta = n\pi + \frac{5\pi}{6}, \quad n \in Z$$

(d) $4\sin^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$

$$\Rightarrow \quad \sin\theta = \sin\left(\pm\frac{\pi}{6}\right) \ \therefore \ \theta = n\pi \pm \frac{\pi}{6}, \quad n \in Z$$

Example 4.19 Solve the following to find general solution :

(a) $2\cos^2\theta + 3\sin\theta = 0$ (b) $\cos 4x = \cos 2x$

(c) $\cos 3x = \sin 2x$ (d) $\sin 2x + \sin 4x + \sin 6x = 0$

Solution :

(a) $2\cos^2\theta + 3\sin\theta = 0, \quad \Rightarrow \quad 2\left(1 - \sin^2\theta\right) + 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0, \quad \Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow \quad \sin\theta = -\frac{1}{2}$ or $\sin\theta = 2,$ Since $\sin\theta = 2$ is not possible.

$$\therefore \quad \sin\theta = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$$

$$\therefore \quad \theta = n\pi + (-1)^n \cdot \frac{7\pi}{6}, \ n \in Z$$

(b) $\cos 4x = \cos 2x$ i.e., $\cos 4x - \cos 2x = 0$

$\Rightarrow \quad -2\sin 3x \sin x = 0$

$\Rightarrow \quad \sin 3x = 0 \quad$ or $\quad \sin x = 0$

$\Rightarrow \quad 3x = n\pi \quad$ or $\quad x = n\pi$

$\Rightarrow \quad x = \dfrac{n\pi}{3} \quad$ or $\quad x = n\pi \qquad n \in Z$

(c) $\cos 3x = \sin 2x \Rightarrow \cos 3x = \cos\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow \quad 3x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right) \qquad n \in Z$

Taking positive sign only, we have $3x = 2n\pi + \dfrac{\pi}{2} - 2x$

$\Rightarrow \quad 5x = 2n\pi + \dfrac{\pi}{2} \qquad \Rightarrow \quad x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}$

Now taking negative sign, we have

$$3x = 2n\pi - \frac{\pi}{2} + 2x \quad \Rightarrow \quad x = 2n\pi - \frac{\pi}{2} \quad n \in Z$$

(d) $\quad \sin 2x + \sin 4x + \sin 6x = 0$

or $\quad (\sin 6x + \sin 2x) + \sin 4x = 0$

or $\quad 2\sin 4x \cos 2x + \sin 4x = 0$

or $\quad \sin 4x\,[2\cos 2x + 1] = 0$

$\therefore \quad \sin 4x = 0 \quad$ or $\cos 2x = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$

$\Rightarrow \quad 4x = n\pi \quad$ or $2x = 2n\pi \pm \dfrac{2\pi}{3}$, $n \in Z$

$x = \dfrac{n\pi}{4} \quad$ or $\quad x = n\pi \pm \dfrac{\pi}{3} \qquad n \in Z$

CHECK YOUR PROGRESS 4.7

1. Find the general value of θ satisfying :

(i) $\sin\theta = \dfrac{\sqrt{3}}{2}$ (ii) $\operatorname{cosec}\theta = \sqrt{2}$

(iii) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (iv) $\sin\theta = -\dfrac{1}{\sqrt{2}}$

2. Find the general value of θ satisfying :

(i) $\cos\theta = -\dfrac{1}{2}$ (ii) $\sec\theta = -\dfrac{2}{\sqrt{3}}$

(iii) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (iv) $\sec\theta = -\sqrt{2}$

3. Find the general value of θ satisfying :

(i) $\tan\theta = -1$ (ii) $\tan\theta = \sqrt{3}$ (iii) $\cot\theta = -1$

4. Find the general value of θ satisfying :

(i) $\sin 2\theta = \dfrac{1}{2}$ (ii) $\cos 2\theta = \dfrac{1}{2}$ (iii) $\tan 3\theta = \dfrac{1}{\sqrt{3}}$

(iv) $\cos 3\theta = -\dfrac{\sqrt{3}}{2}$ (v) $\sin^2\theta = \dfrac{3}{4}$ (vi) $\sin^2 2\theta = \dfrac{1}{4}$

(vii) $4\cos^2\theta = 1$ (viii) $\cos^2 2\theta = \dfrac{3}{4}$

5. Find the general solution of the following :

(i) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ (ii) $4\cos^2\theta - 4\sin\theta = 1$

(iii) $\cot\theta + \tan\theta = 2\,\text{cosec}\,\theta$

LET US SUM UP

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B,$

  $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

  $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}, \quad \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

  $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}, \quad \cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

  $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

  $2\cos A\cos B = \cos(A+B) - \cos(A-B)$

  $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

  $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

  $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

  $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

- $\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$

  $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$

- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

- $\sin^2 A = \dfrac{1-\cos 2A}{2}$, $\cos^2 A = \dfrac{1+\cos 2A}{2}$, $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

- $\sin 3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$

  $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

- $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$, $\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

- $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$, $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

  $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

- $\sin\theta = \sin\alpha \quad\Rightarrow\quad \theta = n\pi + (-1)^n\alpha, \quad n \in Z$

- $\cos\theta = \cos\alpha \quad\Rightarrow\quad \theta = 2n\pi \pm \alpha, \quad n \in Z$

- $\tan\theta = \tan\alpha \quad\Rightarrow\quad \theta = n\pi + \alpha, \quad n \in Z$

SUPPORTIVE WEB SITES

http://mathworld.wolfram.com/ Trigonometric_functions.html
http://en.wikipedia.org/wiki/Trigonometric_functions

TERMINAL EXERCISE

1. Prove that $\tan(A+B)\times\tan(A-B)=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 B-\sin^2 A}$

2. Prove that $\cos\theta-\sqrt{3}\sin\theta=2\cos\left(\theta+\dfrac{\pi}{3}\right)$

3. If $A+B=\dfrac{\pi}{4}$

   Prove that $(1+\tan A)(1+\tan B)=2$ and $(\cot A-1)(\cos B-1)=2$

4. Prove each of the following :

   (i) $\dfrac{\sin(A-B)}{\cos A\cos B}+\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}=0$

   (ii) $\cos\left(\dfrac{\pi}{10}-A\right)\cdot\cos\left(\dfrac{\pi}{10}+A\right)+\cos\left(\dfrac{2\pi}{5}-A\right)\cdot\cos\left(\dfrac{2\pi}{5}+A\right)=\cos 2A$

   (iii) $\cos\dfrac{2\pi}{9}\cdot\cos\dfrac{4\pi}{9}\cdot\cos\dfrac{9\pi}{9}=-\dfrac{1}{8}$

   (iv) $\cos\dfrac{13\pi}{45}+\cos\dfrac{17\pi}{45}+\cos\dfrac{43\pi}{45}=0$

   (v) $\tan\left(A+\dfrac{\pi}{6}\right)+\cot\left(A-\dfrac{\pi}{6}\right)=\dfrac{1}{\sin 2A-\sin\dfrac{\pi}{3}}$

   (vi) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$ (vii) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan 2\theta+\sec 2\theta$

   (viii) $\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right)^2=\tan^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

   (ix) $\cos^2 A+\cos^2\left(A+\dfrac{\pi}{3}\right)+\cos^2\left(A-\dfrac{\pi}{3}\right)=\dfrac{3}{2}$

(x) $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \dfrac{\tan 8A}{\tan 2A}$ (xi) $\cos\dfrac{\pi}{30}\cos\dfrac{7\pi}{30}\cos\dfrac{11\pi}{30}\cos\dfrac{13\pi}{30} = \dfrac{11}{16}$

(xii) $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = -\dfrac{1}{2}$

5. Find the general value of 'θ' satisfying

(a) $\sin\theta = \dfrac{1}{\sqrt{2}}$ (b) $\sin\theta = \dfrac{\sqrt{3}}{2}$

(c) $\sin\theta = -\dfrac{1}{\sqrt{2}}$ (d) $\operatorname{cosec}\theta = \sqrt{2}$

6. Find the general value of 'θ' satisfying

(a) $\cos\theta = \dfrac{1}{2}$ (b) $\sec\theta = \dfrac{2}{\sqrt{3}}$

(c) $\cos\theta = \dfrac{-\sqrt{3}}{2}$ (d) $\sec\theta = -2$

7. Find the general value of 'θ' satisfying

(a) $\tan\theta = 1$ (b) $\tan\theta = -1$ (c) $\cot\theta = -\dfrac{1}{\sqrt{3}}$

8. Find the general value of 'θ' satisfying

(a) $\sin^2\theta = \dfrac{1}{2}$ (b) $4\cos^2\theta = 1$ (c) $2\cot^2\theta = \operatorname{cosec}^2\theta$

9. Solve the following for θ :

(a) $\cos p\theta = \cos q\theta$ (b) $\sin 9\theta = \sin\theta$ (c) $\tan 5\theta = \cot\theta$

10. Solve the following for θ :

(a) $\sin m\theta + \sin n\theta = 0$ (b) $\tan m\theta + \cot n\theta = 0$

(c) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (d) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

ANSWERS

CHECK YOUR PROGRESS 4.1

1. (a) (i) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{21}{221}$

2. (a) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

CHECK YOUR PROGRESS 4.2

1. (i) $\dfrac{\cos^2 A - \sin^2 A}{2}$ (ii) $-\dfrac{1}{4}$

2. (c) $-\dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$

CHECK YOUR PROGRESS 4.3

1. (a) $\sin 5\theta - \sin\theta$; (b) $\cos 2\theta - \cos 6\theta$

   (c) $\cos\dfrac{\pi}{3} + \cos\dfrac{\pi}{6}$ (d) $\sin\dfrac{\pi}{2} + \sin\dfrac{\pi}{6}$

2. (a) $2\sin 5\theta \cos\theta$ (b) $2\cos 5\theta \cdot \sin 2\theta$

   (c) $2\sin 3\theta \cdot \sin\theta$ (d) $2\cos 6\theta \cdot \cos\theta$

CHECK YOUR PROGRESS 4.4

2. (a) $\dfrac{24}{25}$ (b) $\dfrac{120}{169}$ (c) $\dfrac{2016}{4225}$

3. (a) $\dfrac{161}{289}$ (b) $\dfrac{-7}{25}$ (c) $\dfrac{119}{169}$

4. (a) $\dfrac{24}{7}$ (b) $\dfrac{2ab}{b^2 - a^2}$

5. 1

CHECK YOUR PROGRESS 4.5

2. (a) $\dfrac{22}{27}$ (b) $\dfrac{\left(3pq^2 - 4p^3\right)}{q^3}$

3. (a) $\frac{23}{27}$ (b) $\frac{4c^3 - 3cd^2}{d^3}$

CHECK YOUR PROGRESS 4.6

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{\left(4-\sqrt{2}-\sqrt{6}\right)}}{2\sqrt{2}}$

3. (a) $\frac{\sqrt{2-\sqrt{2}}}{2}$ (b) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (c) $\sqrt{2}-1$

CHECK YOUR PROGRESS 4.7

1. (i) $\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

   (iii) $\theta = n\pi + (-1)^n \frac{4\pi}{3}, n \in Z$ (iv) $\theta = n\pi + (-1)^n \frac{5\pi}{4}, n \in Z$

2. (i) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$ (ii) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in Z$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$ (iv) $\theta = 2n\pi \pm \frac{3\pi}{4}, n \in Z$

3. (i) $\theta = n\pi + \frac{3\pi}{4}, n \in Z$ (ii) $\theta = n\pi + \frac{\pi}{3}, n \in Z$

   (iii) $\theta = n\pi - \frac{\pi}{4}, n \in Z$

4. (i) $\theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in Z$ (ii) $\theta = n\pi \pm \frac{\pi}{6}, n \in Z$

   (iii) $\theta = \frac{n\pi}{3} + \frac{\pi}{18}, n \in Z$ (iv) $\theta = \frac{2n\pi}{3} \pm \frac{5\pi}{18}, n \in Z$

   (v) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (vi) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, n \in Z$

   (vii) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (viii) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, n \in Z$

5. (i) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in Z$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

TERMINAL EXERCISE

5. (a) $\theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in Z$

(b) $\theta = n\pi + (-1)^n \dfrac{\pi}{3}, n \in Z$

(c) $\theta = n\pi + (-1)^n \dfrac{5\pi}{4}, n \in Z$

(d) $\theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in Z$

6. (a) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in Z$

(b) $\theta = 2n\pi \pm \dfrac{\pi}{6}, n \in Z$

(c) $\theta = 2n\pi \pm \dfrac{5\pi}{6}, n \in Z$

(d) $\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in Z$

7. (a) $\theta = n\pi + \dfrac{\pi}{4}, n \in Z$

(b) $\theta = n\pi + \dfrac{3\pi}{4}, n \in Z$

(c) $\theta = n\pi + \dfrac{2\pi}{3}, n \in Z$

8. (a) $\theta = n\pi \pm \dfrac{\pi}{4}, n \in Z$

(b) $\theta = n\pi \pm \dfrac{\pi}{3}, n \in Z$

(c) $\theta = n\pi \pm \dfrac{\pi}{4}, n \in Z$

9. (a) $\theta = \frac{2n\pi}{p \mp q}, n \in Z$

(b) $\theta = \frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{10}, n \in Z$

(c) $\theta = (2n+1)\frac{\pi}{12}, n \in Z$

10. (a) $\theta = \frac{(2k+1)\pi}{m-n}$ or $\frac{2k\pi}{m+n}, k \in I$

(b) $\theta = \frac{(2k+1)\pi}{2(m-n)}, k \in Z$

(c) $\theta = (2n+1)\frac{\pi}{4}$ or $2n\pi \pm \frac{2\pi}{3}, n \in Z$

(d) $\theta = \frac{2n\pi}{5}$ or $\theta = n\pi \pm \frac{\pi}{2}, n \in Z$ or $\theta = (2n-1)\pi, n \in Z$

5

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RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

In earlier lesson, we have learnt about trigonometric functions of real numbers, relations between them, drawn the graphs of trigonometric functions, studied the characteristics from their graphs, studied about trigonometric functions of sum and difference of real numbers, and deduced trigonometric functions of multiple and sub-multiples of real numbers.

In this lesson, we shall try to establish some results which will give the relationship between sides and angles of a triangle and will help in finding unknown parts of a triangle.

OBJECTIVES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

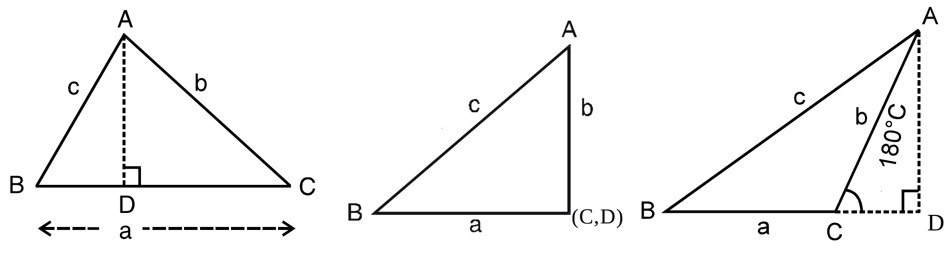
5.1 SINE FORMULA

In a ΔABC, the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

Prove that in any triangle, the lengths of the sides are proportional to the sines of the angles opposite to the sides,

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof : In ΔABC, in Fig. 5.1 [(i), (ii) and (iii)], BC = a, CA = b and AB = c and $\angle$C is acute angle in (i), right angle in (ii) and obtuse angle in (iii).



(i) (ii) (iii)

Fig. 5.1

Draw AD prependicular to BC (or BC produced, if need be)

In $\Delta ABC, \ \dfrac{AD}{AB} = \sin B$ or $\dfrac{AD}{c} = \sin B \Rightarrow AD = c \sin B$ .....(i)

In $\quad \Delta ADC, \ \dfrac{AD}{AC} = \sin C$ in Fig 5.1 (i)

or, $\quad \dfrac{AD}{b} = \sin C \quad \Rightarrow \quad AD = b \sin C$ ....(ii)

In Fig. 5.1 (ii), $\dfrac{AD}{AC} = 1 = \sin \dfrac{\pi}{2} = \sin C$ and $\dfrac{AD}{AB} = \sin B$

$AD = b \sin C$ and $AD = c \sin B$.

and in Fig. 5.1 (iii), $\dfrac{AD}{AC} = \sin(\pi - C) = \sin C$ and $\dfrac{AD}{AB} = \sin B$

or $\quad \dfrac{AD}{b} = \sin C$ or $AD = b \sin C$ and $AD = c \sin B$

Thus, in all three figures, $AD = b \sin C$ and $AD = c \sin B$ ....(iii)

From (iii) we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{....(iv)}$$

Similarly, by drawing prependiculars from C on AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{.....(v)}$$

From (iv) and (v), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{.....(A)}$$

(A) is called the sine-formula

Note : (A) is sometimes written as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \qquad \text{....(A')}$$

The relations (A) and (A') help us in finding unknown angles and sides, when some others are given.

Let us take some examples :

Example 5.1 Prove that $a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2}$, using sine-formula.

Solution : We know that, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (say)

$$\Rightarrow \qquad a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

$$\therefore \qquad \text{R.H.S.} = k(\sin B+\sin C)\cdot\sin\frac{A}{2}$$

$$= k\cdot 2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

Now $\qquad \frac{B+C}{2}=90^\circ-\frac{A}{2} \qquad (\because\ A+B+C=\pi)$

$$\therefore \qquad \sin\frac{B+C}{2}=\cos\frac{A}{2}$$

$$\therefore \qquad \text{R.H.S.} = 2k\cos\frac{A}{2}\cdot\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

$$= k\cdot\sin A\cdot\cos\frac{B-C}{2} = a\cdot\cos\frac{B-C}{2} = \text{L.H.S}$$

Example 5.2 Using sine formula, prove that

$$a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}$$

Solution : We have $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (say)

$$\Rightarrow \qquad a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

$$\therefore \qquad \text{R.H.S} = 2k(\sin B-\sin C)\cdot\cos^2\frac{A}{2}$$

$$= 2k\cdot 2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}\cdot\cos^2\frac{A}{2}$$

$$= 4k \sin\frac{A}{2}\cdot\sin\frac{B-C}{2}\cdot\ \cos^2\frac{A}{2} = 2a \sin\frac{B-C}{2}\cdot\ \cos\frac{A}{2}$$

$$= 2a \sin\frac{B+C}{2}\cdot\ \sin\frac{B-C}{2} \quad = a(\cos C - \cos B)\ =\ \text{L.H.S.}$$

Example 5.3 In any triangle ABC, show that

$$a\sin A - b\sin B = c\sin(A-B)$$

Solution : We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$$\text{L.H.S.}\ =\ k\sin A\cdot\sin A - k\sin B\cdot\sin B\ = k\left[\sin^2 A - \sin^2 B\right]$$

$$= k\sin(A+B)\cdot\sin(A-B)$$

$$A + B = \pi - C \Rightarrow\ \sin(A+B) = \sin C$$

$\therefore$ $$\text{L.H.S.}\ = k\sin C\cdot\sin(A-B)\ = c\sin(A-B) = \text{R.H.S.}$$

Example 5.4 In any triangle, show that

$$a(b\cos C - c\cos B) = b^2 - c^2$$

Solution : We have, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$$\text{L.H.S.}\ =\ k\sin A(k\sin B\cos C - k\sin C\cos B)\ = k^2\cdot\sin A\left[\sin(B-C)\right]$$

$$= k^2\cdot\sin(B+C)\cdot\sin(B-C) \qquad \left[\because\ \sin A = \sin(B+C)\right]$$

$$= k^2\left(\sin^2 B - \sin^2 C\right)\ = k^2\sin^2 B - k^2\sin^2 C\ = b^2 - c^2 = \text{R.H.S}$$

CHECK YOUR PROGRESS 5.1

1. Using sine-formula, show that each of the following hold :

(i) $\dfrac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}} = \dfrac{a-b}{a+b}$ (ii) $b\cos B + c\cos C = a\cos(B-C)$

(iii) $a\sin\dfrac{B-C}{2} = (b-c)\cos\dfrac{A}{2}$ (iv) $\dfrac{b+c}{b-c} = \tan\dfrac{B+C}{2}\cdot\cot\dfrac{B-C}{2}$

(v) $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

2. In any triangle if $\frac{a}{\cos A} = \frac{b}{\cos B}$, prove that the tiangle is isosceles.

5.2 COSINE FORMULA

In any triangle, prove that

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Proof :

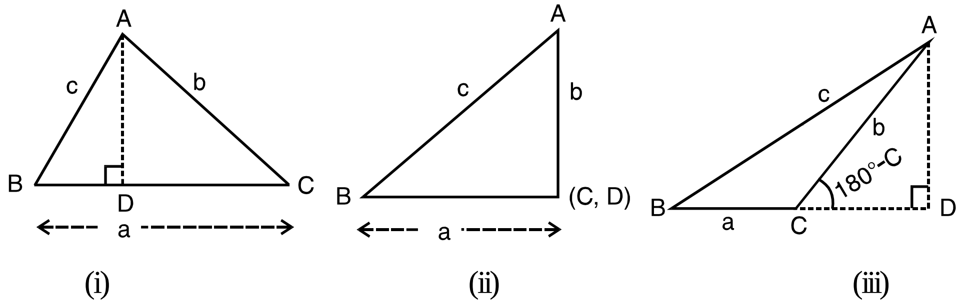


Fig. 5.2

Three cases arise :

(i) When $\angle C$ is acute (ii) When $\angle C$ is a right angle

(iii) When $\angle C$ is obtuse

Let us consider these one by one :

Case (i) When $\angle C$ is acute, $\frac{AD}{AC} = \sin C \quad \Rightarrow \quad AD = b \sin C$

Also $BD = BC - DC = a - b\cos C \quad \left[\because \frac{DC}{b} = \cos C\right]$

From Fig. 5.2 (i) $c^2 = (b \sin C)^2 + (a - b\cos C)^2$

$$= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab\cos C \; = a^2 + b^2 - 2\,ab\,\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$$

Case (ii) When $\angle C = 90°$, $c^2 = AD^2 + BD^2 = b^2 + a^2$

As $C = 90° \Rightarrow \cos C = 0 \;\therefore\; c^2 = b^2 + a^2 - 2ab \cdot \cos C$

$$\Rightarrow \quad \cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

Case (iii) When $\angle C$ is obtuse

$$\frac{AD}{AC} = \sin(180° - C) = \sin C$$

$\therefore$ $AD = b \sin C$

Also, $BD = BC + CD = a + b\cos(180° - C)$

$= a - b\cos C \quad \therefore\ c^2 = (b\sin C)^2 + (a - b\cos C)^2$

$$= a^2 + b^2 - 2ab\cos C \Rightarrow \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\therefore$ In all the three cases, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Similarly, it can be proved that $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Let us take some examples to show its application.

Example 5.5 In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution : We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},\ \cos B = \frac{c^2 + a^2 - b^2}{2ac},\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \qquad \text{L.H.S.} = \frac{b^2 + c^2 - a^2}{2\,abc} + \frac{c^2 + a^2 - b^2}{2\,abc} + \frac{a^2 + b^2 - c^2}{2\,abc}$$

$$= \frac{1}{2\,abc}\left[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2\right]$$

$$= \frac{a^2 + b^2 + c^2}{2\,abc} = \text{R.H.S.}$$

Example 5.6 If $\angle A = 60°$, show that in ΔABC

$$(a + b + c)(b + c - a) = 3bc$$

Solution : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ .....(i) Here $A = 60° \Rightarrow \cos A = \cos 60° = \dfrac{1}{2}$

$\therefore$ (i) becomes $\frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc} \quad \Rightarrow \quad b^2 + c^2 - a^2 = bc$

or $\quad b^2 + c^2 + 2bc - a^2 = 3bc \quad$ or $\quad (b+c)^2 - a^2 = 3bc$

or $\quad (b+c+a)(b+c-a) = 3bc$

Example 5.7 If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of the triangle.

Solution : Here a = 3 cm, b = 5 cm, c = 7 cm

We know that in a triangle, the angle opposite to the largest side is greatest

$\therefore \quad \angle C$ is the greatest angle. $\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$= \frac{9 + 25 - 49}{30} = \frac{-15}{30} = \frac{-1}{2}$$

$\therefore \quad \cos C = \frac{-1}{2} \quad \Rightarrow \quad C = \frac{2\pi}{3}$

$\therefore$ The greatest angle of the triangle is $\frac{2\pi}{3}$ or $\quad 120°$.

Example 5.8 In ΔABC, if $\angle A = 60°$, prove that $\frac{b}{c+a} + \frac{c}{a+b} = 1$.

Solution : $\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad$ or $\quad \cos 60° = \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$

$\therefore \quad b^2 + c^2 - a^2 = bc \quad$ or $\quad b^2 + c^2 = a^2 + bc \quad$ .....(i)

$$\text{L.H.S.} = \frac{b}{c+a} + \frac{c}{a+b} = \frac{ab + b^2 + c^2 + ac}{(c+a)(a+b)}$$

$$= \frac{ab + ac + a^2 + bc}{(c+a)(a+b)} \quad \text{[Using (i)]}$$

$$= \frac{ab + a^2 + ac + bc}{(c+a)(a+b)} = \frac{a(a+b) + c(a+b)}{(a+c)(a+b)} = \frac{(a+c)(a+b)}{(a+c)(a+b)} = 1$$

MODULE - I
Sets, Relations and Functions

Notes

CHECK YOUR PROGRESS 5.2

1. In any triangle ABC, show that

(i) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

(ii) $\left(a^2-b^2+c^2\right)\tan B=\left(b^2-c^2+a^2\right)\tan C=\left(c^2-a^2+b^2\right)\tan A$

(iii) $\frac{k}{2}(\sin 2A+\sin 2B+\sin 2C)=\frac{a^2+b^2+c^2}{2abc}$

where $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

(iv) $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0$

2. The sides of a triangle are $a=9$ cm, $b=8$ cm, $c=4$ cm. Show that

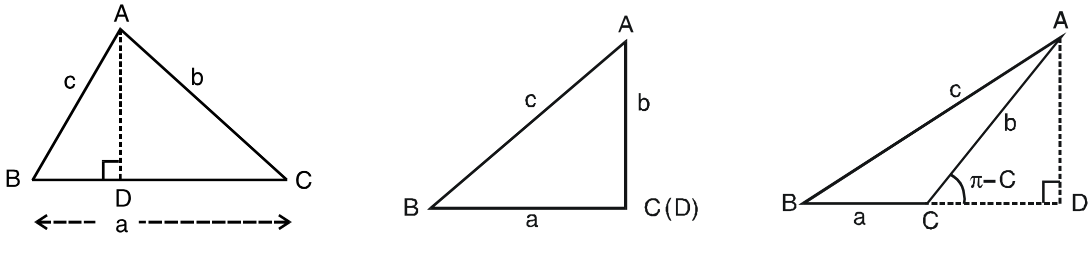
$$6\cos C = 4 + 3\cos B.$$

5.3 PROJECTION FORMULA

In ΔABC, if $BC=a$, $CA=b$ and $AB=c$, then prove that

(i) $a=b\cos C+c\cos B$ (ii) $b=c\cos A+a\cos C$ (iii) $c=a\cos B+b\cos A$

Proof :



(i) (ii) (iii)

Fig. 5.3

As in previous result, three cases arise. We will discuss them one by one.

(i) When $\angle C$ is acute :

In Δ ADB, $\frac{BD}{c}=\cos B \quad\Rightarrow\quad BD=c\cos B$

In Δ ADC, $\frac{DC}{b}=\cos C \quad\Rightarrow\quad DC=b\cos C$

$a = BD + DC = c\cos B + b\cos C$, $\therefore$ $a = c\cos B + b\cos c$

(ii) When $\angle C = 90°$

$$a = BC = \frac{BC}{AB} \cdot AB = \cos B \cdot c \; = c\cos B + 0$$

$$= c\cos B + b\cos 90° \quad (\because \cos 90° = 0) \; = c\cos B + b\cos C$$

(iii) When $\angle C$ is obtuse

In Δ ADB, $\dfrac{BD}{c} = \cos B \Rightarrow BD = c\cos B$

In Δ ADC, $\dfrac{CD}{b} = \cos(\pi - C) = -\cos C \quad \Rightarrow \quad CD = -b\cos C$

In Fig. 5.3 (iii),

$$BC = BD - CD \Rightarrow a = c\cos B - (-b\cos C) \; = c\cos B + b\cos C$$

Thus in all cases, $a = b\cos C + c\cos B$

Similarly, we can prove that

$$b = c\cos A + a\cos C \quad \text{and} \quad c = a\cos B + b\cos A$$

Let us take some examples, to show the application of these results.

Example 5.9 In any triangle ABC, show that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$$

Solution : L.H.S. $= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$$

$$= c + b + a \; = a + b + c = \text{R.H.S.}$$

Example 5.10 In any Δ ABC, prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution : L.H.S. $= \dfrac{1 - 2\sin^2 A}{a^2} - \dfrac{1 - 2\sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \qquad \left(\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = k\right)$$

= R. H. S.

Example 5.11 In Δ ABC, if $a\cos A = b\cos B$, where $a \neq b$ prove that Δ ABC is a right angled triangle.

Solution : $a\cos A = b\cos B$, $\therefore$ $a\left[\frac{b^2+c^2-a^2}{2bc}\right]=b\left[\frac{a^2+c^2-b^2}{2ac}\right]$

or $a^2\left(b^2+c^2-a^2\right)=b^2\left(a^2+c^2-b^2\right)$

or $a^2b^2+a^2c^2-a^4=a^2b^2+b^2c^2-b^4$

or $c^2\left(a^2-b^2\right)=\left(a^2-b^2\right)\left(a^2+b^2\right)$

$\Rightarrow$ $c^2=a^2+b^2$ $\therefore$ Δ ABC is a right triangle.

Example 5.12 If a = 2, b = 3, c = 4, find cos A, cos B and cos C.

Solution : $$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{9+16-4}{2\times3\times4}=\frac{21}{24}=\frac{7}{8}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ac}=\frac{16+4-9}{2\times4\times2}=\frac{11}{16}$$

and $$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{4+9-16}{2\times2\times3}=\frac{-3}{12}=\frac{-1}{4}$$

CHECK YOUR PROGRESS 5.3

1. If a = 3, b = 4 and c = 5, find cos A, cos B and cos C.
2. The sides of a triangle are 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm. Find the smallest angle of the triangle.
3. If a : b : c = 7 : 8 : 9, prove that
   cos A : cos B : cos C = 14 : 11 : 6.
4. If the sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Show that the greatest angle of the triangle is 120°.
5. In a triangle, b cos A = a cos B, prove that the triangle is isosceles.
6. Deduce sine formula from the projection formula.

LET US SUM UP

It is possible to find out the unknown elements of a triangle, if the relevent elements are given by using

Sine-formula :

(i) $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Cosine foumulae :

(ii) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{c^2+a^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Projection formulae :

$a = b\cos C + c\cos B,\ b = c\cos A + a\cos C,\ c = a\cos B + b\cos A$

SUPPORTIVE WEB SITES

www.mathopenref.com/trianglesideangle.html
http://en.wikipedia.org/wiki/Solution_of_triangles
www.themathpage.com/abookI/propI-18-19.htm

TERMINAL EXERCISE

In a triangle ABC, prove the following (1-10) :

1. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

2. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

3. $\dfrac{b^2-c^2}{a^2}\cdot\sin 2A + \dfrac{c^2-a^2}{b^2}\cdot\sin 2B + \dfrac{a^2-b^2}{c^2}\cdot\sin 2C = 0$

4. $\dfrac{c^2+a^2}{b^2+c^2} = \dfrac{1+\cos B\cos(C-A)}{1+\cos A\cos(B-C)}$

5. $\dfrac{c-b\cos A}{b-c\cos A} = \dfrac{\cos B}{\cos C}$

6. $\dfrac{a-b\cos C}{c-b\cos A} = \dfrac{\sin C}{\sin A}$

7. $(a+b+c)\left[\tan\dfrac{A}{2}+\tan\dfrac{B}{2}\right] = 2c\cot\dfrac{C}{2}$

8. $\sin\dfrac{A-B}{2} = \dfrac{a-b}{c}\cos\dfrac{C}{2}$

9. (i) $b\cos B + c\cos C = a\cos(B-C)$ (ii) $a\cos A + b\cos B = c\cos(A-B)$

10. $b^2 = (c-a)^2\cos^2\dfrac{B}{2} + (c+a)^2\sin^2\dfrac{B}{2}$

11. In a triangle, if $b = 5,\ c = 6,\ \tan\dfrac{A}{2} = \dfrac{1}{\sqrt{2}}$, then show that $a = \sqrt{41}$.

12. In any Δ ABC, show that $\dfrac{\cos A}{\cos B} = \dfrac{b-a\cos C}{a-b\cos C}$

ANSWERS

CHECK YOUR PROGRESS 5.3

1. $\cos A = \dfrac{4}{5}$

   $\cos B = \dfrac{3}{5}$

   $\cos C = \text{zero}$

2. The smallest angle of the triangle is 30°.

6

31len06

SEQUENCES AND SERIES

Succession of numbers of which one number is designated as the first, other as the second, another as the third and so on gives rise to what is called a sequence. Sequences have wide applications. In this lesson we shall discuss particular types of sequences called arithmetic sequence, geometric sequence and also find arithmetic mean (A.M), geometric mean (G.M) between two given numbers. We will also establish the relation between A.M and G.M.

Let us consider the following problems :

(a) A man places a pair of newly born rabbits into a warren and wants to know how many rabbits he would have over a certain period of time. A pair of rabbits will start producing offsprings two months after they were born and every following month one new pair of rabbits will appear. At the beginning the man will have in his warren only one pair of rabbits, during the second month he will have the same pair of rabbits, during the third month the number of pairs of rabbits in the warren will grow to two; during the fourth month there will be three pairs of rabbits in the warren. Thus, the number of pairs of rabbits in the consecutive months are :

1, 1, 2, 3, 5, 8, 13, ...

(b) The recurring decimal $0.\bar{3}$ can be written as a sum

$$0.\bar{3} = 0.3 + 0.03 + 0.003 + 0.0003 \ldots$$

(c) A man earns Rs.10 on the first day, Rs. 30 on the second day, Rs. 50 on the third day and so on. The day to day earning of the man may be written as 10, 30, 50, 70, 90, $\cdots$

We may ask what his earnings will be on the 10^{th} day in a specific month.

Again let us consider the following sequences:

(1) 2, 4, 8, 16, $\cdots$ (2) $\frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

(3) 0.01, 0.0001, 0.000001, $\cdots$

In these three sequences, each term except the first, progressess in a definite order but different from the order of other three problems. In this lesson we will discuss those sequences whose term progressess in a definite order.

OBJECTIVES

After studying this lesson, you will be able to :

- describe the concept of a sequence (progression);
- define an A.P. and cite examples;
- find common difference and general term of a A.P;
- find the fourth quantity of an A.P. given any three of the quantities a, d, n and t_n;
- calculate the common difference or any other term of the A.P. given any two terms of the A.P;
- derive the formula for the sum of 'n' terms of an A.P;
- calculate the fourth quantity of an A.P. given three of S, n, a and d;
- insert A.M. between two numbers;
- solve problems of daily life using concept of an A.P;
- state that a geometric progression is a sequence increasing or decreasing by a definite multiple of a non-zero number other than one;
- identify G.P.'s from a given set of progessions;
- find the common ratio and general term of a G.P;
- calculate the fourth quantity of a G.P when any three of the quantities t_n, a, r and n are given;
- calculate the common ratio and any term when two of the terms of the G.P. are given;
- write progression when the general term is given;
- derive the formula for sum of n terms of a G.P;
- calculate the fourth quantity of a G.P. if any three of a, r, n and S are given;
- derive the formula for sum (S_∞) of infinite number of terms of a G.P. when $|r| < 1$;
- find the third quantity when any two of S_∞, a and r are given;
- convert recurring decimals to fractions using G.P;
- insert G.M. between two numbers; and
- establish relationship between A.M. and G.M.

EXPECTED BACKGROUND KNOWLEDGE

- Laws of indices
- Simultaneous equations with two unknowns.
- Quadratic Equations.

6.1 SEQUENCE

A sequence is a collection of numbers specified in a definite order by some assigned law, whereby a definite number a_n of the set can be associated with the corresponding positive integer n. The different notations used for a sequence are.

1. $a_1, a_2, a_3, ..., a_n, ...$ 2. a_n , $n = 1, 2, 3, ...$ 3. $\{a_n\}$

Let us consider the following sequences :

1. 1, 2, 4, 8, 16, 32, ... 2. 1, 4, 9, 16, 25, ...

3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ 4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$

In the above examples, the expression for n^{th} term of the sequences are as given below :

(1) $a_n = 2^{n-1}$ (2) $a_n = n^2$ (3) $a_n = \frac{n}{n+1}$ (4) $a_n = \frac{1}{n}$

for all positive integer n.

Also for the first problem in the introduction, the terms can be obtained from the relation

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1},\ \text{n} \geq 3$$

A finite sequence has a finite number of terms. An infinite sequence contains an infinite number of terms.

6.2 ARITHMETIC PROGRESSION

Let us consider the following examples of sequence, of numbers :

(1) $2, 4, 6, 8, \cdots$ (2) $1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$

(3) $10, 8, 6, 4, \cdots$ (4) $-\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \cdots$

Note that in the above four sequences of numbers, the first terms are respectively 2, 1, 10, and $-\frac{1}{2}$. The first term has an important role in this lesson. Also every following term of the sequence has certain relation with the first term. What is the relation of the terms with the first term in Example (1) ? First term = 2, Second term = 4 = 2 + 1 × 2

Third term = 6 = 2 + 2 × 2

Fourth term = 8 = 2 + 3 × 2 and so on.

The consecutive terms in the above sequence are obtained by adding 2 to its preceding term. i.e., the difference between any two consecutive terms is the same.

A finite sequence of numbers with this property is called an arithmetic progression.
A sequence of numbers with finite terms in which the difference between two consecutive terms is the same non-zero number is called the Arithmetic Progression or simply A. P.

The difference between two consecutive terms is called the common defference of the A. P. and is denoted by 'd'.

In general, an A. P. whose first term is a and common difference is d is written as $a, a + d, a + 2d, a + 3d, \cdots$

Also we use t_n to denote the nth term of the progression.

6.2.1 GENERAL TERM OF AN A. P.

Let us consider A. P. $a, a + d, a + 2d, a + 3d, \cdots$

Here, first term $(t_1) = a$

second term $(t_2) = a + d = a + (2 - 1)\, d,$

third term $(t_3) = a + 2d = a + (3 - 1)\, d$

By observing the above pattern, nth term can be written as: $t_n = a + (n - 1)\, d$

Hence, if the first term and the common difference of an A. P. are known then any term of A. P. can be determined by the above formula.

Note.:

(i) If the same non-zero number is added to each term of an A. P. the resulting sequence is again an A. P.

(ii) If each term of an A. P. is multiplied by the same non-zero number, the resulting sequence is again an A. P.

Example 6.1 Find the 10th term of the A. P.: 2, 4, 6, ...

Solution : Here the first term $(a) = 2$ and common difference $d = 4 - 2 = 2$

Using the formula $t_n = a + (n - 1)\, d$, we have

$$t_{10} = 2 + (10 - 1)\, 2 = 2 + 18 = 20$$

Hence, the 10th term of the given A. P. is 20.

Example 6.2 The 10th term of an A. P. is – 15 and 31st term is –57, find the 15th term.

Solution : Let a be the first term and d be the common difference of the A. P. Then from the formula: $t_n = a + (n - 1)\, d$, we have

$$t_{10} = a + (10 - 1)\, d = a + 9d \text{ and } t_{31} = a + (31 - 1)\, d = a + 30\, d$$

We have, $a + 9d = -15$...(1), $a + 30d = -57$ ...(2)

Solve equations (1) and (2) to get the values of a and d.

Subtracting (1) from (2), we have

$$21d = -57 + 15 = -42 \qquad \therefore\ d = \frac{-42}{21} = -2$$

Again from (1), $a = -15 - 9d = -15 - 9(-2) = -15 + 18 = 3$

Now $t_{15} = a + (15 - 1)d = 3 + 14(-2) = -25$

Example 6.3 Which term of the A. P.: 5, 11, 17, ... is 119 ?

Solution : Here $a = 5$, $d = 11 - 5 = 6$

$$t_n = 119$$

We know that $t_n = a + (n-1)\,d$

$$\Rightarrow \quad 119 = 5 + (n-1) \times 6 \quad \Rightarrow \quad (n-1) = \frac{119-5}{6} = 19$$

$$\therefore \quad n = 20$$

Therefore, 119 is the 20th term of the given A. P.

Example 6.4 Is 600 a term of the A. P.: 2, 9, 16, ...?

Solution : Here, $a = 2$, and $d = 9 - 2 = 7$.

Let 600 be the n^{th} term of the A. P. We have $t_n = 2 + (n-1)\,7$

According to the question,

$$2 + (n-1)\,7 = 600 \qquad \therefore\ (n-1)\,7 = 598$$

or $$n = \frac{598}{7} + 1 \qquad \therefore\ n = 86\frac{3}{7}$$

Since n is a fraction, it cannot be a term of the given A. P. Hence, 600 is not a term of the given A. P.

Example 6.5 If $a + b + c = 0$ and $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are. in A. P., then prove that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are also in A. P.}$$

Solution. : Since $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A. P., therefore

$$\frac{b}{c+a}-\frac{a}{b+c}=\frac{c}{a+b}-\frac{b}{c+a}$$

or, $$\left(\frac{b}{c+a}+1\right)-\left(\frac{a}{b+c}+1\right)=\left(\frac{c}{a+b}+1\right)-\left(\frac{b}{c+a}+1\right)$$

or, $$\frac{a+b+c}{c+a}-\frac{a+b+c}{b+c}=\frac{a+b+c}{a+b}-\frac{a+b+c}{c+a}$$

or, $$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a} \quad (\text{Since } a+b+c \neq 0)$$

or, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

CHECK YOUR PROGRESS 6.1

1. Find the n^{th} term of each of the following A. P's. :

   (a) 1, 3, 5, 7, $\cdots$ (b) 3, 5, 7, 9, $\cdots$

2. If $t_n = 2n + 1$, then find the A. P.

3. Which term of the A. P. $2\frac{1}{2}, 4, 5\frac{1}{2}$, ... is 31? Find also the 10^{th} term?

4. Is – 292 a term of the A. P. 7, 4, 1, –2, ...?

5. The m^{th} term of an A. P. is n and the n^{th} term is m. Show that its $(m + n)^{th}$ term is zero.

6. Three numbers are in A. P. The difference between the first and the last is 8 and the product of these two is 20. Find the numbers.

7. The n^{th} term of a sequence is $na + b$. Prove that the sequence is an A. P. with common difference a.

6.3 TO FIND THE SUM OF FIRST *n* TERMS IN AN A. P.

Let a be the first term and d be the common difference of an A. P. Let l denote the last term, i.e., the n^{th} term of the A. P. Then, $l = t_n = a + (n-1)d$ ... (i)

Let S_n denote the sum of the first n terms of the A. P. Then

$$S_n = a + (a + d) + (a + 2d) + ... + (l - 2d) + (l - d) + l \quad ... \text{(ii)}$$

Reversing the order of terms in the R. H. S. of the above equation, we have

$$S_n = l + (l - d) + (l - 2d) + ... + (a + 2d) + (a + d) + a \quad ... \text{(iii)}$$

Adding (ii) and (iii) vertically, we get

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots \text{ containing } n \text{ terms} = n\,(a + l)$$

i.e., $S_n = \frac{n}{2}(a + l)$

Also $S_n = \frac{n}{2}[2a + (n - 1)\,d]$ [From (i)]

It is obvious that $t_n = S_n - S_{n-1}$

Example 6.6 Find the sum of $2 + 4 + 6 + \ldots\ n$ terms.

Solution.: Here $a = 2$, $d = 4 - 2 = 2$

Using the formula $S_n = \frac{n}{2}[2a + (n - 1)\,d]$, we get

$$S_n = \frac{n}{2}[2 \times 2 + (n - 1)\,2] = \frac{n}{2}[2 + 2n] = \frac{2n\,(n + 1)}{2} = n\,(n + 1)$$

Example 6.7 The 35th term of an A. P. is 69. Find the sum of its 69 terms.

Solution. Let a be the first term and d be the common difference of the A. P.

We have $t_{35} = a + (35 - 1)\,d = a + 34\,d$.

$\therefore \quad a + 34\,d = 69 \qquad \ldots$ (i)

Now by the formula, $S_n = \frac{n}{2}[2a + (n - 1)\,d]$

We have $S_{69} = \frac{69}{2}[2a + (69 - 1)\,d]$

$= 69\,(a + 34d)$ [using (i)]

$= 69 \times 69 = 4761$

Example 6.8 The first term of an A. P. is 10, the last term is 50. If the sum of all the terms is 480, find the common difference and the number of terms.

Solution : We have: $a = 10$, $l = t_n = 50$, $S_n = 480$.

By substituting the values of a, t_n and S_n in the formulae

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ and } t_n = a + (n-1)d, \text{ we get}$$

$$480 = \frac{n}{2}\left[20 + (n-1)\,d\right] \qquad \cdots \text{(i)}$$

$$50 = 10 + (n-1)\,d \qquad \cdots \text{(ii)}$$

From (ii), $(n-1)\,d = 50 - 10 = 40$ $\qquad \cdots$ (iii)

From (i), we have $480 = \frac{n}{2}(20+40)$ $\qquad$ using (i)

or, $\quad 60n = 2 \times 480 \quad \therefore \quad n = \frac{2 \times 480}{60} = 16$

From (iii),

$\therefore \quad d = \frac{40}{15} = \frac{8}{3}$ $\qquad$ (as $n-1 = 16-1 = 15$)

Example 6. 9 Let the n^{th} term and the sum of n terms of an A. P. be p and q respectively. Prove that its first term is $\left(\frac{2q - pn}{n}\right)$.

Solution: In this case, $t_n = p$ and $S_n = q$

Let a be the first term of the A. P.

Now, $S_n = \frac{n}{2}(a + t_n)$ $\quad$ or, $\quad \frac{n}{2}(a+p) = q$

or, $\quad a + p = \frac{2q}{n}$ $\quad$ or, $\quad a = \frac{2q}{n} - p$ $\quad \therefore \quad a = \frac{2q - pn}{n}$

CHECK YOUR PROGRESS 6.2

1. Find the sum of the following A. P's.

   (a) 8, 11, 14, 17, $\cdots$ up to 15 terms $\qquad$ (b) 8, 3, –2, –7, –12, $\cdots$ up to n terms.

2. How many terms of the A. P.: 27, 23, 19, 15, ... have a sum 95?

3. A man takes an interest-free loan of Rs. 1740 from his friend agreeing to repay in monthly instalments. He gives Rs. 200 in the first month and diminishes his monthly instalments by Rs. 10 each month. How many months will it take to repay the loan?

4. How many terms of the progression 3, 6, 9, 12, $\cdots$

   must be taken at the least to have a sum not less than 2000?

5. In a children potato race, n potatoes are placed 1 metre apart in a straight line. A competitor starts from a point in the line which is 5 metre from the nearest potato. Find an expression for the total distance run in collecting the potatoes, one at a time and bringing them back one at a time to the starting point. Calculate the value of n if the total distance run is 162 metres.

6. If the sum of first n terms of a sequence be $an^2 + bn$, prove that the sequence is an A. P. and find its common difference ?

6.4 ARITHMETIC MEAN (A. M.)

When three numbers a, A and b are in A. P., then A is called the arithmetic mean of numbers a and b. We have, $A - a = b - A$

or, $$A = \frac{a+b}{2}$$

Thus, the required A. M. of two numbers a and b is $\frac{a+b}{2}$. Consider the following A. P :

3, 8, 13, 18, 23, 28, 33.

There are five terms between the first term 3 and the last term 33. These terms are called *arithmetic means* between 3 and 33. Consider another A. P. : 3, 13, 23, 33. In this case there are two arithmetic means 13, and 23 between 3 and 33.

Generally any number of arithmetic means can be inserted between any two numbers a and b. Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between a and b, then.

$a, A_1, A_2, A_3 \ldots, A_n, b$ is an A. P.

Let d be the common difference of this A. P. Clearly it contains $(n + 2)$ terms

$\therefore$ $b = (n + 2)^{\text{th}}$ term

$= a + (n + 1)\, d$

$\therefore$ $$d = \frac{b-a}{n+1}$$

Now, $A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$ ...(i)

$$A_2 = a+2d \Rightarrow A_2 = \left[a+\frac{2(b-a)}{n+1}\right] \qquad \text{...(ii)}$$

$$\vdots$$

$$A_n = a+nd \Rightarrow An \left[a+\frac{n(b-a)}{n+1}\right] \qquad \text{... }(n)$$

These are required n arithmetic means between a and b.

Adding (i), (ii), ..., (n), we get

$$A_1 + A_2 + \ldots + A_n = na + \cdots + \frac{b-a}{n+1}[1+2+\ldots n]$$

$$= na + \left(\frac{b-a}{n+1}\right)\left(\frac{n(n+1)}{2}\right) = na + \frac{n(b-a)}{2} = \frac{n(a+b)}{2}$$

$= n$ [Single A. M. between a and b]

Example 6.10 Insert five arithmetic means between 8 and 26.

Solution : Let A_1, A_2, A_3, A_4 and A_5 be five arithmetic means between 8 and 26.

Therefore, 8, A_1, A_2, A_3, A_4, A_5, 26 are in A. P. with $a = 8$, $b = 26$, $n = 7$

We have $26 = 8 + (7-1)\,d \quad \therefore \quad d = 3$

$\therefore \quad A_1 = a + d = 8 + 3 = 11$, $A_2 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 17$, $A_4 = a + 4d = 20$, $A_5 = a + 5d = 23$

Hence, the five arithmetic means between 8 and 26 are 11, 14, 17, 20 and 23.

Example 6.11 The 'n', A. M's between 20 and 80 are such that the ratio of the first mean and the last mean is 1 : 3. Find the value of n.

Solution : Here, 80 is the $(n+2)^{th}$ term of the A. P., whose first term is 20. Let d be the common difference.

$\therefore \quad 80 = 20 + (n+2-1)\,d \quad$ or, $\quad 80 - 20 = (n+1)\,d \quad$ or, $\quad d = \dfrac{60}{n+1}$

The first A. M. $= 20 + \dfrac{60}{n+1} = \dfrac{20n+20+60}{n+1} = \dfrac{20n+80}{n+1}$

The last A. M. $= 20 + n \times \dfrac{60}{n+1} = \dfrac{80n+20}{n+1}$

We have $\frac{20n+80}{n+1} : \frac{80n+20}{n+1} = 1:3$ or, $\frac{n+4}{4n+1} = \frac{1}{3}$

or, $4n + 1 = 3n + 12$ or, $n = 11$

$\therefore$ The number of A. M's between 20 and 80 is 11.

CHECK YOUR PROGRESS 6.3

1. Prove that if the number of terms of an A. P. is odd then the middle term is the A. M. between the first and last terms.

2. Between 7 and 85, m number of arithmetic means are inserted so that the ratio of $(m-3)^{th}$ and m^{th} means is 11 : 24. Find the value of m.

3. Prove that the sum of n arithmetic means between two numbers is n times the single A. M. between them.

4. If the A. M. between p^{th} and q^{th} terms of an A. P., be equal and to the A. M. between r^{th} and s^{th} terms of the A. P., then show that $p + q = r + s$.

6.5 GEOMETRIC PROGRESSION

Let us consider the following sequence of numbers :

(1) $1, 2, 4, 8, 16, \cdots$ (2) $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \cdots$

(3) $1, -3, 9, -27, \cdots$ (4) $x, x^2, x^3, x^4, \cdots$

If we see the patterns of the terms of every sequence in the above examples each term is related to the leading term by a definite rule.

For Example (1), the first term is 1, the second term is twice the first term, the third term is 2^2 times of the leading term.

Again for Example (2), the first term is 3, the second term is $\frac{1}{3}$ times of the first term, third term is $\frac{1}{3^2}$ times of the first term.

A sequence with this property is called a *gemetric progression*.

A sequence of numbers in which the ratio of any term to the term which immediately precedes is the same non zero number (other than1), is called a geometric progression or simply G. P. This ratio is called the common ratio.

Thus, $\frac{\textit{Second term}}{\textit{First term}} = \frac{\textit{Third term}}{\textit{Second term}} =$ ..... is called the common ratio of the geometric progression.

Examples (1) to (4) are geometric progressions with the first term 1, 3, 1, x and with common ratio $2, \frac{1}{3}, -3$, and x respectively.

The most general form of a G. P. with the first term a and common ratio r is $a, ar, ar^2, ar^3, \ldots$

6.5.1 GENERAL TERM

Let us consider a geometric progression with the first term a and common ratio r. Then its terms are given by $a, ar, ar^2, ar^3, \ldots$

In this case, $t_1 = a = ar^{1-1}$ $\quad t_2 = ar = ar^{2-1}$

$t_3 = ar^2 = ar^{3-1}$ $\quad t_4 = ar^3 = ar^{4-1}$

... ...

On generalisation, we get the expression for the n^{th} term as $t_n = ar^{n-1}$ ... (A)

6.5.2 SOME PROPERTIES OF G. P.

(i) If all the terms of a G. P. are multiplied by the same non-zero quantity, the resulting series is also in G. P. The resulting G. P. has the same common ratio as the original one.

If $a, b, c, d, \ldots$ are in G. P.

then $ak, bk, ck, dk \ldots$ are also in G. P. $\quad (k \neq 0)$

(ii) If all the terms of a G. P. are raised to the same power, the resulting series is also in G. P.

Let $a, b, c, d \ldots$ are in G. P.

the $a^k, b^k, c^k, d^k, \ldots$ are also in G. P. $\quad (k \neq 0)$

The common ratio of the resulting G. P. will be obtained by raising the same power to the original common ratio.

Example 6.12 Find the 6^{th} term of the G. P.: 4, 8, 16, ...

Solution : In this case the first term $(a) = 4$ Common ratio $(r) = 8 \div 4 = 2$

Now using the formula $t_n = ar^{n-1}$, we get $t_6 = 4 \times 2^{6-1} = 4 \times 32 = 128$

Hence, the 6^{th} term of the G. P. is 128.

Example 6.13 The 4^{th} and the 9^{th} term of a G. P. are 8 and 256 respectively. Find the G. P.

Solution : Let a be the first term and r be the common ratio of the G. P., then

$t_4 = ar^{4-1} = ar^3, t_9 = ar^{9-1} = ar^8$

According to the question, $ar^8 = 256$ $\quad \cdots (1)$

and $\quad ar^3 = 8$ $\quad \cdots (2)$

$\therefore \quad \dfrac{ar^8}{ar^3} = \dfrac{256}{8}$ or $\quad r^5 = 32 = 2^5 \quad \therefore\ r = 2$

Again from (2), $a \times 2^3 = 8 \quad \therefore \quad a = \dfrac{8}{8} = 1$

Therefore, the G. P. is 1, 2, 4, 8, 16, ...

Example 6.14 Which term of the G. P.: 5, –10, 20, – 40, ... is 320?

Solution : In this case, $a = 5$; $r = \dfrac{-10}{5} = -2$.

Suppose that 320 is the n^{th} term of the G. P.

By the formula, $t_n = ar^{n-1}$, we get $t_n = 5.\ (-2)^{n-1}$

$\therefore \quad 5.\ (-2)^{n-1} = 320 \qquad$ (Given)

$\therefore \quad (-2)^{n-1} = 64 = (-2)^6$

$\therefore \quad n - 1 = 6 \ \therefore \quad n = 7$ Hence, 320 is the 7^{th} term of the G. P.

Example 6.15 If $a, b, c,$ and d are in G. P., then show that $(a + b)^2$, $(b + c)^2$, and $(c + d)^2$ are also in G. P.

Solution. Since $a, b, c,$ and d are in G. P., $\therefore \quad \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c}$

$\therefore \quad b^2 = ac,\ c^2 = bd,\ ad = bc \qquad$ ...(1)

Now, $(a + b)^2 (c + d)^2 = \left[(a+b)(c+d)\right]^2 = (ac + bc + ad + bd)^2$

$= (b^2 + c^2 + 2bc)^2 \qquad$ ...[Using (1)]

$= \left[(b+c)^2\right]^2$

$\therefore \quad \dfrac{(c+d)^2}{(b+c)^2} = \dfrac{(b+c)^2}{(a+b)^2}$ Thus, $(a + b)^2, (b + c)^2, (c + d)^2$ are in G. P.

CHECK YOUR PROGRESS 6.4

1. The first term and the common ratio of a G. P. are respectively 3 and $-\dfrac{1}{2}$. Write down the first five terms.

2. Which term of the G. P. 1, 2, 4, 8, 16, ... is 1024? Is 520 a term of the G. P.?

3. Three numbers are in G. P. Their sum is 43 and their product is 216. Find the numbers in proper order.

4. The n^{th} term of a G. P. is 2×3^n for all n. Find (a) the first term (b) the common ratio of the G. P.

6.6 SUM OF *n* TERMS OF A G. P.

Let a denote the first term and r the common ratio of a G. P. Let S_n represent the sum of first n terms of the G. P. Thus, $S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$ ... (1)

Multiplying (1) by r, we get $r S_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n$ ... (2)

$(1) - (2) \Rightarrow S_n - rS_n = a - ar^n$ or $S_n (1 - r) = a(1 - r^n)$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} \qquad \text{....(A)}$$

$$= \frac{a(r^n-1)}{r-1} \qquad \text{....(B)}$$

Either (A) or (B) gives the sum up to the n^{th} term when $r \neq 1$. It is convenient to use formula (A) when $|r| < 1$ and (B) when $|r| > 1$.

Example 6.16 Find the sum of the G. P.: 1, 3, 9, 27, ... up to the 10^{th} term.

Solution : Here the first term (a) = 1 and the common ratio $(r) = \frac{3}{1} = 3$

Now using the formula, $S_n = \frac{a\left(r^n - 1\right)}{r-1}$, ($\because r > 1$) we get $S_{10} = \frac{1.(3^{10}-1)}{3-1} = \frac{3^{10}-1}{2}$

Example 6.17 Find the sum of the G. P.: $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, \ , \ldots, 81$

Solution : Here, $a = \frac{1}{\sqrt{3}}; r = \sqrt{3}$ and $t_n = l = 81$

Now $t_n = 81 = \frac{1}{\sqrt{3}}(\sqrt{3})^{n-1} = (\sqrt{3})^{n-2}$

$$\therefore \quad (\sqrt{3})^{n-2} = 3^4 = (\sqrt{3})^8 \quad \therefore n - 2 = 8 \text{ or } n = 10$$

$$\therefore \quad S_n = \frac{\frac{1}{\sqrt{3}}\left[\sqrt{3}^{10} - 1\right]}{\sqrt{3}-1} = \frac{\left(\sqrt{3}\right)^{10} - 1}{3-\sqrt{3}}$$

Example 6.18 Find the sum of the G. P.: 0.6, 0.06, 0.006, 0.0006, $\cdots$ to n terms.

Solution. Here, $a = 0.6 = \frac{6}{10}$ and $r = \frac{0.06}{0.6} = \frac{1}{10}$

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we have $\qquad [\because\ r < 1]$

$$S_n = \frac{\frac{6}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}} = \frac{6}{9}\left(1-\frac{1}{10^n}\right) = \frac{2}{3}\left(1-\frac{1}{10^n}\right)$$

Hence, the required sum is $\frac{2}{3}\left(1-\frac{1}{10^n}\right)$.

Example 6.19 How many terms of the G. P.: 64, 32, 16, $\cdots$ has the sum $127\frac{1}{2}$?

Solution : Here, $a = 64, r = \frac{32}{64} = \frac{1}{2}$ (<1) and $S_n = 127\frac{1}{2} = \frac{255}{2}$.

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we get

$$S_n = \frac{64\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \Rightarrow \frac{64\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = \frac{255}{2} \quad \text{... (given)}$$

or $\quad 128\left[1-\left(\frac{1}{2}\right)^n\right] = \frac{255}{2}$ or $1-\left(\frac{1}{2}\right)^n = \frac{255}{256}$

or $\quad \left(\frac{1}{2}\right)^n = 1-\frac{255}{256} = \frac{1}{256} = \left(\frac{1}{2}\right)^8 \qquad \therefore \qquad n = 8$

Thus, the required number of terms is 8.

Example 6.20 Find the sum of the following sequence :

2, 22, 222, ... to n terms.

Solution : Let S denote the sum. Then

$$S = 2 + 22 + 222 + \cdots \text{ to } n \text{ terms} = 2\,(1 + 11 + 111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\,(9 + 99 + 999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+ \quad ...to\ n\ terms)\right\}$$

$$= \frac{2}{9}\left\{(10-10^2+10^3+...to\ n\ terms)-(1+1+1+ \quad ...to\ n\ terms)\right\}$$

$$= \frac{2}{9}\left\{\frac{(10^n-1)}{10-1}-n\right\} \qquad [\because 10-10^2+10^3+\cdots \text{ is a G P with } r = -10<1]$$

$$= \frac{2}{9}\left\{\frac{10^n-1-9n}{9}\right\} = \frac{2}{81}\left(10^n-1-9n\right)$$

Example 6. 21 Find the sum up to n terms of the sequence:

$$0.7,\ 0.77,\ 0.777,\ \cdots$$

Solution : Let S denote the sum, then

$$S = 0.7 + 0.77 + 0.777 + \cdots \text{ to } n \text{ terms}$$

$$= 7(0.1 + 0.11 + 0.111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,(0.9 + 0.99 + 0.999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,\{(1–0.1) + (1–0.01) + (1 – 0.001) + \cdots \text{ to } n \text{ terms}\}$$

$$= \frac{7}{9}\,\{(1 + 1 + 1 + ...\ n \text{ terms}) – (0.1 + 0.01 + 0.001 + \cdots \text{ to } n \text{ terms})\}$$

$$= \frac{7}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots to\ n\ terms\right)\right\}$$

$$= \frac{7}{9}\left\{ n - \frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}} \right\} \qquad \text{(Since r < 1)}$$

$$= \frac{7}{9}\left\{ n - \frac{1}{9}\left[1-\frac{1}{10^n}\right]\right\} = = \frac{7}{9}\left\{\frac{9n-1+10^{-n}}{9}\right\} = \frac{7}{81}\left[9n-1+10^{-n}\right]$$

CHECK YOUR PROGRESS 6.5

1. Find the sum of each of the following G. P's :

   (a) 6, 12, 24, ... to 10 terms (b) $1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} -$ ... to 20 terms.

2. How many terms of the G. P. $8, 16, 32, 64, \cdots$ have their sum 8184 ?

3. Show that the sum of the G. P. $a + b + ... + l$ is $\frac{bl-a^2}{b-a}$

4. Find the sum of each of the following sequences up to n terms.

   (a) 8, 88, 888, ... (b) 0.2, 0.22, 0.222, ...

6.7 INFINITE GEOMETRIC PROGRESSION

So far, we have found the sum of a finite number of terms of a G. P. We will now learn to find out the sum of infinitely many terms of a G P such as. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$

We will proceed as follows: Here $a = 1,\ r = \frac{1}{2}$.

The n^{th} term of the G. P. is $t_n = \frac{1}{2^{n-1}}$ and sum to n terms

i.e., $$S_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}} < 2.$$

So, no matter, how large n may be, the sum of n terms is never more than 2.

So, if we take the sum of all the infinitely many terms, we shall not get more than 2 as answer.

Also note that the recurring decimal 0.3 is really 0.3 + 0.03 + 0.003 + 0.0003 + ...

i.e., 0.3 is actually the sum of the above infinite sequence.

On the other hand it is at once obvious that if we sum infinitely many terms of the G. P. 1, 2, 4, 8, 16, ... we shall get a infiinite sum.

So, sometimes we may be able to add the infinitely many terms of G. P. and sometimes we may not. We shall discuss this question now.

6.7.1 SUM OF INFINITE TERMS OF A G. P.

Let us consider a G. P. with infinite number of terms and common ratio r.

Case 1 : We assume that $|r| > 1$

The expression for the sum of n terms of the G. P. is then given by

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a r^n}{r - 1} - \frac{a}{r - 1} \qquad \text{... (A)}$$

Now as n becomes larger and larger r^n also becomes larger and larger. Thus, when n is infinitely large and $|r| > 1$ then the sum is also infinitely large which has no importance in Mathematics. We now consider the other possibility.

Case 2 : Let $|r| < 1$

Formula (A) can be written as $S = \frac{a\,(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$

Now as n becomes infinitely large, r^n becomes infinitely small, i.e., as $n \to \infty$, $r^n \to 0$, then

the above expression for sum takes the form $S = \frac{a}{1 - r}$

Hence, the sum of an infinite G. P. with the first term a and common ratio r is given by

$$S = \frac{a}{1 - r}, \text{ when } |r| < 1 \qquad \text{...(i)}$$

Example 6.22 Find the sum of the infinite G. P. $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \cdots$

Solution : Here, the first term of the infinite G. P. is $a = \frac{1}{3}$, and $r = \frac{-\frac{2}{9}}{\frac{1}{3}} = -\frac{2}{3}$.

Here, $|r| = \left| -\frac{2}{3} \right| = \frac{2}{3} < 1$

$\therefore$ Using the formula for sum $S = \dfrac{a}{1-r}$ we have $S = \dfrac{\frac{1}{3}}{1-\left(\frac{-2}{3}\right)} = \dfrac{\frac{1}{3}}{1+\frac{2}{3}} = \dfrac{1}{5}$

Hence, the sum of the given G. P. is $\dfrac{1}{5}$.

Example 6.23 Express the recurring decimal $0.\bar{3}$ as an infinite G. P. and find its value in rational form.

Solution. $0.\bar{3}$ = 0.3333333 ...

= 0.3 + 0.03 + 0.003 + 0.0003 + ...

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$$

The above is an infinite G. P. with the first term $a = \dfrac{3}{10}$ and $r = \dfrac{\frac{3}{10^2}}{\frac{3}{10}} = \dfrac{1}{10} < 1$

Hence, by using the formula $S = \dfrac{a}{1-r}$, we get $0.\bar{3} = \dfrac{\frac{3}{10}}{1-\frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \dfrac{3}{9} = \dfrac{1}{3}$

Hence, the recurring decimal $0.\bar{3} = \dfrac{1}{3}$.

Example 6.24 The distance travelled (in cm) by a simple pendulum in consecutive seconds are 16, 12, 9, ... How much distance will it travel before coming to rest ?

Solution : The distance travelled by the pendulum in consecutive seconds are, 16, 12, 9, ... is an infinite geometric progression with the first term $a = 16$ and r $= \dfrac{12}{16} = \dfrac{3}{4} < 1$.

Hence, using the formula $S = \dfrac{a}{1-r}$ we have

$$S = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64$$

$\therefore$ Distance travelled by the pendulum is 64 cm.

Example 6.25 The sum of an infinite G. P. is 3 and sum of its first two terms is $\frac{8}{3}$. Find the first term.

Solution: In this problem $S = 3$. Let a be the first term and r be the common ratio of the given infinite G. P.

Then according to the question. $a + ar = \frac{8}{3}$

or, $\quad 3a(1+r) = 8 \qquad \ldots (1)$

Also from $S = \frac{a}{1-r}$, we have $3 = \frac{a}{1-r}$

or, $\quad a = 3(1-r) \qquad \ldots (2)$

From (1) and (2), we get.

$3.3\,(1 - r)\,(1 + r) = 8$

or, $\quad 1 - r^2 = \frac{8}{9}$ or, $\quad r^2 = \frac{1}{9}$

or, $\quad r = \pm\frac{1}{3}$

From (2), $a = 3\left(1 \mp \frac{1}{3}\right) = 2$ or 4 according as $r = \pm\frac{1}{3}$.

CHECK YOUR PROGRESS 6.6

(1) Find the sum of each of the following inifinite G. P's :

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \infty$ (b) $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots \infty$

2. Express the following recurring decimals as an infinite G. P. and then find out their values as a rational number. (a) $0.\overline{7}$ (b) $0.3\overline{15}$

3. The sum of an infinite G. P. is 15 and the sum of the squares of the terms is 45. Find the G.P.

4. The sum of an infinite G. P. is $\frac{1}{3}$ and the first term is $\frac{1}{4}$. Find the G.P.

6.8 GEOMETRIC MEAN (G. M.)

If a, G, b are in G. P., then G is called the geometric mean between a and b.

If three numbers are in G. P., the middle one is called the geometric mean between the other two.

If $a, G_1, G_2, \ldots, G_n, b$ are in G. P.,

then $G_1, G_2, \ldots G_n$ are called n G. M.'s between a and b.

The geometric mean of n numbers is defined as the n^{th} root of their product.

Thus if $a_1, a_2, \ldots, a_n$ are n numbers, then their

$$\text{G. M.} = (a_1, a_2, \ldots a_n)^{\frac{1}{n}}$$

Let G be the G. M. between a and b, then a, G, b are in G. P $\therefore \dfrac{G}{a} = \dfrac{b}{G}$

or, $G^2 = ab$ or, $\qquad G = \sqrt{ab}$

$\therefore$ Geometric mean $= \sqrt{\text{Product of extremes}}$

Given any two positive numbers a and b, any number of geometric means can be inserted between them Let $a_1, a_2, a_3 \ldots, a_n$ be n geometric means between a and b.

Then $a_1, a_1, a_3, \ldots\ldots. a_n, b$ is a G. P.

Thus, b being the $(n + 2)^{th}$ term, we have

$$b = a\,r^{n+1}$$

or, $\quad r^{n+1} = \dfrac{b}{a}$ or, $\quad r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

Hence, $a_1 = ar = a \times \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}, \; a_2 = ar^2 = a \times \left(\dfrac{b}{a}\right)^{\frac{2}{n+1}}$

$\ldots \qquad \ldots$

$\ldots \qquad \ldots$

$$a_n = ar^n = a \times \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Further we can show that the product of these n G. M.'s is equal to n^{th} power of the single geometric mean between a and b.

Multiplying $a_1 . a_2, \ldots a_n$, we have

$$a_1, a_2 \cdots a_n = a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}+\cdots+\frac{n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{1+2+\cdots+n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n \left[\frac{b}{a}\right]^{\frac{n}{2}} = (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n = \mathrm{G}^n = (\text{single G. M. between } a \text{ and } b)^n$$

Example 6.26 Find the G. M. between $\frac{3}{2}$ and $\frac{27}{2}$

Solution : We know that if a is the G. M. between a and b, then $G = \sqrt{ab}$

$\therefore$ G. M. between $\frac{3}{2}$ and $\frac{27}{2} = \sqrt{\frac{3}{2} \times \frac{27}{2}} = \frac{9}{2}$

Example 6.27 Insert three geometric means between 1 and 256.

Solution : Let G_1, G_2, G_3, be the three geometric means between 1 and 256.

Then 1, G_1, G_2, G_3, 256 are in G. P.

If r be the common ratio, then $t_5 = 256$ i.e, $ar^4 = 256 \Rightarrow 1.\ r^4 = 256$

or, $r^2 = 16$ or, $r = \pm 4$

When $r = 4$, $G_1 = 1.\ 4 = 4$, $G_2 = 1.\ (4)^2 = 16$ and $G_3 = 1.\ (4)^3 = 64$

When $r = -4$, $G_1 = -4$, $G_2 = (1)\ (-4)^2 = 16$ and $G_3 = (1)\ (-4)^3 = -64$

$\therefore$ G.M. between 1 and 256 are 4, 16, 64, or, – 4, 16, –64.

Example 6.28 If 4, 36, 324 are in G. P. insert two more numbers in this progression so that it again forms a G. P.

Solution : G. M. between 4 and 36 = $\sqrt{4 \times 36} = \sqrt{144} = 12$

G. M. between 36 and $324 = \sqrt{36 \times 324} = 6 \times 18 = 108$

If we introduce 12 between 4 and 36 and 108 betwen 36 and 324, the numbers 4, 12, 36, 108, 324 form *a* G. P.

$\therefore$ The two new numbers inserted are 12 and 108.

Example 6.29 Find the value of n such that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

Solution : If x be G. M. between a and b, then $x = a^{\frac{1}{2}} \times b^{\frac{1}{2}}$

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{\frac{1}{2}} b^{\frac{1}{2}} \text{ or, } \quad a^{n+1}+b^{n+1} = \left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right)\left(a^n+b^n\right)$$

$$\text{or, } \quad a^{n+1}+b^{n+1} = a^{n+\frac{1}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{n+\frac{1}{2}} \text{ or, } a^{n+1} - a^{n+\frac{1}{2}} . b^{\frac{1}{2}} = a^{\frac{1}{2}} b^{n+\frac{1}{2}} - b^{n+1}$$

$$\text{or, } \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \text{ or, } \quad a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$$

$$\text{or, } \quad \frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1 \text{ or, } \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = \left(\frac{a}{b}\right)^0$$

$$\therefore \quad n+\frac{1}{2} = 0 \text{ or, } n = \frac{-1}{2}$$

6.8.1 RELATIONSHIP BETWEEN A. M. AND G.M.

Let a and b be the two numbers.

Let A and G be the A. M. and G. M. respectively between a and b

$$\therefore \quad A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab}}{2} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$$

$$\therefore \quad A > G$$

Example 6.30 The arithemetic mean between two numbers is 34 and their geometric mean is 16. Find the numbers.

Solution : Let the numbers be a and b. Since A. M. between a and b is 34,

$$\therefore \quad \frac{a+b}{2} = 34, \text{ or, } a+b = 68 \qquad \text{... (1)}$$

Since G. M. between a and b is 16,

$$\therefore \quad \sqrt{ab} = 16 \text{ or, } ab = 256 \text{ we know that } (a-b)^2 = (a+b)^2 - 4\,ab \qquad \cdots(2)$$

$$= (68)^2 - 4 \times 256 = 4624 - 1024 = 3600$$

$\therefore \quad a - b = \sqrt{3600} = 60 \qquad \cdots(3)$

Adding (1) and (3), we get, $2a = 128 \quad \therefore \quad a = 64$

Subtracting (3) from (1), we get

$2b = 8 \quad$ or, $\quad b = 4$

$\therefore$ Required numbers are 64 and 4.

Example 6.31 The arithmetic mean between two quantities b and c is a and the two geometric means between them are g_1 and g_2. Prove that $g_1^3 + g_2^3 = 2\,abc$

Solution : The A. M. between b and c is a $\therefore \quad \dfrac{b+c}{2} = a$, or, $\; b + c = 2a$

Again g_1 and g_2 are two G. M.'s between b and c $\therefore \quad b, g_1, g_2, c$ are in G. P.

If $\quad r$ be the common ratio, then $c = br^3$ or, $\; r = \left(\dfrac{c}{b}\right)^{\frac{1}{3}}$

$$g_1 = br = b\left(\frac{c}{b}\right)^{\frac{1}{3}} \quad \text{and} \quad g_2 = br^2 = b\left(\frac{c}{b}\right)^{\frac{2}{3}}$$

$$\therefore \quad g_1^3 + g_2^3 = b^3\left[\left(\frac{c}{b}\right) + \left(\frac{c}{b}\right)^2\right] = b^3 \times \frac{c}{b}\left(1 + \frac{c}{b}\right) = b^2 c \times \left(\frac{b+c}{b}\right)$$

$= bc\,(2a) \qquad$ [since $b + c = 2a$]

$= 2abc$

Example 6.32 The product of first three terms of a G. P. is 1000. If we add 6 to its second term and 7 to its 3rd term, the three terms form an A. P. Find the terms of the G. P.

Solution : Let $t_1 = \dfrac{a}{r}, t_2 = a$ and $t_3 = ar$ be the first three terms of G. P.

Then, their product $= \dfrac{a}{r}\,.\,a.\,ar = 1000$ or, $\; a^3 = 1000$, or, $\; a = 10$

By the question, t_1, $t_2 + 6$, $t_3 + 7$ are in A. P. $\qquad$ ...(1)

i.e. $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A. P.

$\therefore \quad (a + 6) - \frac{a}{r} = (ar + 7) - (a + 6)$ or, $\quad 2(a+6) = \frac{a}{r} + (ar+7)$

or, $\quad 2(10+6) = \frac{10}{r} + (10r+7) \qquad$ [using (1)]

or, $\quad 32r = 10 + 10r^2 + 7r \quad$ or, $\quad 10r^2 - 25r + 10 = 0$

$$\therefore \quad r = \frac{25 \pm \sqrt{625 - 400}}{20} = \frac{25 \pm 15}{20} = 2, \frac{1}{2}$$

When $a = 10$, $r = 2$. then the terms are $\frac{10}{2}$, $10(2)$ i.e., 5, 10, 20

When $a = 10$, $r = \frac{1}{2}$ then the terms are $10(2)$, 10, $10\left(\frac{1}{2}\right)$ i.e., 20, 10, 5

CHECK YOUR PROGRESS 6.7

1. Insert 8 G. M.'s between 8 and $\frac{1}{64}$.
2. If a_1 is the first of n geometric means between a and b, show that $a_1^{n+1} = a^n b$
3. If G is the G. M. between a and b, prove that $\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}$
4. If the A. M. and G. M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$
5. If A and G are respectvely arithmetic and geometric means between two numbers a and b, then show that A > G.
6. The sum of first three terms of a G. P. is $\frac{13}{12}$ and their product is –1. Find the G. P.
7. The product of three terms of a G. P. is 512. If 8 is added to first and 6 is added to second term, the numbers form an A. P., Find the numbers.

LET US SUM UP

- A sequence in which the difference of two cousecutive terms is always constant ($\neq 0$) is called an Arithmetic Progression (A. P.)

- The general term of an A. P.
  $a, a + d, a + 2d, \ldots$ is given by $t_n = a + (n-1)\,d$
- $S_{n,}$ the sum of the first n terms of the A.P a, a+d, a+2d,… is given by
  $$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
  $$= \frac{n}{2}\ (a + l), \text{ where } l = a + (n-1)\,d.$$
- $t_n = S_n - S_{n-1}$
- An arithmetic mean between a and b is $\dfrac{a+b}{2}$.
- A sequence in which the ratio of two consecutive terms is always constant ($\neq 0$) is called a Geometric Progression (G. P.)
- The n^{th} term of a G. P.: $a, ar, ar^2, \ldots$ is ar^{n-1}
- Sum of the first n terms of a G. P.: $a, ar, ar^2, \ldots$ is
  $$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{for } |\,r\,| > |$$
  $$= \frac{a(1-r^n)}{1-r} \quad \text{for } |\,r\,| < 1$$
- The sums of an infintite G. P. $a, ar, ar^2, \ldots$ is given by
  $$S = \frac{a}{1-r} \quad \text{for } |\,r\,| < 1$$
- Geometric mean G between two numbers a and b is $\sqrt{ab}$
- The arithmetic mean A between two numbers a and b is always greater than the corresponding Geometric mean G i.e., A > G.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=_cooC3yG_p0

http://www.youtube.com/watch?v=pXo0bG4iAyg

http://www.youtube.com/watch?v=dIGLhLMsy2U

http://www.youtube.com/watch?v=cYw4MFWsB6c

http://www.youtube.com/watch?v=Uy_L8tnihDM

http://www.bbc.co.uk/education/asguru/maths/13pure/03sequences/index.shtml

TERMINAL EXERCISE

1. Find the sum of all the natural numbers between 100 and 200 which are divisible by 7.

2. The sum of the first n terms of two A. P.'s are in the ratio $(2n - 1) : (2n + 1)$. Find the ratio of their 10^{th} terms.

3. If a, b, c are in A. P. then show that $b + c, c + a, a + b$ are also in A. P.

4. If $a_1, a_2, ..., a_n$ are in A. P., then prove that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$$

5. If $(b - c)^2, (c - a)^2, (a - b)^2$ are in A. P., then prove that

$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$, are also in A. P.

6. If the p^{th}, q^{th} and r^{th} terms are P, Q, R respectively. Prove that $P (Q - R) + Q (R - P) + r (P - Q) = 0$.

7. If a, b, c are in G. P. then prove that $a^2 b^2 c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$

8. If a, b, c, d are in G. P., show that each of the following form a G. P. :

(a) $(a^2 - b^2), (b^2 - c^2), (c^2 - d^2)$ (b) $\frac{1}{a^2 + b^2}, \frac{1}{b^2 + c^2}, \frac{1}{c^2 - d^2}$

9. If x, y, z are the p^{th}, q^{th} and r^{th} terms of a G. P., prove that $x^{q-r} y^{r-p} z^{p-q} = 1$

10. If a, b, c are in A. P. and x, y, z are in G. P. then prove that $x^{b-c} y^{c-a} z^{a-b} = 1$

11. If the sum of the first n terms of a G. P. is represented by S_n, then prove that

$S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$

12. If p, q, r are in A. P. then prove that the p^{th}, q^{th} and r^{th} terms of a G. P. are also in G. P.

13. If $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} \ldots + \frac{1}{2^{n-1}}$, find the least value of n such that

$$2 - S_n < \frac{1}{100}$$

14. If the sum of the first n terms of a G. P. is S and the product of these terms is p and the sum of their reciprocals is R, then prove that $p^2 = \left(\frac{S}{R}\right)^n$

ANSWERS

CHECK YOUR PROGRESS 6.1

1. (a) $2n-1$ (b) $2n+1$ 2. 3, 5, 7, 9, ... 3. 20, 16
4. no 5. $m + n$ 6. 10, 6, 2,

CHECK YOUR PROGRESS 6.2

1. (a) 435 (b) $\frac{n}{2}\left[21-5n^2\right]$ 2. 5 3. 12
4. 37 5. $n^2+9n, 9$ 6. $2a$

CHECK YOUR PROGRESS 6. 3

2. 5

CHECK YOUR PROGRESS 6. 4

1. $3,-\frac{3}{2},\frac{3}{4},-\frac{3}{8},\frac{3}{16}$ 2. 11^{th}, no 3. 36, 6, 1 or 1, 6, 36
4. (a) 6 (b) 3

CHECK YOUR PROGRESS 6.5

1. (a) 6138 (b) $\frac{2}{3}\left[1-\frac{1}{2^{20}}\right]$ 2. 10.

4. (a) $\frac{80}{81}\left(10^n-1\right)-\frac{8n}{9}$ (b) $\frac{2n}{9}-\frac{2}{81}\left[1-\frac{1}{10^n}\right]$

CHECK YOUR PROGRESS 6. 6

1. (a) $\frac{3}{2}$ (b) $\frac{13}{24}$ 2. (a) $\frac{7}{9}$ (b) $\frac{52}{165}$

3. $5,\frac{10}{3},\frac{20}{9},\frac{40}{27},\ldots\infty$

4. $\frac{1}{4},\frac{1}{4^2},\frac{1}{4^3},\frac{1}{4^4},\ldots\infty$

CHECK YOUR PROGRESS 6.7

1. $4,2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},$ 6. $\frac{4}{3},-1,\frac{3}{4}$ ... or $\frac{3}{4},-1,\frac{4}{3}$ ... 7. 4, 8, 16

TERMINAL EXERCISE

1. 2107 2. 37 : 39

7

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SOME SPECIAL SEQUENCES

Suppose you are asked to collect pebbles every day in such a way that on the first day if you collect one pebble, second day you collect double of the pebbles that you have collected on the first day, third day you collect double of the pebbles that you have collected on the second day, and so on. Then you write the number of pebbles collected daywise, you will have a sequence, $1, 2, 2^2, 2^3, ...$

From a sequence we derive a series. The series corresponding to the above sequence is

$1 + 2 + 2^2 + 2^3 + ...$

One well known series is Fibonacci series 1 + 1 + 2 + 3 + 5+ 8 + 13 + ...

In this lesson we shall study some special types of series in detail.

OBJECTIVES

After studying this lesson, you will be able to :

- define a series;
- calculate the terms of a series for given values of n from t_n;
- evaluate $\sum n, \sum n^2, \sum n^3$ using method of differences and mathematical induction; and
- evaluate simple series like 1.3 + 3.5 + 5.7 +..... n terms.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a sequence
- Concept of A. P. and G. P., sum of n terms.
- Knowldge of converting recurring decimals to fractions by using G. P.

7.1 SERIES

An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 ..., u_n$ ... is a sequence of numbers. The above series is denoted by $\sum_{r=1}^{n} u_r$. If n is finite then the series is a finite series, otherwise the series is infinite. Thus we find that a series is associated to a sequence. Thus a series is a sum of terms arranged in order, according to some definite law.

Consider the following sets of numbers :

(a) 1, 6, 11, ..., (b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12} \dots$

(c) 48, 24, 12, ..., (d) $1^2, 2^2, 3^2, \dots$

(a), (b), (c), (d) form sequences, since they are connected by a definite law. The series associated with them are :

$$1 + 6 + 11 + \dots, \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \dots, 48 + 24 + 12 + \dots, 1^2 + 2^2 + 3^2 + \dots$$

Example 7.1 Write the first 6 terms of each of the following sequences, whose nth term is given by

(a) $T_n = 2n + 1$, (b) $a_n = n^2 - n + 1$ (c) $f_n = (-1)^n . 5^n$

Hence find the series associated to each of the above sequences.

Solution : (a) $T_n = 2n + 1$, For $n = 1$, $T_1 = 2.1 + 1 = 3$, For $n = 2$, $T_2 = 2.2 + 1 = 5$

For $n = 3$, $T_3 = 2.3 + 1 = 7$, For $n = 4$, $T_4 = 2.4 + 1 = 9$

For $n = 5$, $T_5 = 2.5 + 1 = 11$, For $n = 6$, $T_6 = 2.6 + 1 = 13$

Hence the series associated to the above sequence is $3 + 5 + 7 + 9 + 11 + 13 + \dots$

(b) $a_n = n^2 - n + 1$, For $n = 1$, $a_1 = 1^2 - 1 + 1 = 1$

For $n = 2$, $a_2 = 2^2 - 2 + 1 = 3$, For $n = 3$, $a_3 = 3^2 - 3 + 1 = 7$

For $n = 4$, $a_4 = 4^2 - 4 + 1 = 13$,For $n = 5$, $a_5 = 5^2 - 5 + 1 = 21$

For $n = 6$, $a_6 = 6^2 - 6 + 1 = 31$

Hence the series associated to the above sequence is $1 + 3 + 7 + 13 + \dots$

(c) Here $f_n = (-1)^n 5^{n,}$ For $n = 1$, $f_1 = (-1)^1 5^1 = -5$

For $n = 2$, $f_2 = (-1)^2 5^2 = 25$, For $n = 3$, $f_3 = (-1)^3 5^3 = -125$

For $n = 4$, $f_4 = (-1)^4 5^4 = 625$, For $n = 5$, $f_5 = (-1)^5 5^5 = -3125$

For $n = 6$, $f_6 = (-1)^6 5^6 = 15625$

The corresponding series relative to the sequence

$f_n = (-1)^n 5^n$ is $-5 + 25 - 125 + 625 - 3125 + 15625 -$

Example 7.2 Write the n^{th} term of each of the following series :

(a) $-2 + 4 - 6 + 8 - ....$ (b) $1 - 1 + 1 - 1 + ...$

(c) $4 + 16 + 64 + 256 + ...$ (d) $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$

Solution : (a) The series is $-2 + 4 - 6 + 8$ ....

Here the odd terms are negative and the even terms are positive. The above series is obtained by multiplying the series. $-1 + 2 - 3 + 4 - ....$ by 2

$\therefore$ $T_n = 2(-1)^n n = (-1)^n 2n$

(b) The series is $1 - 1 + 1 - 1 + 1 - ......$

$\therefore$ $T_n = (-1)^{n+1}$

(c) The series is $4 + 16 + 64 + 256 + ...$

The above series can be writen as $4 + 4^2 + 4^3 + 4^4 + ...$

i.e., n^{th} term, $T_n = 4^n$.

(d) The series is $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$ *i.e.*, $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + ...$

$\therefore$ n^{th} term is $T_n = \sqrt{n+1}$.

CHECK YOUR PROGRESS 7.1

1. Write the first 6 terms of each of the following series, whose n^{th} term is given by

(a) $T_n = \dfrac{n(n+1)(n+2)}{6}$ (b) $a_n = \dfrac{n^2 - 1}{2n - 3}$

2. If $A_1 = 1$ and $A_2 = 2$, find A_6 if $A_n = \dfrac{A_{n-1}}{A_{n-2}}, (n > 2)$

3. Write the n^{th} term of each of the following series:

(a) $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \cdots$ (b) $3 - 6 + 9 - 12 + ...$

7.2 SUM OF THE POWERS OF THE FIRST *n* NATURAL NUMBERS

(a) The series of first n natural numbers is

$1 + 2 + 3 + 4 + ... + n.$

Let $S_n = 1 + 2 + 3 + ... + n$

This is an arithmetic series whose the first term is 1, the common difference is 1 and the number of terms is n. $\therefore$ $S_n = \frac{n}{2}\left[2.1+(n-1)1\right] = \frac{n}{2}\left[2n-1\right]$

i.e., $S_n = \frac{n(n+1)}{2}$

$\therefore$ We can write $\sum n = \frac{n(n+1)}{2}$

(b) Determine the sum of the squares of the first n natural numbers.

Let $S_n = 1^2 + 2^2 + 3^2 + \ldots. + n^2$

Consider the identity : $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

By giving the values for $n = 1, 2, 3, \ldots., n-1, n$ in the above identity, we have.

$1^3 - 0^3 = 3.1^2 - 3.1 + 1$

$2^3 - 1^3 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

…… ……

…… ……

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding these we get

$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3\,(1 + 2 + 3 + \ldots + n) +$

$(1 + 1 + 1 + \ldots\ n \text{ times})$

or, $n^3 = 3\,S_n - 3\left(\frac{n\,(n+1)}{2}\right) + n \ldots. \left[\because \sum n = \frac{n\,(n+1)}{2}\right]$

or, $3\,S_n = n^3 + \frac{3n\,(n+1)}{2} - n = n\,(n^2 - 1) + \frac{3n}{2}\,(n+1)$

$= n\,(n+1)\left[n - 1 + \frac{3}{2}\right] = \frac{n\,(n+1)\,(2n+1)}{2}$

$\therefore$ $S_n = \frac{n\,(n+1)\,(2n+1)}{6}$ i.e., $\sum n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(c) Determine the sum of the cubes of the first n natural numbers.

Here $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$

Consider the identity : $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

Putting successively 1, 2, 3, ... for n we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$

$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$

$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$

$$\ldots \qquad \ldots$$

$$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1$$

Adding these, we get

$$n^4 - 0^4 = 4(1^3 + 2^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4\,(1 + 2 + 3 + \ldots + n) - (1 + 1 + \ldots\ n \text{ times})$$

$$\Rightarrow \qquad n^4 = 4.S_n - 6\left(\frac{n\,(n+1)\,(2n+1)}{6}\right) + 4n\,\frac{n+1}{2} - n$$

$$\Rightarrow \quad 4S_n = n^4 + n(n+1)\,(2n+1) - 2n\,(n+1) + n$$

$$= n^4 + n\,(2n^2 + 3n + 1) - 2n^2 - 2n + n$$

$$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n = n^4 + 2n^3 + n^2 = n^2\,(n^2 + 2n + 1)$$

i.e., $4S_n = n^2\,(n+1)^2$

$$\therefore \qquad S_n = \frac{n^2\,(n+1)^2}{4} = \left(\frac{n\,(n+1)}{2}\right)^2$$

$$\therefore \qquad \sum n^3 = \left[\frac{n\,(n+1)}{2}\right]^2 \quad \text{or,} \quad \sum n^3 = \left(\sum n\right)^2$$

Note : *In problems on finding sum of the series, we shall find the nth term of the series (t_n) and then use* $S_n = \sum t_n$.

Example 7.3 Find the sum of first n terms of the series 1.3 + 3.5 + 5.7 + ...

Solution :

Let $S_n = 1.3 + 3.5 + 5.7 + \ldots$

The n^{th} term of the series

$$t_n = \{n^{th} \text{ term of } 1, 3, 5, \ldots\} \times \{n^{th} \text{ term of } 3, 5, 7, \ldots\}$$

$$= (2n-1)\,(2n+1) = 4n^2 - 1$$

$$S_n = \sum t_n = \sum \left[4n^2 - 1\right]$$

$$= 4\sum n^2 - \sum(1) = 4\,\frac{n(n+1)(2n+1)}{6} - n$$

$$= \frac{2n(n+1)(2n+1) - 3n}{3} = \frac{n}{3}\left[2(2n^2 + 3n + 1) - 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n - 1\right]$$

Example 7.4 Find the sum of first n terms of the series

$$1.2^2 + 2.3^2 + 3.4^2 + \ldots \ldots$$

Solution : Here $t_n = n\{2 + (n-1)\}^2 = n(n+1)^2 = n.(n^2 + 2n + 1)$

i.e., $t_n = n^3 + 2n^2 + n$

Let $S_n = 1.2^2 + 2.3^2 + 2.3^2 + 3.4^2 + \ldots. + n.(n+1)^2.$

$\therefore$ $S_n = \sum t_n = \sum (n^3 + 2n^2 + n) = \sum n^3 + 2\sum n^2 + \sum n.$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{n(n+1)}{2}$$

$$= n(n+1)\left\{\frac{n(n+1)}{4} + \frac{2n+1}{3} + \frac{1}{2}\right\}$$

$$= \frac{n(n+1)}{12}(3n^2 + 11n + 10) = \frac{1}{12}n(n+1)(n+2)(3n+5)$$

Example 7.5 Find the sum of first n terms of the series

$$2.\,3.\,5 + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$$

Solution : Let $S_n = 2.\,3.\,5. + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$

n^{th} term of the series

$t_n =$ {n^{th} term of 2, 3, 4, ...} × {n^{th} term of 3, 5, 7, ...} × {n^{th} term of 5, 7, 9, ...}

$= (n+1) \times (2n+1) \times (2n+3)$

$= (n+1)\,[4n^2 + 8n + 3] = 4n^3 + 12n^2 + 11n + 3$

$\therefore$ $S_n = \sum t_n = \sum\,[4n^3 + 12n^2 + 11n + 3]$

$= 4\sum n^3 + 12\sum n^2 + 11\sum n + \sum(3)$

$$= 4\frac{n^2 (n+1)^2}{4} + \frac{12\, n\,(n+1)\,(2n+1)}{6} + \frac{11\, n\,(n+1)}{2} + 3n$$

$$= n^2 (n+1)^2 + 2n\,(n+1)\,(2n+1) + \frac{11n\,(n+1)}{2} + 3n$$

$$= \frac{n}{2}\left[2n\,(n+1)^2 + 4\,(n+1)\,(2n+1) + 11\,(n+1) + 6\right]$$

$$= \frac{n}{2}\left[2n\,(n^2+2n+1) + 4\,(2n^2+3n+1) + 11n + 17\right]$$

$$= \frac{n}{2}\left[2n^3 + 12n^2 + 25n + 21\right]$$

Example 7.6 Find the sum of first n terms of the following series :

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

Solution : $$t_n = \frac{1}{(2n-1)\,(2n+1)}$$

$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Now putting successively for $n = 1, 2, 3, \ldots.$

$$t_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$\cdots \qquad \cdots$$

$$t_n = \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

Adding, $$t_1 + t_2 + \cdots + t_n = \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{n}{(2n+1)}$$

CHECK YOUR PROGRESS 7.2

1. Find the sum of first n terms of each of the following series :

   (a) $1 + (1 + 3) + (1 + 3 + 5) + ...$

   (b) $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

   (c) $(1) + (1 + 3) + (1 + 3 + 3^2) + (1 + 3 + 3^2 + 3^3) + ...$

2. Find the sum of n terms of the series. whose nth term is $n(n + 1)(n + 4)$

3. Find the sum of the series $1.\ 2.\ 3 + 2.\ 3.\ 4. + 3.\ 4.\ 5 + \cdots$ upto n terms

LET US SUM UP

- An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 .... u_n, ...$ is a sequence of numbers

- $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

- $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

- $$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

- $S_n = \Sigma t_n$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Sequence_and_series
http://mathworld.wolfram.com/Series.html

TERMINAL EXERCISE

1. Find the sum of each of the following series :

   (a) $2 + 4 + 6 + ...$ up to 40 terms.

(b) $2 + 6 + 18 + \ldots$ up to 6 terms.

2. Sum each of the following series to n terms :

(a) $1 + 3 + 7 + 15 + 31 + \ldots..$

(b) $\dfrac{1}{1.3.5} + \dfrac{1}{3.5.7} + \dfrac{1}{5.7.9} + \ldots..$

(c) $\dfrac{3}{1.4} + \dfrac{5}{4.9} + \dfrac{7}{9.16} + \dfrac{9}{16.25} + \ldots.$

3. Find the sum of first n terms of the series $1^2 + 3^2 + 5^2 + \ldots.$

4. Find the sum to n terms of the series $5 + 7 + 13 + 31 + \ldots.$

5. Find the sum to n terms of the series $1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \cdots$

6. Find the sum of $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

7. Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \ldots\ldots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \ldots\ldots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

ANSWERS

CHECK YOUR PROGRESS 7.1

1. (a) 1, 4, 10, 20, 35, 56 (b) $0, 3, \frac{8}{3}, 3, \frac{24}{7}, \frac{35}{9}$ 2. $\frac{1}{2}$

3. (a) $(-1)^n \frac{1}{n}$ (b) $(-1)^{n+1}\, 3n$

CHECK YOUR PROGRESS 7.2

1. (a) $\frac{1}{6} n\,(n+1)\,(2n+1)$ (b) $\frac{n}{3n+1}$ (c) $\frac{1}{4}(3^{n+1} - 2n - 3)$

2. $\frac{n\,(n+1)}{12}\left[3n^2 + 23n + 34\right]$ 3. $\frac{1}{4} n\,(n+1)(n+2)(n+3)$

TERMINAL EXERCISE

1. (a) 1640 (b) 728

2. (a) $2^{n+1} - n - 2$ (b) $\frac{1}{12} - \frac{1}{4\,(2n+1)\,(2n+3)}$ (c) $1 - \frac{1}{(n+1)^2}$

3. $\frac{n}{3}(4n^2 - 1)$ 4. $\frac{1}{2}(3^n + 8n - 1)$

5. $\frac{5}{4} + \frac{15}{16}\left[1 - \frac{1}{5^{n-1}}\right] - \frac{3n-2}{4.(5^{n-1})}$ 6. $\frac{2n\,(n+1)\,(2n+1)}{3}$

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8

COMPLEX NUMBERS

We started our study of number systems with the set of natural numbers, then the number zero was included to form the system of whole numbers; negative of natural numbers were defined. Thus, we extended our number system to whole numbers and integers.

To solve the problems of the type $p \div q$ we included rational numbers in the system of integers. The system of rational numbers has been extended further to irrational numbers as all lengths cannot be measured in terms of lengths expressed in rational numbers. Rational and irrational numbers taken together are termed as real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. There are no real numbers which satisfy the equation $x^2+1 = 0$ or $x^2 = -1$. In order to solve such equations, i.e., to find square roots of negative numbers, we extend the system of real numbers to a new system of numbers known as complex numbers. In this lesson the learner will be acquinted with complex numbers, its representation and algebraic operations on complex numbers.

OBJECTIVES

After studying this lesson, you will be able to:

- describe the need for extending the set of real numbers to the set of complex numbers;
- define a complex number and cite examples;
- identify the real and imaginary parts of a complex number;
- state the condition for equality of two complex numbers;
- recognise that there is a unique complex number $x + iy$ associated with the point P(x, y) in the Argand Plane and vice-versa;
- define and find the conjugate of a complex number;
- define and find the modulus and argument of a complex number;
- represent a complex number in the polar form;
- perform algebraic operations (addition, subtraction, multiplication and division) on complex numbers;
- state and use the properties of algebraic operations (closure, commutativity, associativity, identity, inverse and distributivity) of complex numbers; and

- state and use the following properties of complex numbers in solving problems:

  (i) $|z| = 0 \Leftrightarrow z = 0$ and $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

  (ii) $|z| = |-z| = |\bar{z}|$ (iii) $|z_1 + z_2| \le |z_1| + |z_2|$

  (iv) $|z_1 . z_2| = |z_1| . |z_2|$ (v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

- to find the square root of a complex number.

EXPECTED BACKGROUND KNOWLEDGE

- Properties of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane.

8.1 COMPLEX NUMBERS

Consider the equation $x^2 + 1 = 0$. ...(A)

This can be written as $x^2 = -1$ or $x = \pm\sqrt{-1}$

But there is no real number which satisfy $x^2 = -1$.In other words, we can say that there is no real number whose square is -1.In order to solve such equations, let us imagine that there exist a number *'i'* which equal to $\sqrt{-1}$.

In 1748, a great mathematician, L. Euler named a number 'i' as *Iota* whose square is -1. This *Iota or 'i'* is defined as imaginary unit. With the introduction of the new symbol *'i'*, we can interpret the square root of a negative number as a product of a real number with i.

Therefore, we can denote the solution of (A) as $x = \pm i$

Thus, $-4 = 4(-1)$

$$\therefore \quad \sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2 . 2^2} = i2$$

Conventionally written as 2i.

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are all examples of complex numbers.

Consider another quadratic equation: $x^2 - 6x + 13 = 0$

This can be solved as under:

$$(x - 3)^2 + 4 = 0 \text{ or, } (x - 3)^2 = -4$$

or, $x - 3 = \pm 2i$ or, $x = 3 \pm 2i$

We get numbers of the form x + yi where x and y are real numbers and $i = \sqrt{-1}$.

> Any number which can be expressed in the form a + bi where a,b are real numbers and $i = \sqrt{-1}$, is called a complex number.

A complex number is, generally, denoted by the letter z.

i.e. z = a + bi, 'a' is called the real part of z and is written as Re (a+bi) and 'b' is called the imaginary part of z and is written as Imag (a + bi).

If a = 0 and b ≠ 0, then the complex number becomes b*i* which is a purely imaginary complex number.

$-7i$, $\frac{1}{2}i$, $\sqrt{3}i$ and πi are all examples of purely imaginary numbers.

If $a \neq 0$ and b = 0 then the complex number becomes 'a' which is a real number.

5, 2.5 and $\sqrt{7}$ are all examples of real numbers.

If a = 0 and b = 0, then the complex number becomes 0 (zero). Hence the real numbers are particular cases of complex numbers.

Example 8.1 Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}.\sqrt{-4}$

Solution: (i) $\sqrt{-36} = \sqrt{36(-1)} = 6i$

(ii) $\sqrt{25}.\sqrt{-4} = 5 \times 2i = 10i$

8.2 POSITIVE INTEGRAL POWERS OF i

We know that

$i^2 = -1,\ i^3 = i^2.i = -1.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1,\ i^5 = (i^2)^2.i = 1.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1,\ i^7 = (i^2)^3(i) = -i,\ i^8 = (i^2)^4 = 1$

Thus, we find that any higher powers of 'i' can be expressed in terms of one of four values i, –1, –i, 1

If n is a positive integer such that n>4, then to find i^n, we first divide n by 4.

Let m be the quotient and r be the remainder.

Then n = 4m + r. where $0 \leq r < 4$.

Thus, $i^n = i^{(4m+r)} = i^{4m}.i^r = (i^4)^m.i^r = i^r$ ($\because i^4=1$)

Note : *For any two real numbers a and b,* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is true only when atleast one of a and b is either 0 or positive.*

If fact $\quad \sqrt{-a} \times \sqrt{-b}$

$= i\sqrt{a} \times i\sqrt{b} = i^2\sqrt{ab}$

$= -\sqrt{ab}$ $\quad$ where a and b are positive real numbers.

Example 8.2 Find the value of $1 + i^{10} + i^{20} + i^{30}$

Solution: $1 + i^{10} + i^{20} + i^{30}$

$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15} = 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$

$= 1 + (-1) + 1 + (-1) = 1 - 1 + 1 - 1 = 0$

Thus, $1 + i^{10} + i^{20} + i^{30} = 0$.

Example 8.3 Express $8i^3 + 6i^{16} - 12i^{11}$ in the form of a + bi

Solution: $8i^3 + 6i^{16} - 12i^{11}$ can be written as $8(i^2).i + 6(i^2)^8 - 12(i^2)^5.i$

$= 8(-1).i + 6(-1)^8 - 12(-1)^5.i = -8i + 6 - 12(-1).i$

$= -8i + 6 + 12i = 6 + 4i$

which is of the form a + bi where 'a' is 6 and 'b' is 4.

CHECK YOUR PROGRESS 8.1

1. Simplify each of the following using 'i'.

   (a) $\sqrt{-27}$ $\quad$ (b) $-\sqrt{-9}$ $\quad$ (c) $\sqrt{-13}$

2. Express each of the following in the form of a + bi

   (a) 5 $\quad$ (b) $-3i$ $\quad$ (c) 0

3. Simplify $10i^3 + 6i^{13} - 12i^{10}$

4. Show that $i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0$ for all $m \in N$.

8.3 CONJUGATE OF A COMPLEX NUMBER

The complex conjugate (or simply conjugate) of a complex number z = a + bi is defined as the complex number a – bi and is denoted by $\bar{z}$.

Thus, if $z = a + bi$ then $\bar{z} = a - bi$.

Note : The conjugate of a complex number is obtained by changing the sing of the imaginary part.

Following are some examples of complex conjugates:

(i) If $z = 2 + 3i$, then $\bar{z} = 2 - 3i$

(ii) If $z = 1 - i$, then $\bar{z} = 1 + i$

(iii) If $z = -2 + 10i$, then $\bar{z} = -2 - 10i$

8.3.1 PROPERTIES OF COMPLEX CONJUGATES

(i) If z is a real number then $z = \bar{z}$ i.e., the conjugate of a real number is the number itself.

For example, let $z = 5$

This can be written as $z = 5 + 0i$

$\therefore \quad \bar{z} = 5 - 0i = 5, \qquad \therefore \quad z = 5 = \bar{z}$.

(ii) If z is a purely imaginary number then $\bar{z} = -z$

For example, if $z = 3i$. This can be written as $z = 0 + 3i$

$\therefore \quad \bar{z} = 0 - 3i = -3i = -z$

$\therefore \quad \bar{z} = -z$.

(iii) Conjugate of the conjugate of a complex number is the number itself.

i.e., $\overline{(\bar{z})} = z$

For example, if $z = a + bi$ then $\bar{z} = a - bi$

Again $\overline{(\bar{z})} = \overline{(a - bi)} = a + bi = z$

$\therefore \quad \overline{(\bar{z})} = z$

Example 8.4 Find the conjugate of each of the following complex numbers:

(i) $3 - 4i$ (ii) $(2 + i)^2$

Solution : (i) Let $z = 3 - 4i$ then $\bar{z} = \overline{3 - 4i} = 3 + 4i$

Hence, $3 + 4i$ is the conjugate of $3 - 4i$.

(iii) Let $z = (2 + i)^2$

i.e. $z = (2)^2 + (i)^2 + 2(2)(i) = 4 - 1 + 4i = 3 + 4i$

Then $\bar{z} = \left(\overline{3 + 4i}\right) = 3 - 4i$

Hence, $3 - 4i$ is the conjugate of $(2 + i)^2$

8.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

Let $z = a + bi$ be a complex number. Let two mutually perpendicular lines xox' and yoy' be taken as x-axis and y-axis respectively, O being the origin.

Let P be any point whose coordinates are (a,b). We say that the complex number $z = a + bi$ is represented by the point P (a , b) as shown in Fig. 8.1

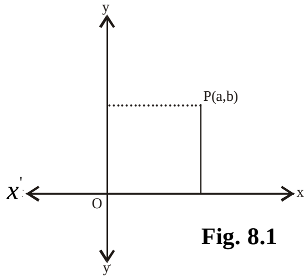


Fig. 8.1

If b = 0, then z is real and the point representing complex number $z = a + 0i$ is denoted by (a, 0). This point (a, 0) lies on the x-axis.

So, xox' is called the real axis. In the Fig. 8.2 the point Q (a, 0) represent the complex number $z = a + 0i$.

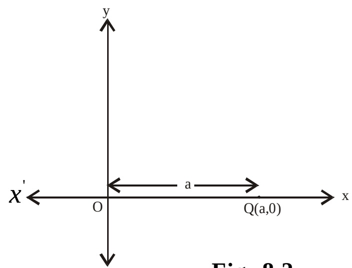


Fig. 8.2

If a = 0, then z is purely imaginary and the point representing complex number $z = 0 + bi$ is denoted by (0, b). The point (0, b) lies on the y-axis.

So, yoy' is called the imaginary axis. In Fig.8.3, the point R (0, b) represents the complex number $z = 0 + bi$.

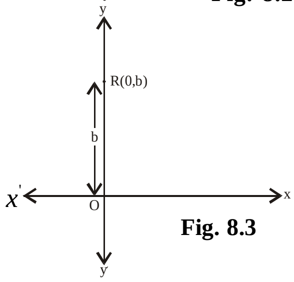


Fig. 8.3

The plane of two axes representing complex numbers as points is called the complex plane or Argand Plane.

The diagram which represents complex number in the Argand Plane is called Argand Diagram.

Example 8.5

Represent complex numbers $2 + 3i, -2-3i, 2-3i$ in the same Argand Plane

Solution: (a) $2+3i$ is represented by the point P (2, 3)

(b) $-2-3i$ is represented by the point Q (–2,–3)

(c) $2-3i$ is represented by the point R (2, –3)

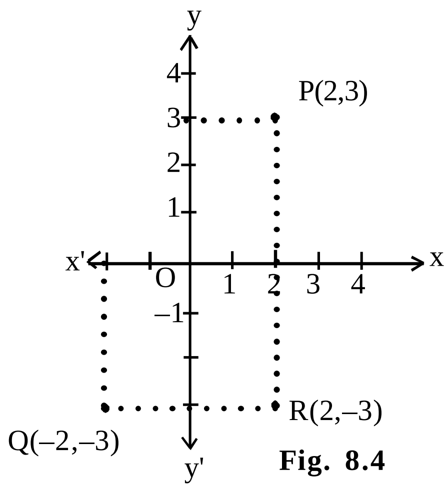


Fig. 8.4

8.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number z = a + bi can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing a + bi. Draw perpendiculars PM and PL on x-axis and y-axis respectively. Let OM = a and MP = b. We have to find the distance of P from the origin.

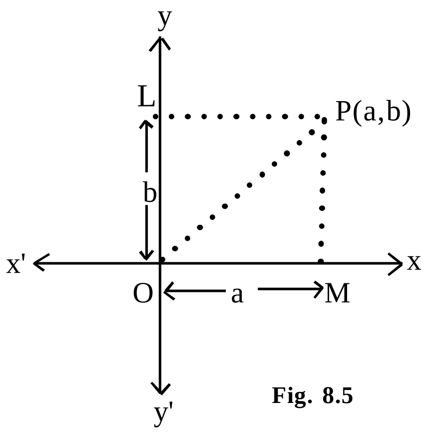


Fig. 8.5

$$\therefore \quad OP = \sqrt{OM^2 + MP^2}$$

$$= \sqrt{a^2 + b^2}$$

OP is called the modulus or absolute value of the complex number a + bi.

$\therefore$ Modulus of any complex number z such that $z = a + bi$, $a \in R$, $b \in R$ is denoted by $|z|$ and is given by $\sqrt{a^2 + b^2}$

$$\therefore \quad |z| = |a + ib| = \sqrt{a^2 + b^2}$$

8.5.1 Properties of Modulus

(a) $|z| = 0 \Leftrightarrow z = 0.$

Proof : Let $z = a + bi$, $a \in R$, $b \in R$

then $|z| = \sqrt{a^2 + b^2}$, $|z| = 0 \Leftrightarrow a^2 + b^2 = 0$

$\Leftrightarrow$ $a = 0$ and $b = 0$ (since a^2 and b^2 both are positive), $\Leftrightarrow$ $z = 0$

(b) $|z| = |\bar{z}|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

Now, $\bar{z} = a - bi$ $\therefore$ $|\bar{z}| = \sqrt{a^2 + (-b^2)} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |\bar{z}|$ ...(i)

(c) $|z| = |-z|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

$-z = -a - bi$ then $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$ ...(ii)

By (i) and (ii) it can be proved that $|z| = |-z| = |\bar{z}|$ ...(iii)

Now, we consider the following examples:

Example 8.6 Find the modulus of z, –z and $\bar{z}$ where z = 1 + 2i

Solution : $z = 1 + 2i$ then $-z = -1-2i$ and $\bar{z} = 1 - 2i$

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5},\ |-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

and $$|\bar{z}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

Thus, $$|z| = |-z| = \sqrt{5} = |\bar{z}|$$

Example 8.7 Find the modulus of the complex numbers shown in an Argand Plane (Fig. 8.6)

Solution: (i) P(4, 3) represents the complex number $z = 4 + 3i$

$\therefore$ $|z| = \sqrt{4^2 + 3^2} = \sqrt{25}$

or $|z| = 5$

(ii) Q(–4, 2) represents the complex number $z = -4 + 2i$

$\therefore$ $|z| = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20}$

or $|z| = 2\sqrt{5}$

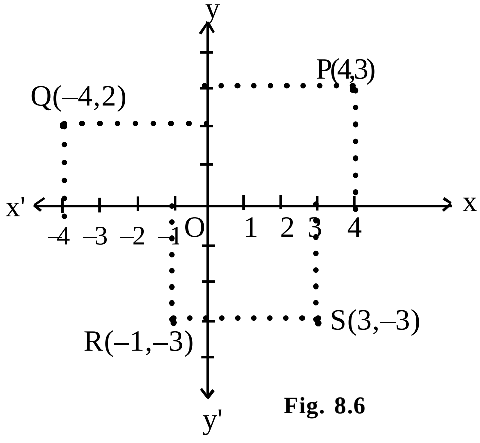


Fig. 8.6

(iii) R(–1, –3) represents the complex number $z = -1 - 3i$

$\therefore \quad |z| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1+9}$

or $\quad |z| = \sqrt{10}$

(iv) S(3, –3) represents the complex number $z = 3 - 3i$

$\therefore \quad |z| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9}$

or $\quad |z| = \sqrt{18} = 3\sqrt{2}$

CHECK YOUR PROGRESS 8.2

1. Find the conjugate of each of the following:

   (a) $-2i$ (b) $-5-3i$ (c) $-\sqrt{2}$ (d) $(-2+i)^2$

2. Represent the following complex numbers on Argand Plane :

   (a) (i) $2+0i$ (ii) $-3+0i$ (iii) $0-0i$ (iv) $3-0i$

   (b) (i) $0+2i$ (ii) $0-3i$ (iii) $4i$ (iv) $-5i$

   (c) (i) $2+5i$ and $5+2i$ (ii) $3-4i$ and $-4+3i$

   (iii) $-7+2i$ and $2-7i$ (iv) $-2-9i$ and $-9-2i$

   (d) (i) $1+i$ and $-1-i$ (ii) $6+5i$ and $-6-5i$

   (iii) $-3+4i$ and $3-4i$ (iv) $4-i$ and $-4+i$

   (e) (i) $1+i$ and $1-i$ (ii) $-3+4i$ and $-3-4i$

   (iii) $6-7i$ and $6+7i$ (iv) $-5-i$ and $-5+i$

3. (a) Find the modulus of following complex numbers :

   (i) 3 (ii) $(i+1)(2-i)$ (iii) $2-3i$ (iv) $4+\sqrt{5}i$

   (b) For the following complex numbers, verify that $|z| = |\bar{z}|$

   (i) $-6+8i$ (ii) $-3-7i$

   (c) For the following complex numbers, verify that $|z| = |-z|$

(i) $14 + i$ (ii) $11 - 2i$

(d) For the following complex numbers, verify that $|z| = |-z| = |\bar{z}|$

(i) $2 - 3i$ (ii) $-6 - i$ (iii) $7 - 2i$

8.6 EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers are equal if and only if their real parts and imaginary parts are respectively equal.

In general $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Example 8.8 For what value of x and y, $5x + 6yi$ and $10 + 18i$ are equal?

Solution : It is given that $5x + 6yi = 10 + 18i$

Comparing real and imaginary parts, we have

$5x = 10$ or $x = 2$

and $6y = 18$ or $y = 3$

For $x = 2$ and $y = 3$, the given complex numbers are equal.

8.7 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (a + c) + (b + d)i$$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$,

then $z_1 + z_2 = [2 + (-4)] + [3 + 5]i = -2 + 8i$.

Example 8.9 Simplify

(i) $(3 + 2i) + (4 - 3i)$ (ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Solution : (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i = 7 - i$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i) = (2 - 3 + 1) + (5 - 7 - 1)i = 0 - 3i$

or $(2 + 5i) + (-3 - 7i) + (1 - i) = -3i$

8.7.1 Geometrical Represention of Addition of Two Complex Numbers

Let two complex numbers z_1 and z_2 be represented by the points P(a, b) and Q(c, d).

Their sum, $z_1 + z_2$ is represented by the point R $(a + c, b + d)$ in the same Argand Plane.

Join OP, OQ, OR, PR and QR.

Draw perpendiculars PM, QN, RL from P, Q, R respectvely on X-axis.

Draw perpendicular PK to RL

In ΔQON

ON = c

and QN = d.

In ΔROL	In ΔPOM
RL = b + d	PM = b
and OL = a + c	OM = a

Also $PK = ML = OL - OM$

$= a + c - a \;\; = c = ON$

$RK = RL - KL = RL - PM$

$= b + d - b = d = QN.$

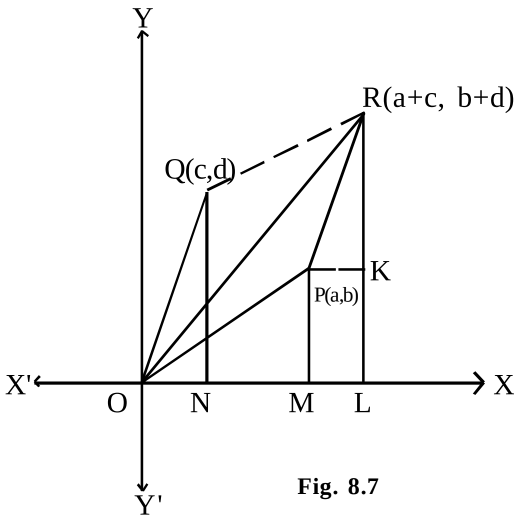


Fig. 8.7

In ΔQON and ΔRPK,

$ON = PK$, $QN = RK$ and $\angle QNO = \angle RKP = 90^o$

$\therefore \qquad \Delta QON \cong \Delta RPK$

$\therefore \qquad OQ = PR$ and $OQ \parallel PR$

$\Rightarrow \qquad$ OPRQ is a parallelogram and OR its diagonal.

Therefore, we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

Example 8.10 Prove that $|z_1 + z_2| \le |z_1| + |z_2|$

Solution: We have proved that the sum of two complex numbers z_1 and z_2 represented by the diagonal of a parallelogram OPRQ (see fig. 8.8).

In $\qquad \Delta OPR$, $OR \le OP + PR$

or $\qquad OR \le OP + OQ$ (since $OQ = PR$)

or $\qquad |z_1 + z_2| \le |z_1| + |z_2|$

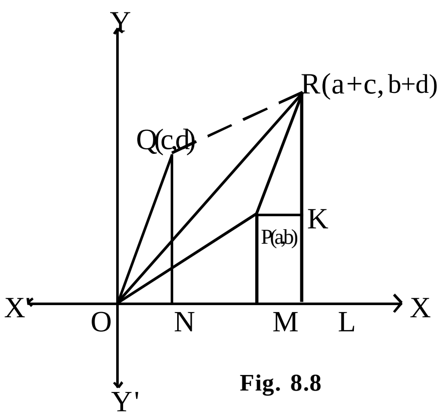


Fig. 8.8

Example 8.11 If $z_1 = 2 + 3i$ and $z_2 = 1 + i$,

verify that $|z_1 + z_2| \le |z_1| + |z_1|$

Solution: $z_1 = 2 + 3i$ and $z_2 = 1 + i$ represented by the points (2, 3) and (1, 1) respectively. Their sum $(z_1 + z_2)$ will be represented by the point (2+1, 3+1) i.e. (3, 4)

Verification

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.6 \text{ approx.}$$

$$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41 \text{ approx.}$$

$$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|z_1| + |z_2| = 3.6 + 1.41 = 5.01$$

$\therefore \quad |z_1 + z_2| \le |z_1| + |z_2|$

8.7.2 Subtraction of Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$\therefore \quad (z_1) - (z_2) = (a + bi) - (c + di) = (a - c) + (b - d)i$

which represents a point $(a - c, b - d)$

$\therefore$ The difference i.e. $z_1 - z_2$ is represented by the point $(a - c, b - d)$.

Thus, to subtract a complex number from another, we subtract corresponding real and imaginary parts separately.

Example 8.12 Find $z_1 - z_2$ if:

$z_1 = 3 - 4i, \qquad z_2 = -3 + 7i$

Solution: $z_1 - z_2 = (3 - 4i) - (-3 + 7i) = (3 - 4i) + (3 - 7i)$

$= (3 + 3) + (-4 - 7)i = 6 + (-11i) \quad = 6 - 11i$

Examle 8.13 What should be added to i to obtain 5 + 4i?

Solution: Let $z = a + bi$ be added to i to obtain $5 + 4i$

$\therefore \quad i + (a + bi) = 5 + 4i$

or, $\quad a + (b + 1)i = 5 + 4i$

Equating real and imaginary parts, we have

$a = 5$ and $b + 1 = 4$ or $b = 3$, $\therefore$ $z = 5 + 3i$ is to be added to i to obtain $5 + 4i$

8.8 PROPERTIES: WITH RESPECT TO ADDITION OF COMPLEX NUMBERS.

1. Closure : The sum of two complex numbers will always be a complex number.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, $a_1, b_1, a_2, b_2 \in R$.

Now, $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex nu mber.

This proves the closure property of complex numbers.

2. Commutative : If z_1 and z_2 are two complex numbers then

$$z_1 + z_2 = z_2 + z_1$$

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Now $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$

$= (a_2 + a_1) + (b_2 + b_1)i$ [commutative property of real numbers]

$= (a_2 + b_2 i) + (a_2 + b_1 i) = z_2 + z_1$

i.e. $z_1 + z_2 = z_2 + z_2$ Hence, addition of complex numbers is commutative.

3. Associative

If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Now $z_1 + (z_2 + z_3) = (a_1 + b_1 i) + \{(a_2 + b_2 i) + (a_3 + b_3 i)\}$

$= (a_1 + b_1 i) + \{(a_2 + a_3) + (b_2 + b_3)i\} = \{a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}i$

$= \{(a_1 + a_2) + (b_1 + b_2)i\} + (a_3 + b_3 i) = \{(a_1 + b_1 i) + (a_2 + b_2 i)\} + (a_3 + b_3 i)$

$= (z_1 + z_2) + z_3$

Hence, the associativity property holds good in the case of addition of complex numbers.

4. Existence of Additive Identitiy

if $z = a + bi$ is any complex number, then $(a + bi) + (0 + 0i) = a + bi$

i.e. $(0 + 0i)$ is called the additive identity for $a + ib$.

5. Existence of Additive Inverse

For every complex number $a + bi$ there exists a unique complex number $-a - bi$ such that $(a + bi) + (-a - bi) = 0 + 0i$. $-a - ib$ is called the additive inverse of $a + ib$.

In general, additive inverse of a complex number is obtained by changing the signs of real and imaginary parts.

CHECK YOUR PROGRESS 8.3

1. Simplify:

(a) $\left(\sqrt{2}+\sqrt{5}i\right)+\left(\sqrt{5}-\sqrt{2}i\right)$ (b) $\dfrac{2+i}{3}+\dfrac{2-i}{6}$

(c) $(1+i)-(1-6i)$ (d) $\left(\sqrt{2}-\sqrt{3}i\right)-(-2-7i)$

2. If $z_1 = (5+i)$ and $z_2 = (6+2i)$, then:

(a) find $z_1 + z_2$ (b) find $z_2 + z_1$ (c) Is $z_1 + z_2 = z_2 + z_1$?

(d) find $z_1 - z_2$ (e) find $z_2 - z_1$ (f) Is $z_1 - z_2 = z_2 - z_1$?

3. If $z_1 = (1+i)$, $z_2 = (1-i)$ and $z_3 = (2+3i)$, then:

(a) find $z_1 + (z_2 + z_3)$ (b) find $(z_1 + z_2) + z_3$

(c) Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? (d) find $z_1 - (z_2 - z_3)$

(e) find $(z_1 - z_2) - z_3$ (f) Is $z_1 - (z_2 - z_3) = (z_1 - z_2) - z_3$?

4. Find the additive inverse of the following:

(a) $12 - 7i$ (b) $4 - 3i$

5. What shoud be added to $(-15+4i)$ to obtain $(3-2i)$?

6. Show that $\overline{\left\{(3+7i)-(5+2i)\right\}} = \overline{(3+7i)} - \overline{(5+2i)}$

8.9 ARGUMENT OF A COMPLEX NUMBER

Let P(a, b) represent the complex number $z = a + bi$, $a \in R$, $b \in R$, and OP makes an angle θ with the positive direction of x-axis. Draw $PM \perp OX$, Let $OP = r$

In right Δ OMP, OM = a, MP = b

$\therefore$ $r\cos\theta = a$, $r\sin\theta = b$

Then $z = a + bi$ can be written as $z = r(\cos\theta + i\sin\theta)$ ...(i)

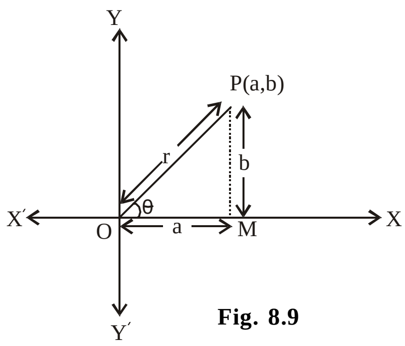


Fig. 8.9

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$ or $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

(i) is known as the polar form of the complex number z, and r and θ are respectively called the modulus and argument of the complex number.

8.10 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then, $z_1.z_2 = (a + bi).(c + di)$

$= a(c + di) + bi(c + di)$

or $= ac + adi + bci + bdi^2$

or $= (ac - bd) + (ad + bc)i.$ [since $i^2 = -1$]

If (a + bi) and (c + di) are two complex numbers, their product is defined as the complex number $(ac - bd) + (ad + bc)i$

Example 8.14 Evaluate: $(1 + 2i)(1 - 3i)$,

Solution:

$$(1 + 2i)(1 - 3i) = \{1 - (-6)\} + (-3 + 2)i = 7 - i$$

8.10.1 Prove that

$$|z_1.z_2| = |z_1| . |z_2|$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad |z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$

Similarly, $|z_2| = r_2$.

Now, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

[Since $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ and
$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$]

$$|z_1.z_2| = r_1r_2\sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)} = r_1r_2$$

$\therefore \qquad |z_1 . z_2| = r_1 r_2 = |z_1|.|z_2|$

and argument of $z_1 z_2 = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Example 8.15 Find the modulus of the complex number (1 + i) (4 – 3i)

Solution: Let z = (1 + i) (4 – 3i)

then $\quad |z| = |(1+i)(4-3i)|$

$\qquad = |(1+i)|.|(4-3i)| \qquad$ (since $|z_1 z_2| = |z_1|.|z_2|$)

But $\quad |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$, $|4-3i| = \sqrt{4^2+(-3)^2} = 5$

$\therefore \qquad |z| = \sqrt{2}.5 = 5\sqrt{2}$

8.11 DIVISION OF TWO COMPLEX NUMBERS

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator. We will explain it through an example.

Let $\quad z_1 = a + bi$ **and** $\mathbf{z_2 = c+di}$**, then.**

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\left(c+di \neq 0\right)$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Thus,
$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 8.16 Divide 3+i by 4–2i

Solution: $\dfrac{3+i}{4-2i} = \dfrac{(3+i)(4+2i)}{(4-2i)(4+2i)}$

Multiplying numerator and denominator by the conjugate of (4 – 2i) we get

$$= \frac{10+10i}{20} = \frac{1}{2} + \frac{1}{2}i$$

Thus, $\dfrac{3+i}{4-2i} = \dfrac{1}{2} + \dfrac{1}{2}i$

8.11.1 Prove that $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Proof: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$|z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$

and $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

Thus, $= \left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}\sqrt{\cos^2(\theta_1 - \theta_2) + \sin^2(\theta_1 - \theta_2)} = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

$\therefore$ Argument of $\left|\dfrac{z_1}{z_2}\right| = \theta_1 - \theta_2 = arg(z_1) - arg(z_2)$

Example 8.17 Find the modulus of the complex number $\frac{2+i}{3-i}$

Solution : Let $z = \frac{2+i}{3-i}$

$$\therefore \qquad |z| = \left|\frac{2+i}{3-i}\right| \qquad = \frac{|2+i|}{|3-i|} \quad \left(\text{since} \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$= \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-1)^2}} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}} \qquad \therefore \qquad |z| = \frac{1}{\sqrt{2}}$$

8.12 PROPERTIES OF MULTIPLICATION OF TWO COMPLEX NUMBERS

1. Closure If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then their product z_1z_2 is also a complex number.

2. Cummutative If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then $z_1z_2 = z_2z_1$.

3. Associativity If $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$ then

$$z_1(z_2.z_3) = (z_1.z_2).z_3$$

4. Existence of Multiplicative Identity: For every non-zero complex number $z_1 = a + bi$ there exists a unique complex number $(1 + 0i)$ such that

$$(a + bi).(1 + 0i) = (1 + 0i)\ (a + bi) = a + bi$$

Let $z_1 = x + yi$ be the multipicative identity of $z = a + bi$ Then $\quad z.z_1 = z.$

i.e. $\qquad (a + bi)\ (x + yi) = a + bi$

or $\qquad (ax - by) + (ay + bx)i = a + bi$

or $\qquad ax - by = a$ and $ay + bx = b$

pr $\qquad x = 1$ and $y = 0$, i.e. $\quad z_1 = x + yi = 1 + 0i$ is the multiplicative identity.

The complex number $1 + 0i$ is the identity for multiplication.

5. Existence of Multiplicative inverse: Multiplicative inverse is a complex number that when multiplied to a given non-zero complex munber yields one. In other words, for every non-zero complex number $z = a + bi$, there exists a unique complex number $(x + yi)$ such that their product is $(1 + 0i)$. i.e. $\quad (a + bi)\ (x + yi) = 1 + 0i$ or $(ax - by) + (bx + ay)i = 1 + 0i$

Equating real and imaging parts, we have

$ax - by = 1$ and $bx + ay = 0$

By cross multiplication

$$\frac{x}{a} = \frac{y}{-b} = \frac{1}{a^2 + b^2} \Rightarrow \quad x = \frac{a}{a^2+b^2} = \frac{\operatorname{Re}(z)}{|z|^2} \text{ and } y = \frac{-b}{a^2+b^2} = -\frac{\operatorname{Im}(z)}{|z|^2}$$

Thus, the multiplicative inverse of a non-zero compelx number $z = (a + bi)$ is

$$x + yi = \left(\frac{\operatorname{Re}(z)}{|z|^2} - \frac{\operatorname{Im}(z)}{|z|^2} i\right) = \frac{\bar{z}}{|z|^2}$$

Example 8. 18 Find the multiplication inverse of $2 - 4i$.

Solution: Let $z = 2 - 4i$ **We have,** $\bar{z} = 2 + 4i$ and $|z|^2 = \left|2^2 + (-4)^2\right| = 20$

$\therefore$ Required multiplicative inverse is $\dfrac{\bar{z}}{|z|^2} = \dfrac{2+4i}{20} = \dfrac{1}{10} + \dfrac{1}{5}i$

6. Distributive Property of Multiplication over Addition

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$

Then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

CHECK YOUR PROGRESS 8.4

1. Simplify each of the following:

   (a) $(1+2i)(\sqrt{2} - i)$ (b) $(\sqrt{2} + i)^2$ (c) $(3 + i)(1 - i)(-1 + i)$

   (d) $(2 + 3i) \div (1 - 2i)$ (e) $(1 + 2i) \div (1 + i)$ (f) $(1 + 0i) \div (3 + 7i)$

2. Compute multiplicative inverse of each of the following complex numbers:

   (a) $3 - 4i$ (b) $\sqrt{3} + 7i$ (c) $\dfrac{3+5i}{2-3i}$

3. If $z_1 = 4 + 3i$, $z_2 = 3 - 2i$ and $z_3 = i + 5$, verify that $z_1(z_2 + z_3) = z_1 z_2 + z_2 z_3$.

4. If $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 2 - i$ then verify that $(z_1.z_2)z_3 = z_1(z_2.z_3)$

8.13 SQUARE ROOT OF A COMPLEX NUMBER

Let $a + ib$ be a complex number and $x + iy$ be its square root

i.e., $\sqrt{a+ib} = x + iy$

$\Rightarrow \quad a + ib = x^2 - y^2 + 2ixy$

Equating real and imaginary parts we have

$x^2 - y^2 = a$ ....(1)

and $2xy = b$ ...(2)

Using the algebraic identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$ we get

$(x^2 + y^2)^2 = a^2 + b^2 \Rightarrow x^2 + y^2 = \sqrt{a^2+b^2}$ ...(3)

From equations (1) and (3), we get

$$\left.\begin{aligned} 2x^2 &= \sqrt{a^2+b^2}+a \Rightarrow x = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} \\ \text{and } 2y^2 &= \sqrt{a^2+b^2}-a \Rightarrow y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{aligned}\right\} \quad ...(4)$$

Out of these four pairs of values of x and y (given by equation (4) we have to choose the values which satisfy (1) and (2) both.

From (2) if b is + ve then both x and y should be of same sign and in that case

$$\sqrt{a+ib} = \sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} + i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and $$-\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} - i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and if b is –ve then x and y should be of opposite sign. Therefore in that case

$$\sqrt{a+ib} = -\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} + i\sqrt{\frac{1}{2}(\sqrt{b^2+b^2}-a)}$$

and $$\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} - i\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}$$

Hence $a + ib$ has two square roots in each case and the two square roots just differ in sign.

Example 8.19 Find the square root of $7 + 24i$

Solution : Let $\sqrt{7+24i} = a + ib$ ...(1)

Squaring both sides, we get $7 + 24i = a^2 - b^2 + 2\,iab$

Comparing real and imaginary parts, we have $a^2 - b^2 = 7$ ...(2)

and $2\ ab = 24 \Rightarrow ab = 12$ ....(3)

Now $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$

$\Rightarrow \quad (a^2 + b^2)^2 = 49 + 4 \times 144$

$\Rightarrow \quad (a^2 + b^2)^2 = 625$

$\Rightarrow \quad a^2 + b^2 = 25$ ...(4)

Solving (2) and (4), we get $2a^2 = 32 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4$

and $2b^2 = 18 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3$

From (3), $ab = 12$ which is +ve $\Rightarrow$ a and b should be of same sign

$\therefore$ Either $a = 4$, $b = 3$ or $= -4$, $b = -3$

Hence, the two square roots of $7 + 24i$ are $4 + 3i$ and $-4 - 3i$

Example 8.20 Find the square root of $-i$

Solution : Let $\sqrt{-i} = a + ib$

$\Rightarrow \quad -i = a^2 - b^2 + 2\ iab$ ...(1)

Equating real and imaginary parts of (1), we get $a^2 - b^2 = 0$ ...(2)

and $2\ ab = -1 \Rightarrow ab = -\dfrac{1}{2}$ ...(3)

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 = 0 + 4\left(\dfrac{1}{4}\right) = 1$

$\Rightarrow \quad a^2 + b^2 = 1$ ...(4)

From (2) and (4), $2b^2 = 1 \Rightarrow b^2 = \dfrac{1}{2} \Rightarrow b = \pm \dfrac{1}{\sqrt{2}}$

and $2a^2 = 1 \Rightarrow a^2 = \dfrac{1}{2} \Rightarrow a = \pm \dfrac{1}{\sqrt{2}}$.

Equation (3) suggests; that a and b should be of opposite sign therefore two square roots of $-i$ are $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ and $-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

CHECK YOUR PROGRESS 8.5

Find the square root of the following complex numbers :

(*i*) $-21 - 20i$ (ii) $-4 - 3i$ (iii) $-48 - 14i$

LET US SUM UP

- $z = a + bi$ is a complex number in the standard form where $a, b \in R$ and $i = \sqrt{-1}$.
- Any higher powers of 'i' can be expressed in terms of one of the four values $i, -1, -i, 1$.
- Conjugate of a complex number $z = a + bi$ is $a - bi$ and is denoted by $\bar{z}$.
- Modulus of a complex number $z = a + bi$ is $\sqrt{a^2 + b^2}$ i.e. $|z| = |a + bi| = \sqrt{a^2 + b^2}$

  (a) $|z| = 0 \Leftrightarrow z = 0$ (b) $|z| = |\bar{z}|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$
- $z = r(\cos\theta + i\sin\theta)$ represents the polar form of a complex number $z = a + bi$ where $r = \sqrt{a^2 + b^2}$ is modulus and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ is its argument.
- Multiplicative inverse of a complex number $z = a + bi$ is $\frac{\bar{z}}{|z|^2}$
- Square root of a complex number is also a complex number.
- Two square roots of a complex number only differ in sign.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=MEuPzvh0roM
http://www.youtube.com/watch?v=kpywdu1afas
http://www.youtube.com/watch?v=bPqB9a1uk_8
http://www.youtube.com/watch?v=SfbjqVyQljk
http://www.youtube.com/watch?v=tvXRaZbIjO8
http://www.youtube.com/watch?v=cWn6g8Qqvs4
http://www.youtube.com/watch?v=Z8j5RDOibV4
http://www.youtube.com/watch?v=dbxJ6LD0344

TERMINAL EXERCISE

1. Find real and imaginary parts of each of the following:

   (a) $2 + 7i$ (b) $3 + 0i$ (c) $-\frac{1}{2}$ (d) $5i$ (e) $\frac{1}{2 + 3i}$

2. Simplify each of the following:

 (a) $\sqrt{-3}.\sqrt{-27}$ (b) $\sqrt{-3}\sqrt{-4}\sqrt{-72}$ (c) $3i^{15} - 5i^{8} + 1$

3. Form the complex numbers whose real and imaginary parts are given in the form of ordered pairs.

 (a) $z(3, -5)$ (b) $z(0, -4)$ (c) $z(8, \pi)$

4. Find the conjugate of each of the following:

 (a) $1 - 2i$ (b) $-1 - 2i$ (c) $6 - \sqrt{2}i$ (d) $4i$ (e) $-4i$

5. Find the modulus of each of the following:

 (a) $1 - i$ (b) $3 + \pi i$ (c) $-\frac{3}{2}i$ (d) $-2 + \sqrt{3}i$

6. Express $7i^{17} - 6i^{6} + 3i^{3} - 2i^{2} + 1$ in the form of $a + bi$.

7. Find the values of x and y if:

 (a) $(x - yi) + 7 - 2i = 9 - i$ (b) $2x + 3yi = 4 - 9i$ (c) $x - 3yi = 7 + 9i$

8. Simplify each of the following:

 (a) $(3 + i) - (1 - i) + (-1 + i)$ (b) $\left(\frac{1}{7}+i\right)-\left(\frac{2}{7}-i\right)+\left(\frac{3}{7}-2i\right)$

9. Write additive inverse and multiplicative inverse of each of the following:

 (a) $3 - 7i$ (b) $11 - 2i$ (c) $\sqrt{3} + 2i$ (d) $1 - \sqrt{2}i$ (e) $\frac{1+5i}{1-i}$

10. Find the modulus of each of the following complex numbers:

 (a) $\frac{1+i}{3-i}$ (b) $\frac{5+2i}{\sqrt{2}+\sqrt{3}i}$ (c) $(3 + 2i)(1 - i)$ (d) $(1 - 3i)(-2i^{3} + i^{2} + 3)$

11. For the following pairs of complex numbers verify that $|z_1 z_2| = |z_2||z_1|$

 (a) $z_1 = 3 - 2i$, $z_2 = 1 - 5i$ (b) $z_1 = 3 - \sqrt{7}i$, $z_2 = \sqrt{3} - i$

12. For the following pairs of complex numbers verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

 (a) $z_1 = 1 + 3i$, $z_2 = 2 + 5i$ (b) $z_1 = -2 + 5i$, $z_2 = 3 - 4i$

13. Find the square root of $2 + 3i$

14. Find the square root of $-2+2\sqrt{-3}$.

15. Find the square root of i.

ANSWERS

CHECK YOUR PROGRESS 8.1

1 (a) $3\sqrt{3}i$ (b) $-3i$ (c) $\sqrt{13}i$

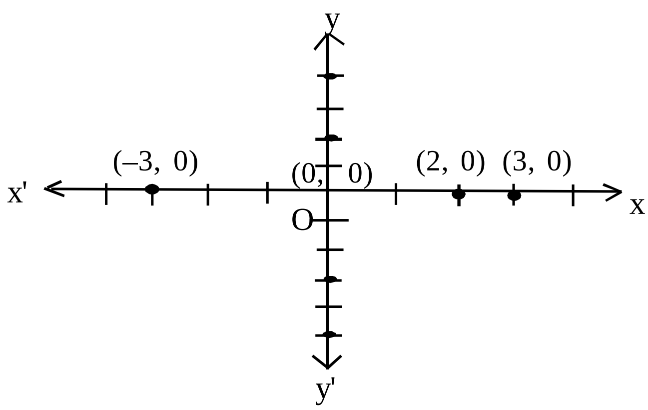
2. (a) $5 + 0i$ (b) $0 - 3i$ (c) $0 + 0i$

3. $12 - 4i$

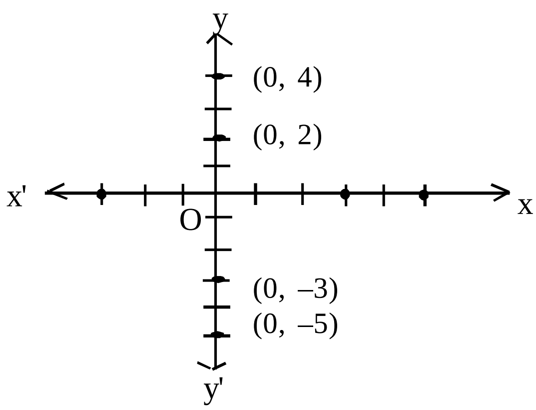
CHECK YOUR PROGRESS 8.2

1 (a) $2i$ (b) $-5 + 3i$ (c) $-\sqrt{2}$ (d) $3 + 4i$

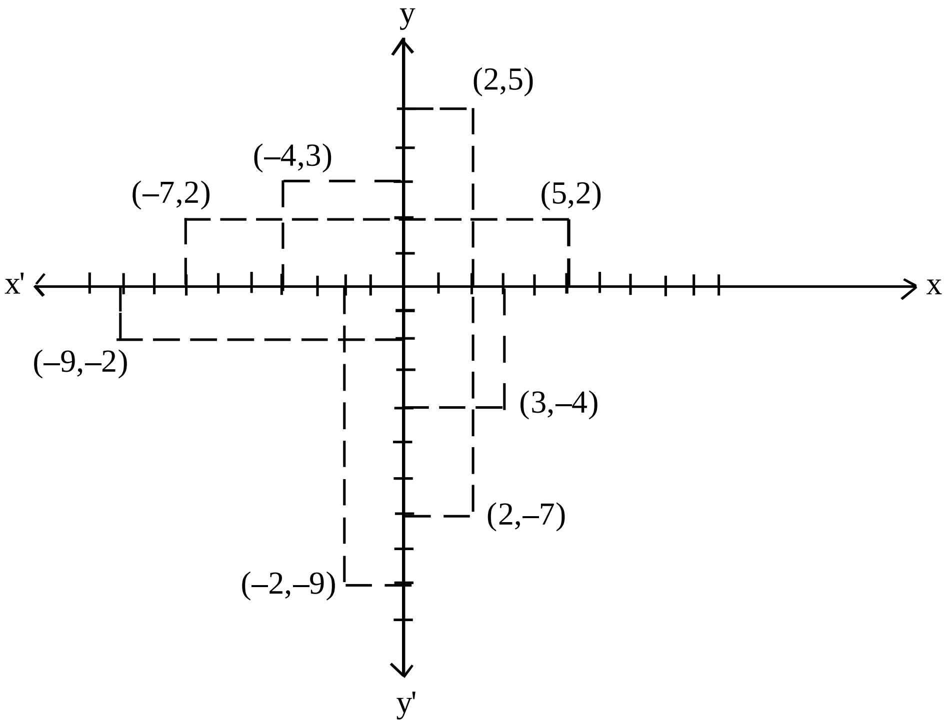
2. **(a)**



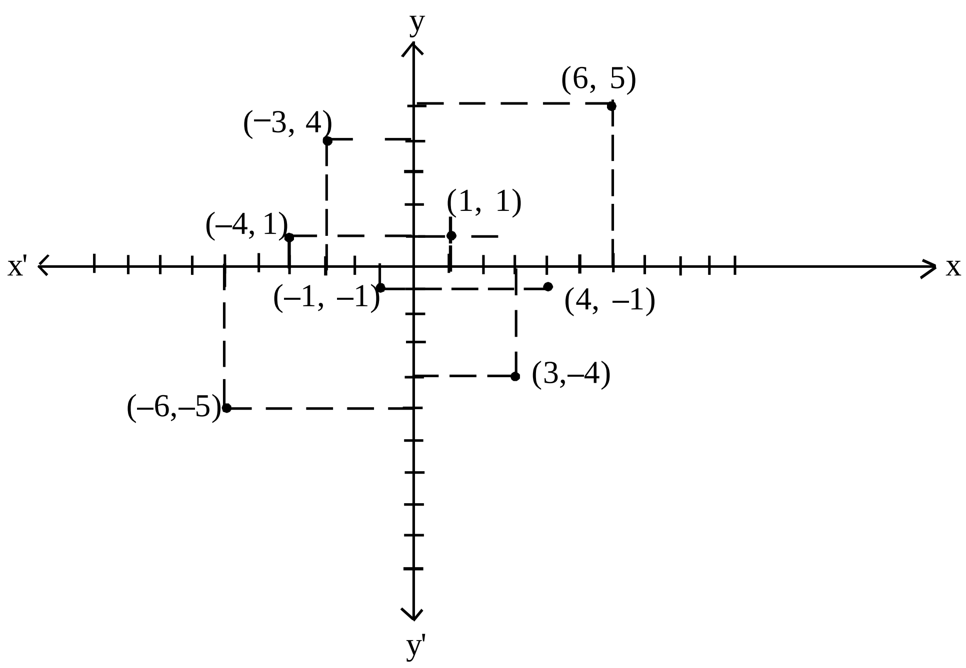
(b)



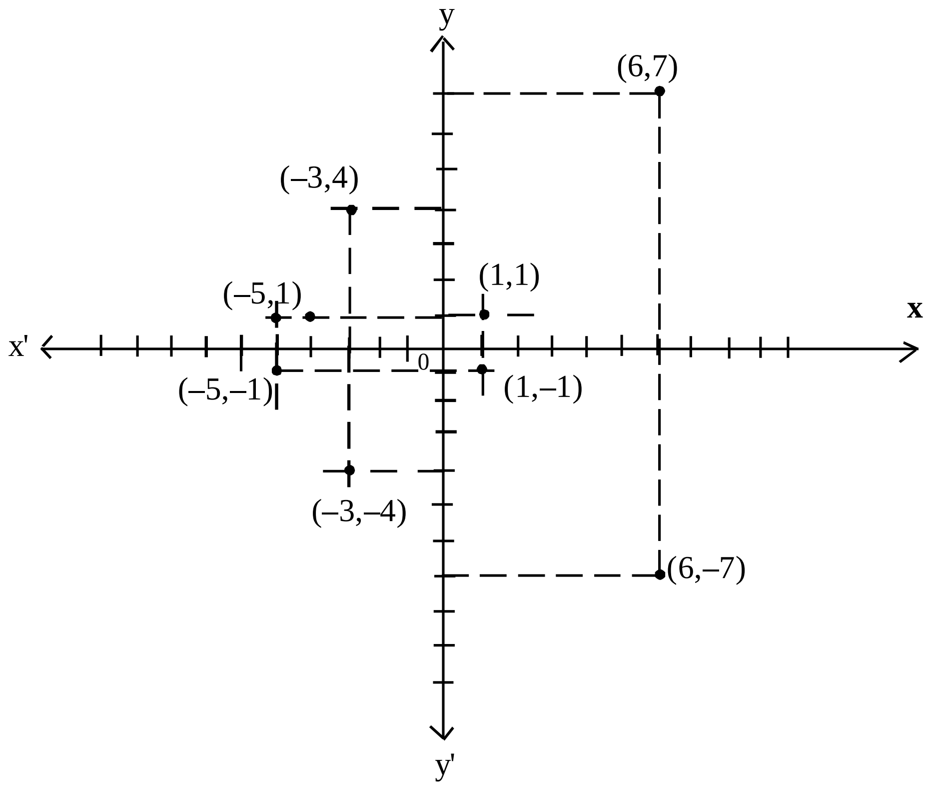
(c)



(d)



(e)



3. (a) (i) 3 (ii) $\sqrt{10}$ (iii) $\sqrt{13}$ (iv) $\sqrt{21}$

CHECK YOUR PROGRESS 8.3

1. (a) $\left(\sqrt{2}+\sqrt{5}\right)+\left(\sqrt{5}-\sqrt{2}\right)i$ (b) $\frac{1}{6}(6+i)$

   (c) 7i (d) $\sqrt{2}\left(\sqrt{2}+1\right)+(7-\sqrt{3})$

2. (a) 11 + 3i (b) 11 + 3i (c) Yes

   (d) –1 – i (e) 1 + i (f) No

3. (a) 4 + 3i (b) 4 + 3i (c) Yes

   (d) 2 + 5i (e) – 2 – i (f) No.

4. (a) –12 + 7i (b) –4 + 3i

5. 18 – 6i

CHECK YOUR PROGRESS 8.4

1. (a) $\left(\sqrt{2}+2\right)+\left(2\sqrt{2}-1\right)i$ (b) $1+2\sqrt{2}i$

(c) $-2 + 6i$ (d) $\frac{1}{\sqrt{5}}(-4+7i)$

(e) $\frac{1}{2}(3+i)$ (f) $\frac{1}{58}(3-7i)$

2. (a) $\frac{1}{25}(3+4i)$ (b) $\frac{1}{52}(\sqrt{3}-7i)$

(c) $\frac{1}{34}(-9-19i)$

CHECK YOUR PROGRESS 8.5

(i) $2 - 5i, -2 + 5i$ (ii) $\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}}i, \frac{-1}{\sqrt{2}}+\frac{3}{\sqrt{2}}i$

(iii) $1-7i, -1+7i$

TERMINAL EXERCISE

1. (a) 2, 7 (b) 3, 0

(c) $-\frac{1}{2}, 0$ (d) 0, 5

(e) $\frac{2}{13}, -\frac{3}{13}$

2. (a) -9 (b) $-12\sqrt{6}i$

(c) $-4 - 3i$

3. (a) $3 - 5i$ (b) $0 - 4i$

(c) $8+\pi i$

4. (a) $1 + 2i$ (b) $-1 + 2i$

(c) $6+\sqrt{2}i$ (d) $-4i$

(e) $4i$

5. (a) $\sqrt{2}$ (b) $\sqrt{9+\pi^2}$

(c) $\frac{3}{2}$ (d) $\sqrt{7}$

6. $9 + 4i$

7. (a) $x = 2, \ y = -1$ (b) $x = 2, \ y = -3$

(c) $x = 7, \ y = -3$

8. (a) $1 + 3i$ (b) $\frac{2}{7} + 0i$

9. (a) $-3 + 7i, \ \frac{1}{58}(3+7i)$ (b) $-11+2i, \ \frac{1}{125}(-11+2i)$

(c) $-\sqrt{3}-2i, \ \frac{1}{7}\left(\sqrt{3}-2i\right)$ (d) $-1+\sqrt{2}i, \ \frac{1}{3}(1+\sqrt{2}i)$

(e) $2-3i, \ \frac{1}{13}\left(2+3i\right)$

10. (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{5}\sqrt{145}$

(c) $\sqrt{26}$ (d) $4\sqrt{5}$

13. $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}}+\sqrt{\frac{\sqrt{13}-2}{2}}\,i\right)$

14. $1+\sqrt{3}i, -1-\sqrt{3}i$

15. $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

9

QUADRATIC EQUATIONS AND LINEAR INEQUALITIES

Recall that an algebraic equation of the second degree is written in general form as $ax^2 + bx + c = 0,\ a \neq 0$. It is called a quadratic equation in x. The coefficient 'a' is the first or leading coefficient, 'b' is the second or middle coefficient and 'c' is the constant term (or third coefficient). For example, $7x^2 + 2x + 5 = 0$, $\frac{5}{2}x^2 + \frac{1}{2}x + 1 = 0$,

$3x^2 - x = 0$, $x^2 + \frac{1}{2} = 0$, $\sqrt{2}\,x^2 + 7x = 0$, are all quadratic equations.

Some times, it is not possible to translate a word problem in the form of an equation. Let us consider the following situation:

Alok goes to market with Rs. 30 to buy pencils. The cost of one pencil is Rs. 2.60. If x denotes the number of pencils which he buys, then he will spend an amount of Rs. 2.60x. This amount cannot be equal to Rs. 30 as x is a natural number. Thus.

$$2.60\,x < 30 \qquad \ldots \text{(i)}$$

Let us consider one more situation where a person wants to buy chairs and tables with Rs. 50,000 in hand. A table costs Rs. 550 while a chair costs Rs. 450. Let x be the number of chairs and y be the number of tables he buys, then his total cost = Rs.(550 x + 450 y)

Thus, in this case we can write, $550x + 450y \leq 50{,}000$

or $\quad 11x + 9y \leq 1000 \qquad \ldots$ (ii)

Statement (i) involves the sign of inequality '<' and statement (ii) consists of two statements: $11x+9y < 1000$, $11x+9y = 1000$ in which the first one is not an equation: Such statements are called Inequalities. In this lesson, we will discuss linear inequalities and their solution.

We will also discuss how to solve quadratic equations with real and complex coefficients and establish relation between roots and coefficients.

OBJECTIVES

After studying this lesson, you will be able to:

- solve a quadratic equation with real coefficients by factorization and by using quadratic formula;

- find relationship between roots and coefficients;
- form a quadratic equation when roots are given;
- differentiate between a linear equation and a linear inequality;
- state that a planl region represents the solution of a linear inequality;
- represent graphically a linear inequality in two variables;
- show the solution of an inequality by shading the appropriate region;
- solve graphically a system of two or three linear inequalities in two variables;

EXPECTED BACKGROUND KNOWLEDGE

- Real numbers
- Quadratic Equations with real coefficients.
- Solution of linear equations in one or two variables.
- Graph of linear equations in one or two variables in a plane.
- Graphical solution of a system of linear equations in two variables.

9.1 ROOTS OF A QUADRATIC EQUATION

The value which when substituted for the variable in an equation, satisfies it, is called a root (or solution) of the equation.

If α be one of the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0 \qquad \text{... (i)}$$

then $\quad a\alpha^2 + b\alpha + c = 0$

In other words, $x - \alpha$ is a factor of the quadratic equation (i)

In particular, consider a quadratic equation $x^2 + x - 6 = 0$ ...(ii)

If we substitute x = 2 in (ii), we get L.H.S $= 2^2 + 2 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put x = -3 in (ii), we get L.H.S. $= (-3)^2 - 3 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put $x = -1$ in (ii) ,we get L.H.S $= (-1)^2 + (-1) - 6 = -6 \neq 0$ = R.H.S.

$\therefore$ $x = 2$ and $x = -3$ are the only values of x which satisfy the quadratic equation (ii)

There are no other values which satisfy (ii)

$\therefore$ $x = 2$, $x = -3$ are the only two roots of the quadratic equation (ii)

Note: *If α, β be two roots of the quadratic equation*

$ax^2 + bx + c = 0$, $a \neq 0$ ...(A)

then $(x - \alpha)$ and $(x - \beta)$ will be the factors of (A). The given quadratic equation can be written in terms of these factors as $(x - \alpha)(x - \beta) = 0$

9.2 SOLVING QUADRATIC EQUATION BY FACTORIZATION

Recall that you have learnt how to factorize quadratic polynomial of the form $p(x) = ax^2 + bx + c,\ a \neq 0,$ by splitting the middle term and taking the common factors. Same method can be applied while solving a quadratic equation by factorization.

If $x - \frac{p}{q}$ and $x - \frac{r}{s}$ are two factors of the quadratic equation

$$ax^2 + bx + c = 0, a \neq 0 \text{ then } \left(x - \frac{p}{q}\right)\left(x - \frac{r}{s}\right) = 0$$

$\therefore$ either $x = \frac{p}{q}$ or, $x = \frac{r}{s}$

$\therefore$ The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{p}{q}, \frac{r}{s}$

Example 9.1 Using factorization method, solve the quadratic equation: $6x^2 + 5x - 6 = 0$

Solution: The given quadratic equation is $6x^2 + 5x - 6 = 0$ ... (i)

Splitting the middle term, we have $6x^2 + 9x - 4x - 6 = 0$

or, $3x(2x + 3) - 2(2x + 3) = 0$ or, $(2x + 3)(3x - 2) = 0$

$\therefore$ Either $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$ or, $3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{2}, \frac{2}{3}$

Example 9.2 Using factorization method, solve the quadratic equation:

$$3\sqrt{2}x^2 + 7x - 3\sqrt{2} = 0$$

Solution: Splitting the middle term, we have $3\sqrt{2}x^2 + 9x - 2x - 3\sqrt{2} = 0$

or, $3x(\sqrt{2}x + 3) - \sqrt{2}(\sqrt{2}x + 3) = 0$ or, $(\sqrt{2}x + 3)(3x - \sqrt{2}) = 0$

$\therefore$ Either $\sqrt{2}x + 3 = 0 \Rightarrow x = -\frac{3}{\sqrt{2}}$ or, $3x - \sqrt{2} = 0 \Rightarrow x = \frac{\sqrt{2}}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{\sqrt{2}}, \frac{\sqrt{2}}{3}$

Example 9.3 Using factorization method, solve the quadratic equation:

$$(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$$

Solution: The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$

Splitting the middle term, we have

$(a + b)^2 x^2 + 3(a^2 - b^2)x + 3(a^2 - b^2)x + 9(a - b)^2 = 0$

or, $(a + b)x\{(a + b)x + 3(a - b)\} + 3(a - b)\{(a + b)x + 3(a - b)\} = 0$

or, $\{(a + b)x + 3(a - b)\}\{(a + b)x + 3(a - b)\} = 0$

$\therefore$ either $(a + b)x + 3(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

or, $(a + b)x + 3(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

The equal roots of the given quadratic equation are $\dfrac{3(b-a)}{a+b}, \dfrac{3(b-a)}{a+b}$

Alternative Method

The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$

This can be rewritten as

$\{(a + b)x\}^2 + 2.(a + b)x . 3(a - b) + \{3(a - b)\}^2 = 0$

or, $\{(a + b)x + 3(a - b)\}^2 = 0$ or, $x = -\dfrac{3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

$\therefore$ The quadratic equation has equal roots $\dfrac{3(b-a)}{a+b}, \dfrac{3(b-a)}{a+b}$

CHECK YOUR PROGRESS 9.1

1. Solve each of the following quadratic equations by factorization method:

(i) $\sqrt{3}x^2 + 10x + 8\sqrt{3} = 0$ (ii) $x^2 - 2ax + a^2 - b = 0$

(iii) $x^2 + \left(\dfrac{ab}{c} - \dfrac{c}{ab}\right)x - 1 = 0$ (iv) $x^2 - 4\sqrt{2}x + 6 = 0$

9.3 SOLVING QUADRATIC EQUATION BY QUADRATIC FORMULA

Recall the solution of a standard quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ by the **"Method of Completing Squares"**

Roots of the above quadratic equation are given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{D}}{2a}, \qquad = \frac{-b-\sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ **if**

(i) **D>0, the equation will have two real and unequal roots**

(ii) **D=0, the equation will have two real and equal roots and both roots are equal to** $-\dfrac{\mathbf{b}}{\mathbf{2\,a}}$

(iii) **D<0, the equation will have two conjugate complex (imaginary) roots.**

Example 9.4 Examine the nature of roots in each of the following quadratic equations and also verify them by formula.

(i) $x^2 + 9x + 10 = 0$ (ii) $9y^2 - 6\sqrt{2}y + 2 = 0$

(iii) $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

Solution:

(i) The given quadratic equation is $x^2 + 9x + 10 = 0$

Here, $a = 1,\ b = 9$ and $c = 10$

$\therefore$ $D = b^2 - 4ac = 81 - 4.1.10 = 41 > 0.$

$\therefore$ The equation will have two real and unequal roots

Verification: By quadratic formula, we have $x = \dfrac{-9 \pm \sqrt{41}}{2}$

$\therefore$ The two roots are $\dfrac{-9+\sqrt{41}}{2}, \dfrac{-9-\sqrt{41}}{2}$ which are real and unequal.

(ii) The given quadratic equation is $9y^2 - 6\sqrt{2}\,y + 2 = 0$

Here, $D = b^2 - 4ac = (-6\sqrt{2})^2 - 4.9.2 = 72 - 72 = 0$

$\therefore$ The equation will have two real and equal roots.

Verification: By quadratic formula, we have $y = \dfrac{6\sqrt{2} \pm \sqrt{0}}{2.9} = \dfrac{\sqrt{2}}{3}$

$\therefore$ The two equal roots are $\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$.

(iii) The given quadratic equation is $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

Here, $D = (-3)^2 - 4.\sqrt{2}.3\sqrt{2} = -15 < 0$

$\therefore$ The equation will have two conjugate complex roots.

Verification: By quadratic formula, we have t = $\dfrac{3\pm\sqrt{-15}}{2\sqrt{2}} = \dfrac{3\pm\sqrt{15}i}{2\sqrt{2}}$, where $i=\sqrt{-1}$

$\therefore$ Two conjugate complex roots are $\dfrac{3+\sqrt{15}\,i}{2\sqrt{2}}, \dfrac{3-\sqrt{15}\,i}{2\sqrt{2}}$

Example 9.5 Prove that the quadratic equation $x^2 + px - 1 = 0$ has real and distinct roots for all real values of p.

Solution: Here, D = $p^2 + 4$ which is always positive for all real values of p.

$\therefore$ The quadratic equation will have real and distinct roots for all real values of p.

Example 9.6 For what values of k the quadratic equation

$(4k+ 1) x^2 + (k + 1) x + 1 = 0$ will have equal roots ?

Solution: The given quadratic equation is $(4k + 1)x^2 + (k + 1) x + 1 = 0$

Here, $D = (k + 1)^2 - 4.(4k + 1).1$

For equal roots, D = 0 $\therefore$ $(k + 1)^2 - 4 (4k + 1) = 0$

$\Rightarrow$ $k^2 - 14k - 3 = 0$

$\therefore$ $k = \dfrac{14\pm\sqrt{196+12}}{2}$ or $k = \dfrac{14\pm\sqrt{208}}{2} = 7 \pm 2\sqrt{13}$ or $7 + 2\sqrt{13}, 7 - 2\sqrt{13}$

which are the required values of k.

Example 9.7 Prove that the roots of the equation

$x^2 (a^2+ b^2) + 2x (ac+ bd) + (c^2+ d^2) = 0$ are imaginary. But if ad = bc, roots are real and equal.

Solution: The given equation is $x^2 (a^2 + b^2) + 2x (ac + bd) + (c^2 + d^2) = 0$

Discriminant $= 4 (ac + bd)^2 - 4 (a^2 + b^2) (c^2 + d^2)$

$= 8\,abcd - 4(a^2 d^2 + b^2 c^2) = -4 (-2\,abcd + a^2 d^2 + b^2 c^2)$

$= -4 (ad - bc)^2$, <0 for all a, b, c, d

$\therefore$ The roots of the given equation are imaginary.

For real and equal roots, discriminant is equal to zero.

$\Rightarrow$ $-4 (ad - bc)^2 = 0$ or, ad = bc

Hence, if ad=bc, the roots are real and equal.

CHECK YOUR PROGRESS 9.2

1. Solve each of the following quadratic equations by quadratic formula:

(i) $2x^2 - 3x + 3 = 0$ (ii) $-x^2 + \sqrt{2}x - 1 = 0$

(iii) $-4x^2 + \sqrt{5}x - 3 = 0$ (iv) $3x^2 + \sqrt{2}x + 5 = 0$

2. For what values of k will the equation
   $y^2 - 2(1 + 2k)y + 3 + 2k = 0$ have equal roots ?
3. Show that the roots of the equation
   $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real and they can not be equal unless $a = b = c$.

9.4 RELATION BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

You have learnt that, the roots of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Let $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ...(i) and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ ... (ii)

Adding (i) and (ii), we have $\alpha + \beta = \dfrac{-2b}{2a} = \dfrac{-b}{a}$

$\therefore$ **Sum of the roots** $= -\dfrac{coefficient\ of\ x}{coefficient\ of\ x^2} = -\dfrac{b}{a}$ ... (iii)

$$\alpha\ \beta = \frac{+b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ **Product of the roots** $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{c}{a}$ ...(iv)

(iii) and (iv) are the required relationships between roots and coefficients of a given quadratic equation. These relationships helps to find out a quadratic equation when two roots are given.

Example 9.8 If, α, β are the roots of the equation $3x^2 - 5x + 9 = 0$ find the value of:

(a) $\alpha^2 + \beta^2$ (b) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution: (a) It is given that α, β are the roots of the quadratic equation $3x^2 - 5x + 9 = 0$.

$\therefore$ $\alpha + \beta = \dfrac{5}{3}$ ... (i)

and $\alpha\beta = \dfrac{9}{3} = 3$ ... (ii)

Now, $\alpha^2+\beta^2 \quad =(\alpha+\beta)^2-2\alpha\beta \quad = \left(\frac{5}{3}\right)^2 - 2.3$ [By (i) and (ii)]

$$= -\frac{29}{9}$$

(b) Now, $\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{\frac{-29}{9}}{9}$ [By (i) and (ii)]

$$= -\frac{29}{81}$$

Example 9.9 If α, β are the roots of the equation $3y^2 + 4y + 1 = 0$, form a quadratic equation whose roots are α^2, β^2

Solution: It is given that α, β are two roots of the quadratic equation $3y^2 + 4y + 1 = 0$.

$\therefore$ Sum of the roots

i.e., $\alpha + \beta = -\frac{\text{coefficient of y}}{\text{coefficient of } y^2} = -\frac{4}{3}$ ... (i)

Product of the roots i.e., $\alpha\ \beta = \frac{\text{constant term}}{\text{coefficient of } y^2} = \frac{1}{3}$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\ \alpha\ \beta$

$$= \left(-\frac{4}{3}\right)^2 - 2.\frac{1}{3} \quad \text{[By (i) and (ii)]}$$

$$= \frac{16}{9} - \frac{2}{3} = \frac{10}{9}$$

and $\alpha^2\beta^2 = (\alpha\ \beta)^2 = \frac{1}{9}$ [By (i)]

$\therefore$ The required quadratic equation is $y^2 - (\alpha^2 + \beta^2)y + \alpha^2\beta^2 = 0$

or, $y^2 - \frac{10}{9}y + \frac{1}{9} = 0$ or, $9y^2 - 10y + 1 = 0$

Example 9.10 If one root of the equation $ax^2 + bx + c = 0,\ a \neq 0$ be the square of the other, prove that $b^3 + ac^2 + a^2c = 3abc$

Solution: Let α, α^2 be two roots of the equation $ax^2 + bx + c = 0$.

$\therefore \quad \alpha + \alpha^2 = -\dfrac{b}{a}$ ... (i)

and $\quad \alpha . \alpha^2 = \dfrac{c}{a}$

i.e., $\quad \alpha^3 = \dfrac{c}{a}$ . ... (ii)

From (i) we have $\alpha\,(\alpha + 1) = -\dfrac{b}{a}$

or, $\quad \{\alpha\,(\alpha + 1)\}^3 = \left(-\dfrac{b}{a}\right)^3 = -\dfrac{b^3}{a^3}$ or, $\alpha^3(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = -\dfrac{b^3}{a^3}$

or, $\quad \dfrac{c}{a}\left\{\dfrac{c}{a} + 3\left(-\dfrac{b}{a}\right) + 1\right\} = -\dfrac{b^3}{a^3}$ ... [By (i) and (ii)]

or, $\quad \dfrac{c^2}{a^2} - \dfrac{3bc}{a^2} + \dfrac{c}{a} = -\dfrac{b^3}{a^3}$ or, $ac^2 - 3abc + a^2c = -b^3$

or, $b^3 + ac^2 + a^2c = 3abc$, which is the required result.

Example 9.11 Find the condition that the roots of the equation $ax^2 + bx + c = 0$ are in the ratio m : n

Solution: Let $m\alpha$ and $n\alpha$ be the roots of the equation $ax^2 + bx + c = 0$

Now, $\quad m\alpha + n\alpha = -\dfrac{b}{a}$ ... (i)

and $\quad mn\alpha^2 = \dfrac{c}{a}$ ... (ii)

From (i) we have, $\quad \alpha\,(m + n) = -\dfrac{b}{a}$ or, $\alpha^2\,(m + n)^2 = \dfrac{b^2}{a^2}$

or, $\quad \dfrac{c}{a}\,(m + n)^2 = mn\,\dfrac{b^2}{a^2}$ [By (ii)]

or, $\quad ac\,(m+n)^2 = mn\,b^2$, which is the required condition

CHECK YOUR PROGRESS 9.3

1. If α, β are the roots of the equation $ay^2 + by + c = 0$ then find the value of :

(i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ (ii) $\dfrac{1}{\alpha^4}+\dfrac{1}{\beta^4}$

2. If α, β are the roots of the equation $5x^2-6x+3=0$, form a quadratic equation whose roots are:

   (i) α^2, β^2 (ii) $\alpha^3\beta, \alpha\beta^3$

3. If the roots of the equation $ay^2+by+c=0$ be in the ratio 3:4, prove that $12b^2=49\,ac$

4. Find the condition that one root of the quadratic equation $px^2-qx+p=0$ may be 1 more than the other.

9.5 SOLUTION OF A QUADRATIC EQUATION WHEN D < 0

Let us consider the following quadratic equation:

(a) Solve for t : $t^2+3t+4=0$

$$\therefore \quad t=\frac{-3\pm\sqrt{9-16}}{2}=\frac{-3\pm\sqrt{-7}}{2}$$

Here, $D=-7<0$

$$\therefore \quad \text{The roots are } \frac{-3+\sqrt{-7}}{2} \text{ and } \frac{-3-\sqrt{-7}}{2}$$

$$\text{or,} \quad \frac{-3+\sqrt{7}\,i}{2}, \frac{-3-\sqrt{7}\,i}{2}$$

Thus, the roots are complex and conjugate.

(b) Solve for y :

$$-3y^2+\sqrt{5}\,y-2=0$$

$$\therefore \quad y=\frac{-\sqrt{5}\pm\sqrt{5-4(-3).(-2)}}{2(-3)} \text{ or } y=\frac{-\sqrt{5}\pm\sqrt{-19}}{-6}$$

Here, $D=-19<0$

$$\therefore \quad \text{The roots are } \frac{-\sqrt{5}+\sqrt{19}i}{-6}, -\frac{\sqrt{5}-\sqrt{19}i}{-6}$$

Here, also roots are complex and conjugate. From the above examples , we can make the following conclusions:

(i) $D<0$ in both the cases

(ii) Roots are complex and conjugate to each other.

Is it always true that complex roots occur in conjugate pairs ?

Let us form a quadratic equation whose roots are $2+3i$ and $4-5i$

The equation will be $\{x-(2+3i)\}\,\{x-(4-5i)\}=0$

or, $x^2-(2+3i)x-(4-5i)x+(2+3i)(4-5i)=0$

or, $x^2+(-6+2i)x+23+2i=0$, which is an equation with complex coefficients.

> ***Note*** **:** *If the quadratic equation has two complex roots, which are not conjugate of each other, the quadratic equation is an equation with complex coefficients.*

9.6 Fundamental Theorem of Algebra

You may be interested to know as to how many roots does an equation have? In this regard the following theorem known as fundamental theorem of algebra, is stated (without proof). 'A polynomial equation has at least one root'.

As a consequence of this theorem, the following result, which is of immense importance is arrived at.

'A polynomial equation of degree *n* has exactly *n* roots'

CHECK YOUR PROGRESS 9.4

Solve each of the following equations.

1. $-x^2+x+2=0$ 2. $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$ 3. $x^2+\dfrac{1}{\sqrt{2}}x+1=0$

4. $\sqrt{5}x^2+x+\sqrt{5}=0$ 4. $x^2+3x+5=0$

9.7 INEQUALITIES (INEQUATIONS)

Now we will discuss about linear inequalities and their applications from daily life. A statement involving a sign of equality (=) is an equation.

Similarly, a statement involving a sign of inequality, $<$, $>$, $\leq$, or $\geq$ is called an inequalities.

Some examples of inequalities are:

(i) $2x+5>0$ (ii) $3x-7<0$

(iii) $ax+b\geq 0,\ a\neq 0$ (iv) $ax+b\leq c,\ a\neq 0$

(v) $3x+4y\leq 12$ (vi) $x^2-5x+6<0$

(vii) $ax+by+c\geq 0$

(v) and (vii) are inequalities in two variables and all other inequalities are in one variable. (i) to (v) and (vii) are linear inequalities and (vi) is a quadratic inequalities.

In this lesson, we shall study about linear inequalities in one or two variables only.

9.8 SOLUTIONS OF LINEAR INEQUALITIES IN ONE/TWO VARIABLES

Solving an inequalities means to find the value (or values) of the variable (s), which when substituted in the inequalities, satisfies it.

For example, for the inequalities $2.60x<30$ (statement) (i) all values of $x \le 11$ are the solutions. (x is a whole number)

For the inequalities $2x + 16>0$, where x is a real number, all values of x which are > -8 are the solutions.

For the linear inequation in two variables, like $ax + by + c \ge 0$, we shall have to find the pairs of values of x and y which make the given inequalities true.

Let us consider the following situation :

Anil has Rs. 60 and wants to buy pens and pencils from a shop. The cost of a pen is Rs. 5 and that of a pencil is Rs. 3 If x denotes the number of pens and y, the number of pencils which Anil buys, then we have the inequality $5x + 3y \le 60$ ... (i)

Here, $x = 6$, $y = 10$ is one of the solutions of the inequalities (i). Similarly $x = 5$, $y = 11$; $x = 4$, $y = 13$; $x = 10$, $y = 3$ are some more solutions of the inequalities.

In solving inequalities, we follow the rules which are as follows :

1. Equal numbers may be added (or subtracted) from both sides of an inequalities.

   Thus (i) if $a > b$ then $a + c > b + c$ and $a - c > b - c$

   and (ii) if $a \le b$ then $a + d \le b + d$ and $a - d \le b - d$

2. Both sides of an inequalities can be multiplied (or divided) by the same positive number.

   Thus (i) if $a > b$ and $c > 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

   and (ii) if $a \le b$ and $c > 0$ then $ac \le bc$ and $\frac{a}{c} \le \frac{b}{c}$

3. When both sides of an inequalities are multiplied by the same negative number, the sign of inequality gets reversed.

   Thus (i) if $a > b$ and $d < 0$ then $ad < bd$ and $\frac{a}{d} < \frac{b}{d}$

   and (ii) if $a \le b$ and $c < 0$ then $ac \ge bc$ and $\frac{a}{c} \ge \frac{b}{c}$

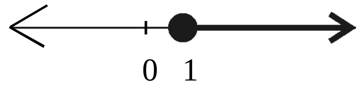
Example 9.12 Solve $\frac{3x-4}{2} \ge \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution: We have

$$\frac{3x-4}{2} \ge \frac{x+1}{4} - 1 \text{ or } \frac{3x-4}{2} \ge \frac{x+3}{4}$$

$$\text{or} \quad 2(3x-4) \ge (x-3) \text{ or } 6x - 8 \ge x - 3 \text{ or } 5x \ge 5 \; or \; x \ge 1$$

The graphical representation of solutions is given in Fig.



Example 9.13 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution: Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60 \text{ or } 110+x \geq 180 \text{ or } x \geq 70$$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 9.14 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution: Let x litres of 30% acid solution is required to be added. Then

Total mixture = (x + 600) litres

Therefore 30% x + 12% of 600 > 15% of (x + 600)

and 30% x + 12% of 600 < 18% of (x + 600)

or $$\frac{30x}{100}+\frac{12}{100}(600)>\frac{15}{100}(x+600)$$

and $$\frac{30x}{100}+\frac{12}{100}(600)<\frac{18}{100}(x+600)$$

or $30x + 7200 > 15x + 9000$

and $30x + 7200 < 18x + 10800$

or $15x > 1800$ and $12x < 3600$

or $x > 120$ and $x < 300$,

i.e. $120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

6.3 GRAPHICAL REPRESENTATION OF LINEAR INEQUALITIES IN ONE OR TWO VARIABLES.

In Section 6.2, while translating word problem of purchasing pens and pencils, we obtained the following linear inequalities in two variables x and y :

$5x + 3y \leq 60$ .... (i)

Let us now find all solutions of this inequation, keeping in mind that x and y here can be only whole numbers.

To start with, let $x = 0$.

Thus, we have $3y \leq 60$ or $y \leq 20$, i,e the values of y corresponding to $x = 0$ can be 0,1,2,3..........., 20 only Thus, the solutions with $x = 0$ are

(0,0), (0,1), (0,2) ....(0, 20)

Similarly the other solutions of the inequalities, when $x = 1,2, ...12$ are

(1,0) (1,1) (1,2) .... (1,18)

(2,0) (2,1) (2,2) .... (2,16)

....

....

(10,0) (10,1) (10,2), (10,3)

(11,0) (11,1)

(12,0)

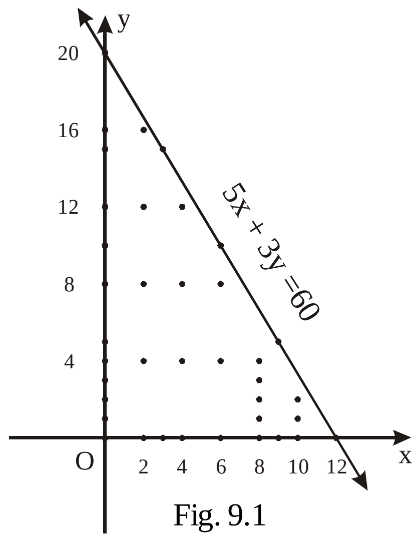


Fig. 9.1

You may note that out of the above ordered pairs, some pairs such as (0,20), (3, 15), (6, 10), (9, 5), (12,0) satisfy the equation $5x + 3y = 60$ which is a part of the given inequation and all other possible solutions lie on **one of the two half planes** in which the line $5x + 3y = 60$, divides the xy - plane.

If we now extend the domain of x and y from whole numbers to real numbers, the inequation $5x + 3y \leq 60$ will represent one of the two half planes in which the line $5x + 3y = 60$, divides the xy-plane.

Thus we can generalize as follows :

If $a,b,c,$ are real numbers, then $ax + by + c = 0$ is called a linear equalities in two variables x and y, where as $ax + by + c \leq 0$ or $ax + by + c \geq 0$, $ax + by + c > 0$ and $ax + by + c < 0$ are called linear inequations in two variables x and y.

The equation $ax + by + c = 0$ is a straight line which divides the xy plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$.

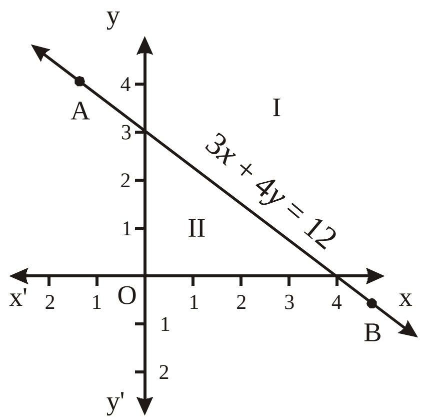


Fig. 9.2

For example $3x + 4y - 12 = 0$ can be represented by line AB, in the xy - plane as shown in Fig. 9.2

The line AB divides the cordinate plane into two half -plane regions :

(i) half plane region I above the line AB

(ii) half plane region II below the line AB. One of the above region represents the inequality $3x + 4y - 12 \le 0$ ...(i) and the other region will be represented by $3x + 4y - 12 \ge 0$. ... (ii)

To identify the half plane represented by inequation (i), we take any arbitrary point, preferably origin, if it does not lie on AB. If the point satisfies the inequation (i), then the half plane in which the arbitrary point lies, is the desired half plane. In this case, taking origin as the arbitrary point we have

$0+0-12 \le 0$ i.e $-12 \le 0$. Thus origin satisfies the inequalities $3x + 4y - 12 \le 0$. Now, origin lies in half plane region II. Hence the inequalty $3x + 4y - 12 \le 0$ represents half plane II and the inequality $3x + 4y - 12 \ge 0$ will represent the half plane I

Example 9.15 Show on graph the region represented by the inequalities $x + 2y \ge 5$.

Solution : The given inequalities is $x + 2y \ge 5$

Let us first take the corresponding linear equation $x + 2y = 5$ and draw its graph with the help of the following table :

x	1	3	5
y	2	1	0

Since (0,0) does not lie on the line AB, so we can select (0,0) as the arbitrary point. Since $0 + 0 \ge 5$ is not true

$\therefore$ The desired half plane is one, in which origin does not lie

$\therefore$ The desired half plane is the shaded one (See Fig. 9.3)

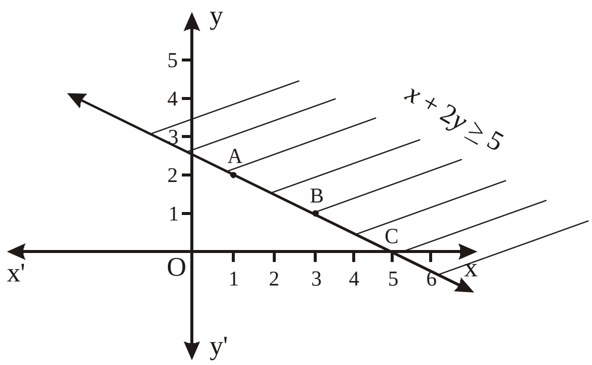


Fig. 9.3

Before taking more examples, it is important to define the following :

(i) **Closed Half Plane:** A half plane is said to be closed half plane if all points on the line separating the two half planes are also included in the solution of the inequation. The Half plane in Example 6.1 is a closed half plane.

(ii) **An Open Half Plane :** A half plane in the xy plane is said to be an open half plane if the points on the line separting the planes are not included in the half plane.

Example 9.16 Draw the graph of inequation $x-5y>0$

Solution : The given inequation is $x-5y>0$

The corresponding linear equation is $x-5y=0$ we have the following table.

x	0	5	–5
y	0	1	–1

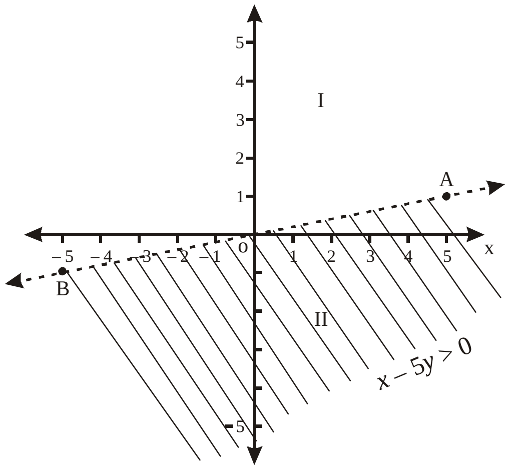


Fig. 9.4

The line AOB divides xy - plane into two half planes I and II. As the line AOB passes through origin, we consider any other arbitrary point (say) P (3,4) which is in half plane I. Let us see whether it satisfies the given inequation $x-5y>0$

$\therefore$ Then $3-5(4)>0$ or $3-20>0$, or $-17>0$ which is not true

$\therefore$ The desired half plane is II

Again the inequation is a **strict** inequation $x-5y>0$

$\therefore$ Line AOB is not a part of the graph and hence has been shown as a dotted line.

Hence, the graph of the given inequation is the shaded region half plane II excluding the line AOB.

Example 9.17 Represent graphically the inequlities $3x-12\geq 0$

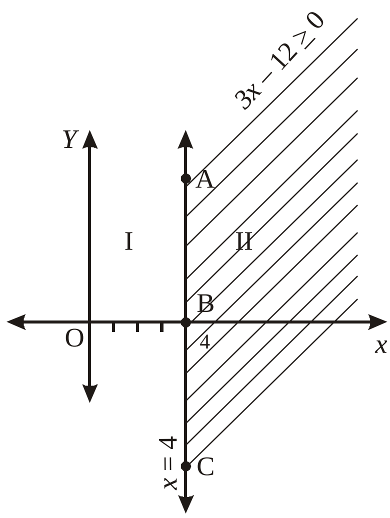


Fig. 9.5

Solution : Given inequation is $3x-12\geq 0$ and the corresponding linear equation is $3x-12=0$ or $x-4=0$ or $x=4$ which is represented by the line ABC on the xy - plane (See Fig. 9.5). Taking (0,0) as the arbitrary point, we can say that $0\neq 4$ and so, half plane II represents the inequation $3x-12\geq 0$

Example 9.18 Solve graphically the inequation $2y+4\geq 0$

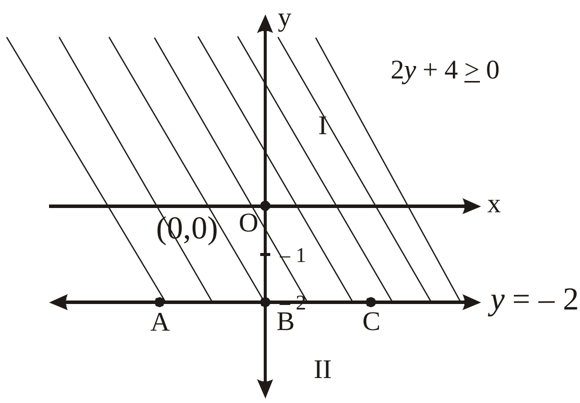


Fig. 9.6

Solution : Here the inequation is

$2\,y + 4 \geq 0$ and the corresponding equation is $2y + 4 = 0$ or $y = -2$

The line ABC represents the line
$y = -2$ which divides the xy - plane into two half planes and the inequation
$2y + 4 \geq 0$ is represented by the half plane I.

CHECK YOUR PROGRESS 9.5

Represent the solution of each of the following inequations graphically in two dimensional plane:

1. $2x + y \geq 8$	2. $x - 2y \leq 0$
3. $3x + 6 \geq 0$	4. $8 - 2y \geq 2$
5. $3y \geq 6 - 2x$	6. $3x \geq 0$
7. $y \leq 4$	8. $y > 2x - 8$
9. $-y < x - 5$	10. $2y \leq 8 - 4x$

6.4 GRAPHICAL SOLUTION OF A SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES.

You already know how to solve a system of linear equations in two variables.

Now, you have also learnt how to solve **linear inequations** in two variables graphically. We will now discuss the technique of finding the solutions of a system of simultaneous linear inequations. By the term solution of a system of simultaneons linear inequations we mean, finding all ordered pairs (x,y) for which each linear inequation of the system is satisfied.

A system of simultaneous inequations may have no solution or an infinite number of solutions represented by the region bounded or unbounded by straight lines corresponding to linear inequations.

We take the following example to explain the technique.

Example 9.19 Solve the following system of inequations graphically:

$$x + y \geq 6\,; \qquad 2x - y \geq 0.$$

Solution : Given inequations are

$$x + y \geq 6 \;..... \text{(i)}$$

and $\quad 2x - y \geq 0 \;..... \text{(ii)}$

We draw the graphs of the lines $x + y = 6$ and $2x - y = 0$ (Fig. 9.7)

The inequation (i) represent the shaded region above the line $x + y = 6$ and inequations (ii) represents the region on the right of the line $2x - y = 0$

The common region represented by the double shade in Fig. 9.7 represents the solution of the given system of linear inequations.

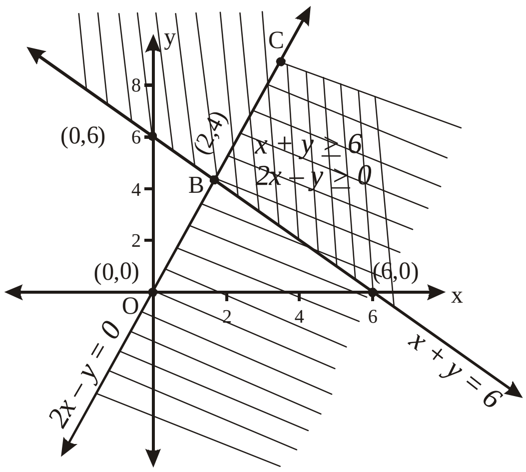


Fig. 9.7

Example 9.20 Find graphically the solution of the following system of linear inequations :

$x + y \le 5, \qquad 4x + y \ge 4,$
$x + 5y \ge 5, \qquad x \le 4, \;\; y \le 3.$

Solution : Given inequations are

$x + y \le 5$ ..... (i)
$4x + y \ge 4$ ..... (ii)
$x + 5y \ge 5$ ..... (iii)
$x \le 4$ ..... (iv)
and $y \le 3$ ..... (v)

We draw the graphs of the lines
$x + y = 5$, $4x + y = 4$, $x + 5y = 5$,
$x = 4$ and $y = 3$ (Fig. 9.8)

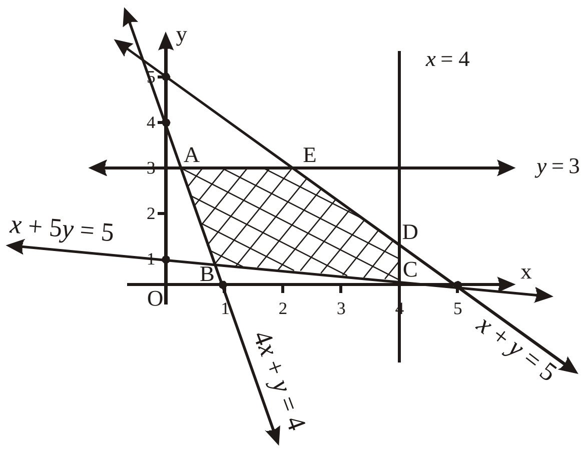


Fig. 9.8

The inequlities (i) represents the region below the line $x + y = 5$. The inequations (ii) represents the region on the right of equation $4x + y = 4$ and the region above the line $x + 5y = 5$ represents the inequation (iii). Similarly after shading the regions for inequations (iv) and (v) we get the common region as the bounded region <u>ABCDE</u> as shown in (Fig. 9.8) The co-ordinates of the points of the shaded region satisfy the given system of inequations and therefore all these points represent solution of the given system.

Example 9.21 Solve graphically the following system of inequations :

$x + 2y \le 3$, $3x + 4y \ge 12$, $x \ge 0$, $y \ge 0$.

Solution : We represent the inequations $x + 2y \le 3$, $3x + 4y \ge 12$, $x \ge 0$, $y \ge 0$ by shading the corresponding regions on the graph as shown in Fig. 9.9

Here we find that there is no common region represented by these inequations.

We thus conclude that there is no solution of the given system of linear inequations.

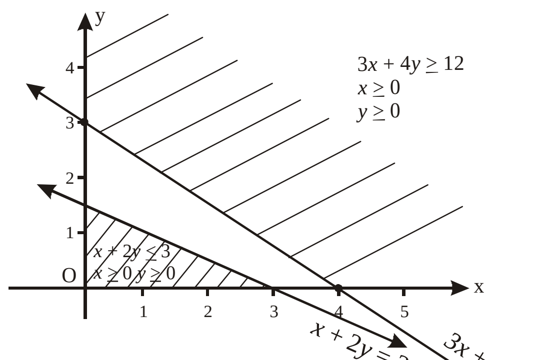


Fig. 9.9

Example 9.22 Solve the following system of linear inequations graphically :

$x - y < 2, \;\; 2x + y < 6; \; x \ge 0, \; y \ge 0.$

Solution : The given inequations are

$x - y < 2$ ... (i)
$2x + y < 6$ ... (ii)
$x \ge 0 \; ; \; y \ge 0$ ... (iii)

After representing the inequations $x - y < 2$, $2x + y < 6$, $x \ge 0$ and $y \ge 0$ on the graph we find the common region which is the bounded region *OABC* as shown in Fig. 9.10

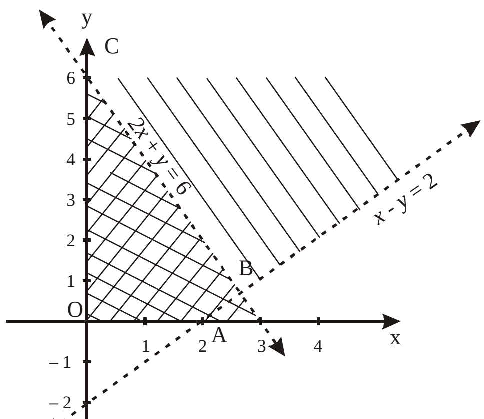


Fig. 9.10

CHECK YOUR PROGRESS 9.6

Solve each of the follwing systems of linear inequations in two variables graphically :

1. $x \geq 3,\ y \geq 1.$
2. $y \geq 2x, y \leq 2.$
3. $2x + y - 3 \geq 0, x - 2y + 1 \leq 0.$
4. $3x + 4y \leq 12,\ 4x+3y \leq 12,\ x \geq 0,\ y \geq 0$
5. $2x + 3y \geq 3,\ 3x + 4y \leq 18,\ 7x - 4y + 14 \geq 0,\ x - 6y \leq 3,\ x \geq 0, y \geq 0$
6. $x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0$
7. $x + y \geq 1;\ 2x + 3y \leq 6,\ x \geq 0,\ y \geq 0.$
8. $x + 3y \geq 10;\ x + 2y \leq 3,\ x - 2y \leq 2,\ x \geq 0\ ;\ y \geq 0$

LET US SUM UP

- Roots of the quadratic equation $ax^2 + bx + c = 0$ are complex and conjugate of each other, when $D < 0$. and $a, b, c \in R$.
- If α, β be the roots of the quadratic equation

  $ax^2 + bx + c = 0$ then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$
- If α and β are the roots of a quadratic equation. then the equation is:

  $x^2 - (\alpha + \beta)x + \alpha\beta = 0$
- The maximum number of roots of an equation is equal to the degree of the equation.
  A statement involving a sign of inequality like, $<, >, \leq, \geq$, is called an inequation.
- The equation $ax + by + c = 0$ is a straight line which divides the xy-plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$
- By the term, solution of a system of simultaneous linear inequalities we mean, finding all values of the ordered pairs (x,y) for which each linear inequalities of the system are satisfied.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=EoCeL4SPIcA
http://www.youtube.com/watch?v=FnrqBgot3jM
http://www.youtube.com/watch?v=-aTy1ED1m5I
http://www.youtube.com/watch?v=YBYu5aZPLeg
http://www.youtube.com/watch?v=2oGsLdAWxlk

TERMINAL EXERCISE

1. Show that the roots of the equation
$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are imaginary, when $a \neq b$

2. Show that the roots of the equation
$bx^2 + (b - c)x = c + a - b$ are always real if those of
$ax^2 + b(2x + 1) = 0$ are imaginary.

3. If α, β be the roots of the equation $2x^2 - 6x + 5 = 0$, find the equation whose roots are:

(i) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ (ii) $\alpha + \dfrac{1}{\beta}, \beta + \dfrac{1}{\alpha}$ (iii) $\alpha^2 + \beta^2, \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solve the following inequalities graphically.

4. $x \geq -2$ 5. $y \leq 2.$ 6. $x < 3$ 7. $y \geq -3$

8. $5 - 3y \geq -4$ 9. $2x - 5 \leq 3.$ 10. $3x - 2y \leq 12$ 11. $\dfrac{x}{3} + \dfrac{y}{5} \geq 1.$

12. $2x - 3y \geq 0$ 13. $x + 2y \leq 0.$

Solve each of the following systems of linear inequalities in two variables graphically.

14. $-1 \leq x \leq 3,\ 1 \leq y \leq 4.$

15. $2x + 3y \leq 6,\ 3x + 2y \leq 6.$

16. $6x + 5y \leq 150,$
$x + 4y \leq 80$
$x \leq 15,\ x \geq 0,\ y \geq 0.$

17. $3x + 2y \leq 24,\ x + 2y \leq 16$
$x + y \leq 10,\ x \geq 0,\ y \geq 0$

18. $x + y \geq 3,\ 7x + 6y \leq 42$
$x \leq 5, \quad y \leq 4$
$x \geq 0, \quad y \geq 0$

Solve that inequalities:

19. $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$

20. $37 - (3x+5) \geq 9x - 8(x-3)$

21. $\dfrac{(2x-1)}{3} \geq \dfrac{(3x-2)}{4} - \dfrac{(2-x)}{5}$

22. $5x + 1 > -24, 5x - 1 < 24$

23. $3x - 7 > 2(x-6), 6 - x > 11 - 2x$

24. $5(2x-7) - 3(2x+3) \leq 0, 2x + 19 \leq 6x + 47.$

25. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be move than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

26. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

ANSWERS

CHECK YOUR PROGRESS 9.1

1. (i) $-2\sqrt{3}, \dfrac{-4}{\sqrt{3}}$ (ii) $a-\sqrt{b},\ a+\sqrt{b}$ (iii) $-\dfrac{ab}{c}, \dfrac{c}{ab}$ (iv) $3\sqrt{2}, \sqrt{2}$

CHECK YOUR PROGRESS 9.2

1. (i) $\dfrac{3\pm\sqrt{15}\,i}{4}$ (ii) $\dfrac{1\pm i}{\sqrt{2}}$ (iii) $\dfrac{\sqrt{5}\pm\sqrt{43}\,i}{8}$ (iv) $\dfrac{-\sqrt{2}\pm\sqrt{58}\,i}{6}$

2. $-1, \dfrac{1}{2}$

CHECK YOUR PROGRESS 9.3

1. (i) $\dfrac{b^2-2ac}{c^2}$ (ii) $\dfrac{(b^2-2ac)^2-2a^2c^2}{c^4}$

2. (i) $25x^2-6x+9=0$ (ii) $625x^2-90x+81=0$

4. $q^2-5p^2=0$

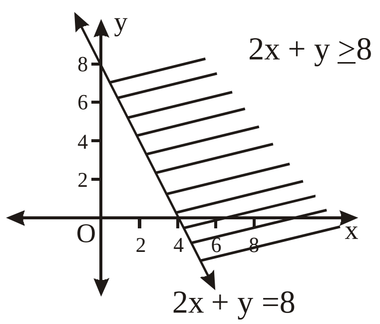
CHECK YOUR PROGRESS 9.4

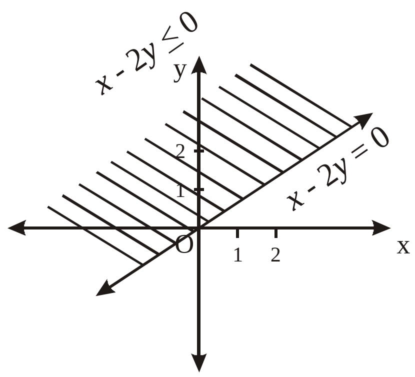
1. $\dfrac{-1\pm\sqrt{7}i}{2}$ 2. $\dfrac{\sqrt{2}\pm\sqrt{34}i}{2}$

3. $\dfrac{-1\pm\sqrt{7}i}{2\sqrt{2}}$ 4. $\dfrac{-1\pm\sqrt{19}i}{2}$

5. $\dfrac{-3\pm\sqrt{11}i}{2}$

CHECK YOUR PROGRESS 9.5

1. 

2. 

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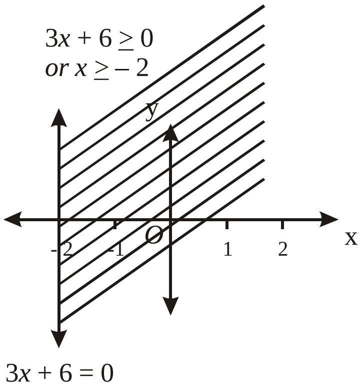
3x + 6 ≥ 0
or x ≥ – 2
y
O
x
-2
-1
1
2
3x + 6 = 0

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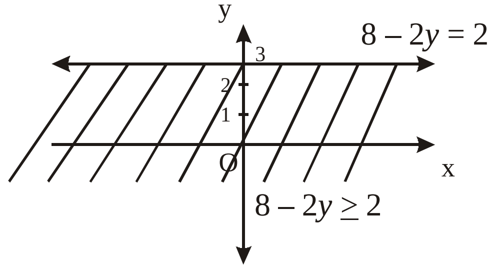
y
8 – 2y = 2
3
2
1
O
x
8 – 2y ≥ 2

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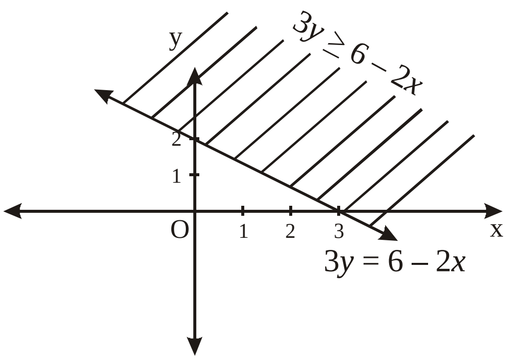
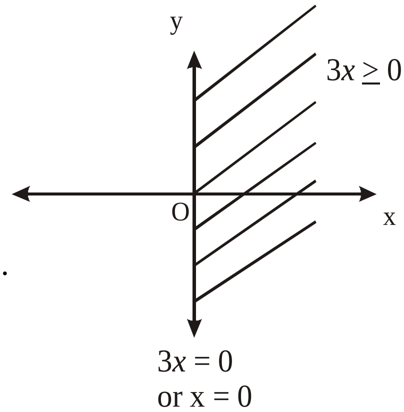
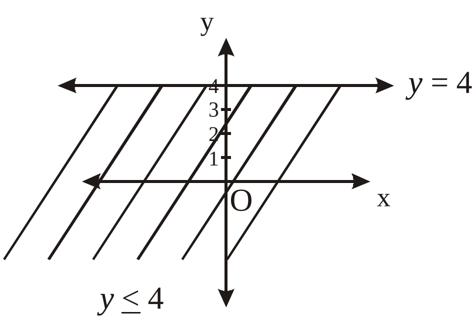

y
3y ≥ 6 – 2x
2
1
O
1
2
3
x
3y = 6 – 2x

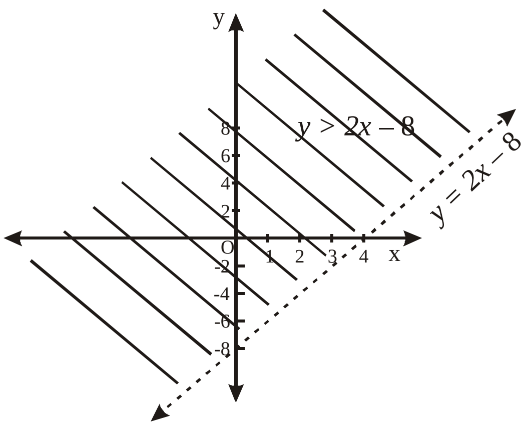
6.


y
3x ≥ 0
O
x
3x = 0
or x = 0

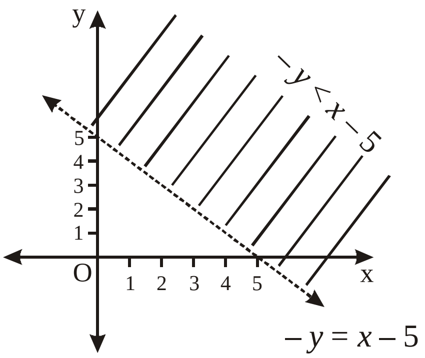
7.


y
y = 4
4
3
2
1
O
x
y ≤ 4

8.


y
y > 2x – 8
y = 2x – 8
8
6
4
2
O
1
2
3
4
x
-2
-4
-6
-8

9.



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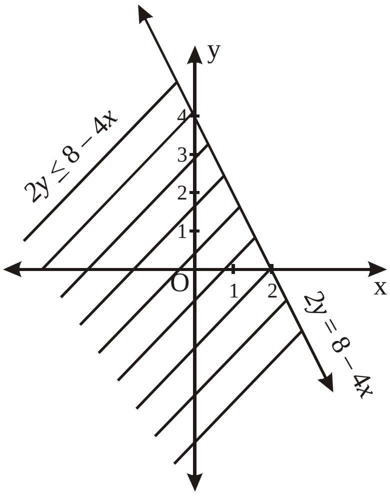

CHECK YOUR PROGRESS 9.6

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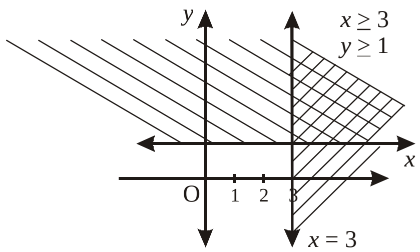

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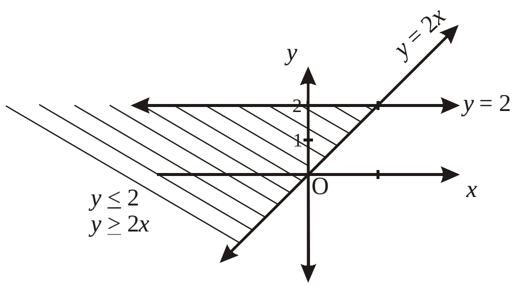

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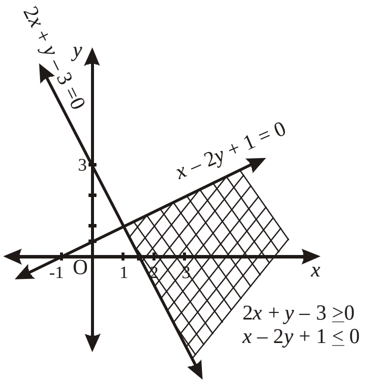
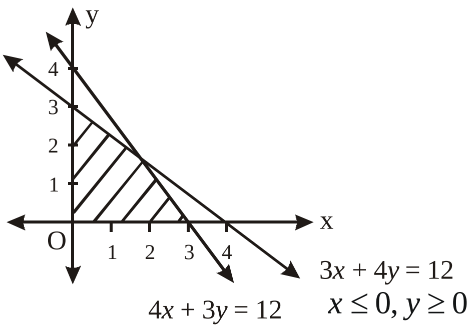


(4)



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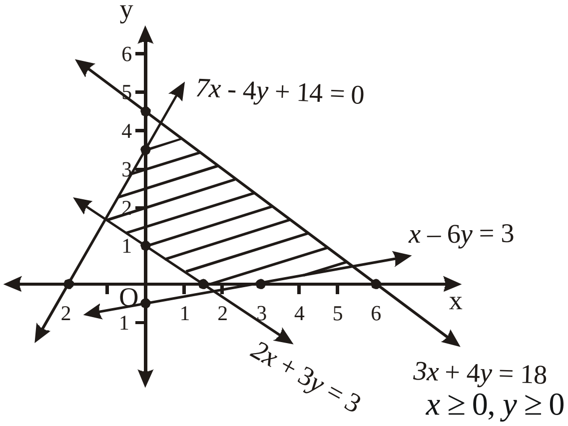

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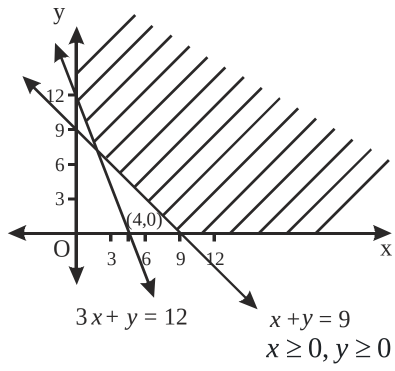

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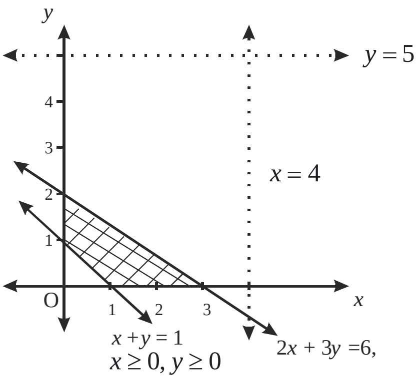

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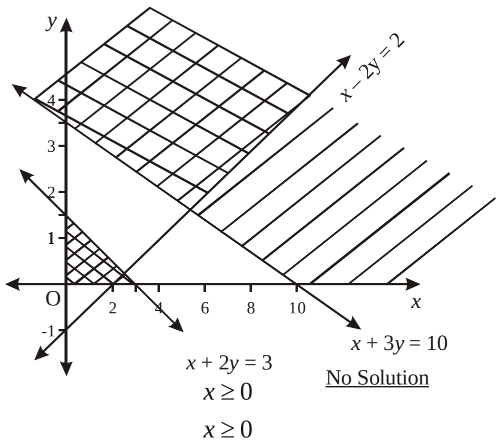
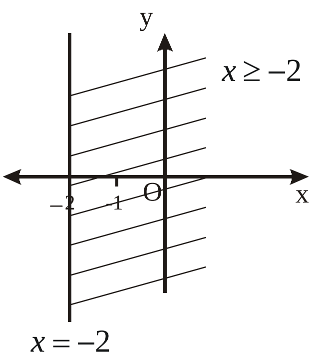


TERMINAL EXERCISE

3. (i) $5x^2 - 8x + 5 = 0$ (ii) $10x^2 - 42x + 49 = 0$ (iii) $25x^2 - 116x + 64 = 0$

4.



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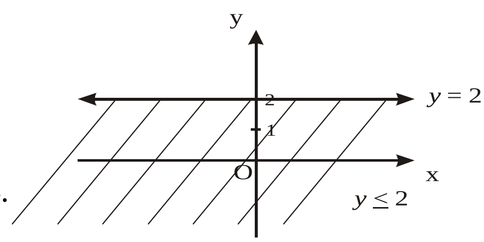

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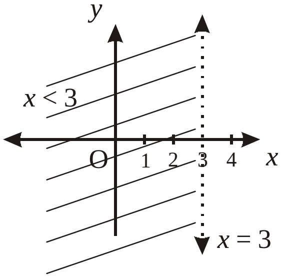

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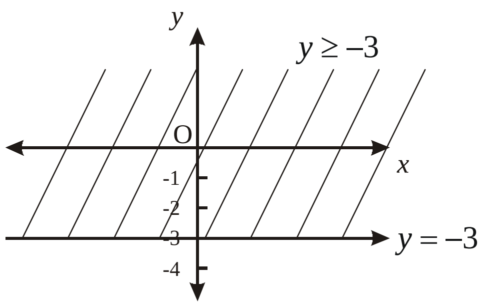

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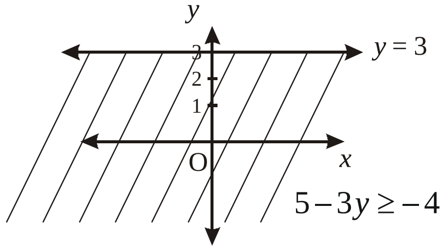

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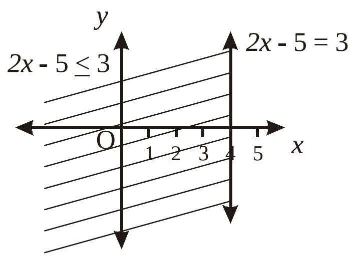

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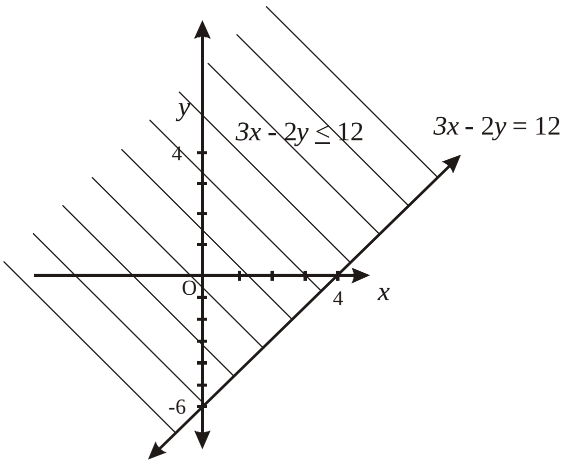

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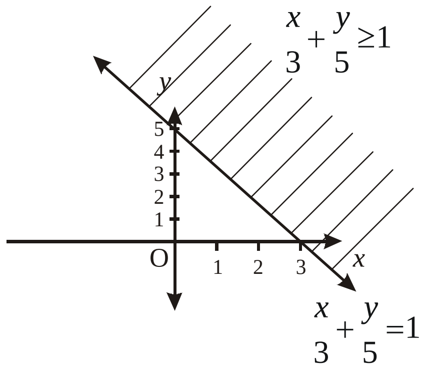

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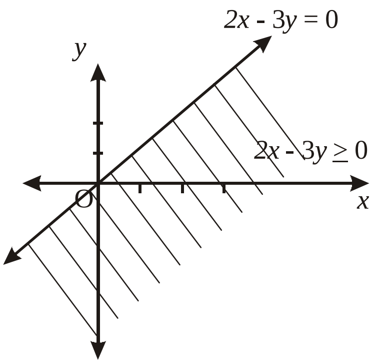

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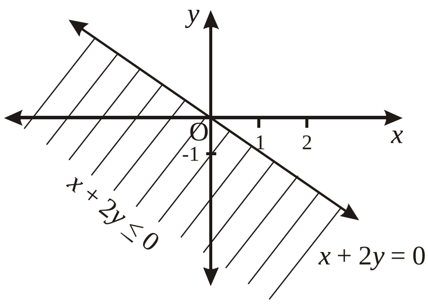

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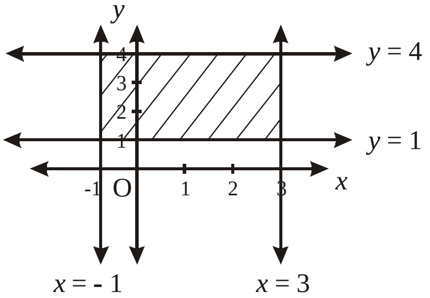

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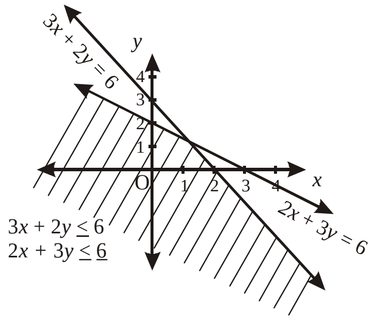

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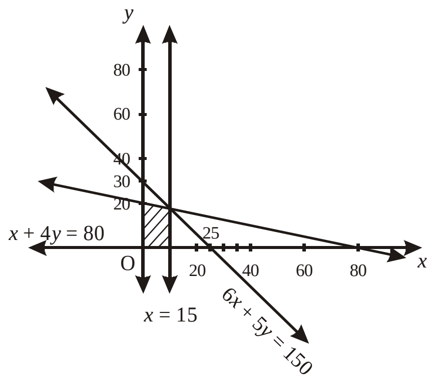

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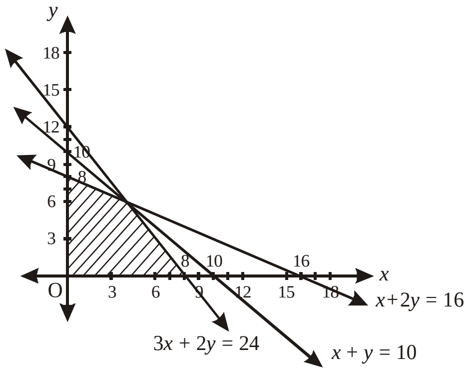

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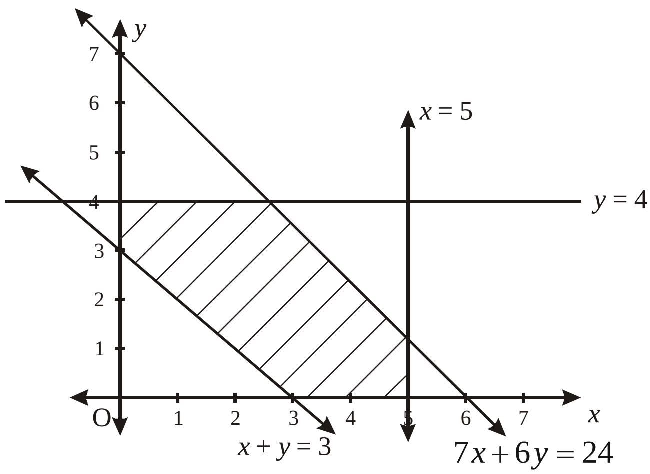

19. $(-\infty, 2]$ 20. $(-\infty, 2]$

21. $(-\infty, 2]$ 22. $(-5, 5)$ 23. $(5, \infty)$ 24. $[-7, 11]$

25. More then 320 litre but less then 1280 litres

26. More then 562.5 litres but less then 900 litres

10

311en10

PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friend just has a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of **deductive** logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations". In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to.

In the present chapter, first of all we shall introduce you with a statement and then we shall introduce the concept of principale of Mathematical induction, which we shall be using in proving some statements.

OBJECTIVES

After studying this lesson, you will be able to:

- To check whether the given sentence is a statement or not.
- state the Principle of Mathematical Induction;
- verify the truth or otherwise of the statement P(n) for n = 1;
- verify P(k+1) is true, assuming that P(k) is true;
- use principle of mathematical induction to establish the truth or otherwise of mathematical statements;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.

10.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true** or **false** are called **statement** or **propositions**. For instance,

"I am 20 years old" and "If $x = 3$, then $x^2 = 9$" are statements, but 'When will you leave ?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. For example, '$x - 5 = 7$' is not a statement, because we don't know what x, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number x' are statements, the first one true and the second one false.

Example 10.1 Which of the following sentences is a statement ?

(i) India has never had a woman President. , (ii) 5 is an even number.

(iii) $x^n > 1$, (iv) $(a + b)^2 = a^2 + 2ab + b^2$

Solution : (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x and n can take.

Now look at (iv). At first glance , you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identity. Therefore, in this case, even though we have not specified the range of values for a and b, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement :

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement P (n) [P stands for proposition].

Then P (1) would be $1 = \frac{1(1+1)}{2}$

Similarly, P (2) would be the statement , $1 + 2 = \frac{2(2+1)}{2}$ and so on.

Let us look at some examples to help you get used to this notation.

Example 10.2 If P (n) denotes $2^n > n-1$, write P (1), P (k) and P (k+1), where $k \in N$.

Solution : Replacing n by 1, k and $k + 1$, respectively in P (n), we get

$P(1): 2^1 > 2 - 1$, i.e., $2 > 1$, $P(k): 2^k > k - 1$

$P(k+1): 2^{k+1} > (k+1) - 1$, i.e., $2^{k+1} > k$

Example 10.3 If P (n) is the statement , '$1 + 4 + 7 + (3n - 2) = \dfrac{n(3n-1)}{2}$

write $P(1)$, $P(k)$ and $P(k+1)$.

Solution : To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till $3 \times 1 - 2$, i.e., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS)of $P(1) = \dfrac{1 \times (3 \times 1 - 1)}{2} = 1$, Therefore, $P(1)$ is $1 = 1$.

Replacing n by 2, we get

$P(2): 1 + 4 = \dfrac{2 \times (3 \times 2 - 1)}{2}$, i.e., $5 = 5$.

Replacing n by k and $k+1$, respectively, we get

$$P(k): 1 + 4 + 7 + \ldots. + (3k - 2) = \frac{k(3k-1)}{2}$$

$$P(k+1): 1 + 4 + 7 + \ldots. + (3k - 2) + [3(k+1) - 2] = \frac{(k+1)[3(k+1)-1]}{2}$$

$$\text{i.e. , } 1 + 4 + 7 + \ldots. + (3k + 1) = \frac{(k+1)[(3k+2)}{2}$$

CHECK YOUR PROGRESS 10.1

1. Determine which of the following are statements :

   (a) $1 + 2 + 4 \quad \ldots\ldots.. + 2^n > 20$ (b) $1 + 2 + 3 + \ldots\ldots. + 10 = 99$

   (c) Chennai is much nicer than Mumbai. (d) Where is Timbuktu ?

   (e) $\dfrac{1}{1 \times 2} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for $n = 5$ (f) $\operatorname{cosec}\theta < 1$

2. Given that $P(n)$: 6 is a factor of $n^3 + 5n$, write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$ where k is a natural number.

3. Write $P(1)$, $P(k)$ and $P(k+1)$, if $P(n)$ is:

   (a) $2^n \geq n + 1$ (b) $(1 + x)^n \geq 1 + nx$

   (c) $n(n+1)(n+2)$ is divisible by 6. (d) $(x^n - y^n)$ is divisible by $(x - y)$.

(e) $(ab)^n = a^n b^n$ (f) is a natural number.

4. Write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$, if $P(n)$ is :

(a) $\frac{1}{1\times 2}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, (b) $1 + 3 + 5 + \ldots\ldots.. + (2n-1) = n^2$

(c) $(1\times 2)+(2\times 3)+\ldots.+n(n+1) < n(n+1)^2$

(d) $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\ldots\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Now ,when you are given a statement like the ones given in Examples 10.2 and 10.3, how would you check whether it is true or false ? One effective method is mathematical induction, which we shall now discuss.

10.2 The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k+1)$ is true; then $P(n)$ must be true for every natural number n.

Note *that condition (ii) above* ***does not*** *say that P(k)* ***is*** *true. It says that* ***whenever*** *P(k) is true, then P(k + 1) is true'.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1 + 1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2 + 1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. It is now clear that if we continue like this, we shall get that $P(11)$ is true.

It is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Example 10.4 Prove that, $1+2+3+\cdots+n=\frac{n}{2}(n+1)$, where n is a natural number.

Solution: We have , $P(n) : 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$

Therefore, $P(1)$ is '$1 = \frac{1}{2}(1+1)$', which is true,. Therefore, $P(1)$ is true.

Let us now see, is $P(k + 1)$ true whenever $P(k)$ is true.

Let us, therefore, assume that $P(k)$ is true, i.e., $1 + 2 + 3 \ldots + k = \frac{k}{2}(k+1)$ ....(i)

Now, $P(k + 1)$ is $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $P(k + 1) = (1 + 2 + 3 \ldots + k) + (k + 1) = \frac{k}{2}(k+1) + (k + 1)$ ....[From (i)]

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \text{RHS of } P(k+1)$$

So, $P(k + 1)$ is true, if we assume that $P(k)$ is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps – the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

Example 10.5 For every natural number n, prove that $\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x+y)$, where $x, y \in N$.

Solution: Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement '$\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x+y)$',

Then $P(1)$ is '$\left(x^{2-1} + y^{2-1}\right)$ is divisible by $(x+y)$', i.e., '$(x+y)$ is divisible by $(x+y)$', which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number k, i.e., $(x^{2k-1} + y^{2k-1})$ is divisible by $(x + y)$.

This means that for some natural number t, $x^{2k-1} + y^{2k-1} = (x+y)t$

Then, $x^{2k-1} = (x+y)t - y^{2k-1}$

We wish to prove that $P(k+1)$ is true, i.e., '$[x^{2(k+1)-1} + y^{2(k+1)-1}]$ is divisible by $(x+y)$' is true.

Now,

$$x^{2(k+1)-1} + y^{2(k+1)-1} = x^{2k+1} + y^{2k+1}$$

$$= x^{2k-1+2} + y^{2k+1}$$

$$= x^{2} \cdot x^{2k-1} + y^{2k+1}$$

$$= x^{2}[(x+y)t - y^{2k-1}] + y$$
$$= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2k+1}$$
$$= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2}y^{2k-1}$$
$$= x^{2}(x+y)t - y^{2k-1}(x^{2} - y^{2})$$
$$= (x+y)[x^{2}t - (x-y)\,y^{2k-1}]$$

which is divisible by $(x + y)$.
Thus, $P\,(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 10.6 Prove that $2^n > n$ for every natural number n.

Solution: We have $P(n) : 2^n > n$.

Therefore, $P(1) : 2^1 > 1$, i.e., $2 > 1$, which is true.

We assume $P(k)$ to be true, that is,

$2^k > k$ ... (i)

We wish to prove that $P(k + 1)$ is true, i.e. $2^{k+1} > k + 1$.

Now, multiplying both sides of (i) by 2, we get, $2^{k+1} > 2k$

$\Rightarrow 2^{k+1} > k + 1$, since $k > 1$. Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say a (as in Example 10.7 below). In such a situation, we replace $P(1)$ by $P(a + 1)$ in the statement of the principle.

Example 10.7 Prove that

$n^2 > 2(n + 1)$ for all $n \geq 3$, where n is a natural number.

Solution: For $n \geq 3$, let us call the given statement, $P(n) : n^2 > 2\,(n + 1)$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$.
We, therefore, see whether $P(3)$ is true.
$P(3) : 3^2 > 2 \times 4$, i.e. $9 > 8$. So, $P\,(3)$ is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is , $k^2 > 2(k + 1)$ .... (i)

We wish to prove that $P\,(k + 1)$ is true.

$P\,(k + 1) : (k + 1)^2 > 2(k + 2)$

LHS of $P\,(k + 1) = (k + 1)^2 = k^2 + 2k + 1$

$> 2\,(k+1) + 2k + 1$ ... [By (i)]

$> 3 + 2k + 1$, since $2\,(k+1) > 3. = 2\,(k+2)$,

Thus, $(k+1)^2 > 2(k+2)$. Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Example 10.8 Using principle of mathematical induction, prove that

$$\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right) \text{ is a natural number for all natural numbers } n.$$

Solution : Let $P(n) : \left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ be a natural number.

$$\therefore P(1) : \left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right) \text{ is a natural number.}$$

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1$, which is a natural number $\therefore$ $P(1)$ is true.

Let $P(k) : \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ... (i)

$$\text{Now } \frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$$

$$=\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+1 \quad \text{...(ii)}$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

$\therefore$ (ii) being sum of natural numbers is a natural number.

$\therefore$ $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(n)$ is true for all natural numbers n.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

CHECK YOUR PROGRESS 10.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number n:

   (a) $1^2+2^2+3^2+........+n^2=\frac{n}{6}(n+1)(2n+1)$

   (b) $1^3+2^3+3^3+........+n^3=(1+2+.....+n)^2$

   (c) $1+3+5+........+(2n-1)=n^2$

   (d) $1+4+7+........+(3n-2)=\frac{n}{2}(3n-1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number n:

   (a) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$

   (b) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

   (c) $(1\times 2)+(2\times 3)+....+n(n+1)=\frac{n(n+1)(n+2)}{3}$

3. For every natural number n, prove that

   (a) n^3+5n is divisible by 6.　　(b) (x^n-1) is divisible by $(x-1)$.

   (c) (n^3+2n) is divisible by 3.　　(d) 4 divides $(n^4+2n^3+n^2)$.

4. Prove the following inequalities for any natural number n:

   (a) $3^n \geq 2n+1$　　(b) $4^{2n} > 15n$　　(c) $1+2+....+n<\frac{1}{8}(2n+1)^2$

5. Prove the following statements using induction:

   (a) $2^n > n^2$ for $n \geq 5$, where n *is any natural number.*

   (b) $\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{2n}>\frac{13}{24}$ for any natural number n greater than 1.

6. Prove that $n(n^2 - 1)$ is divisible by 3 for every natural number n greater than 1.

To prove that a statement $P(n)$ is true for every $n \in N$, both the basic as well as the induction steps must hold.

If even one of these conditions does not hold, then the proof is invalid. For instance, if $P(n)$ is' $(a+b)^n \le a^n + b^n$ ' for all reals a and b, then $P(1)$ is certainly true. But, $P(k)$ being true does not imply the truth of $P(k + 1)$. So, the statement is not true for every natural number n. (For instance, $(2+3)^2 \le 2^2 + 3^2$).

As another example, take $P(n)$ to be $n > \frac{n}{2} + 20$.

In this case, $P(1)$ is not true. But the induction step is true. Since $P(k)$ being true.

$$\Rightarrow k > \frac{k}{2} + 20 \quad \Rightarrow k+1 > \frac{k}{2} + 20 + 1 > \frac{k}{2} + 20 + \frac{1}{2} = \frac{k+1}{2} + 20 \quad \Rightarrow P(k+1) \text{ is true.}$$

LET US SUM UP

- Sentences which are either true or false are called statement or propasitions.
- The word induction means, formulating a general principle (or rule) based on several particular instances.
- The statement of the principle of mathematical indction.

  P (n), a statement involving a natural number n, is true for all $n \ge 1$, where n is a fixed natural number, if

  (i) P (1) is true, and

  (ii) whenever P (k) is true, then P (K+1) is true for $k \in n$

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/13pure/01proof/01proof/05induction/index.shtml
www.mathguru.com/result/**principle-of-mathematical-induction**.aspx
http://en.wikipedia.org/wiki/**Mathematical_induction**

TERMINAL EXERCISE

1. Verify each of the following statements, using the principle of mathematical induction :

   (1) The number of subsets of a set with n elements is 2^n.

   (2) $(a + b)^n > a^n + b^n \ \forall n \ge 2$, where a and b are positive real numbers.

(3) $a + ar + ar^2 + \ldots. + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$,where r > 1 and a is a real number.

(4) $(x^{2n} - 1)$ is divisible by $(x+1)\ \forall x \in N$.

(5) $(10^{2n-1} + 1)$ is a multiple of 11, where $n \in N$

(6) $(4.10^{2n} + 9.10^{2n-1} + 5)$ is a multiple of 99. where $n \in N$

(7) $(1 + x)^n > 1 + nx$, where $x > 0$ and $n \in N$

(8) $1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots + n.2^n = (n-1).2^{n+1}$, where $n \in N$

(9) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, $n \in N$

(10) $\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \cdots + \dfrac{1}{(2n+1)(2n+3)} = \dfrac{n}{3(2n+3)}$ where $n \in N$

ANSWERS

CHECK YOUR PROGRESS 10.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of n, and therefore we are not in a position to decide, if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

   Note that (*f*)is universally false.

2. P(1) : 6 is a factor of $1^3 + 5.1$, P(2) : 6 is a factor of $2^3 + 5.2$

   P(k) : 6 is a factor of $k^3 + 5k$, P(k+1) : 6 is a factor of $(k + 1)^3 + 5(k + 1)$

3. (a) $P(1) : 2 \geq 2$, $P(k) : 2^k \geq k + 1$, $P(k + 1) : 2^{k+1} \geq k + 2$

   (b) $P(1) : 1 + x \geq 1 + x$, $P(k) : (1 + x)^k \geq 1 + kx$

   $P(k + 1)$: $(1 + x)^{k+1} \geq 1 + (k + 1)x$

   (c) $P(1)$: 6 is divisible by 6. $P(k)$: $k(k + 1)(k + 2)$ is divisible by 6.

   $P(k + 1)$: $(k + 1)\ (k + 2)\ (k + 3)$ is divisible by 6

   (d) $P(1)$: $(x - y)$ is divisible by $(x - y)$. $P(k)$: $(x^k - y^k)$ is divisible by $(x - y)$

   $P(k + 1)$: $(x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) $P(1)$: $ab = ab$, $P(k)$: $(ab)^k = a^k b^k$

   P(k+1): $(ab)^{k+1} = a^{k+1}.\ b^{k+1}$

   (f) $P(1) : \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$ is a natural number. $P(k)$: $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is a natural number.

   $P(k + 1)$: $\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$ is a natural number.

4. (a) $P(1)$: $\frac{1}{1 \times 2} = \frac{1}{2}$, $P(2) : \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{2}{3}$

$$P(k): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

$$P(k+1): \frac{1}{1\times 2}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

(b) $P(1):\ 1=1^2,\ \ P(2):\ 1+3=2^2$

$$P(k):\ 1+3+5+\ldots+(2k-1)=k^2$$

$$P(k+1):\ 1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$$

(c) $P(1): 1\times 2<1(2)^2,\ P(2): (1\times 2)+(2\times 3)<2(3)^2$

$$P(k): (1\times 2)+(2\times 3)+\ldots+k(k+1)<k(k+1)^2.$$

$$P(x+1): (1\times 2)+(2\times 3)+\ldots+(k+1)(k+2)<(k+1)(k+2)^2$$

(d) $P(1):\ \frac{1}{1\times 3}=\frac{1}{3},\ \ P(2):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}=\frac{2}{5}$

$$P(k):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

$$P(k+1): \frac{1}{1\times 3}+\frac{1}{3\times 5}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

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PERMUTATIONS AND COMBINATIONS

One day, I wanted to travel from Bangalore to Allahabad by train. There is no direct train from Bangalore to Allahabad, but there are trains from Bangalore to Itarsi and from Itarsi to Allahabad. From the railway timetable I found that there are two trains from Bangalore to Itarsi and three trains from Itarsi to Allahabad. Now, in how many ways can I travel from Bangalore to Allahabad?

There are **counting problems** which come under the branch of Mathematics called **combinatorics.**

Suppose you have five jars of spices that you want to arrange on a shelf in your kitchen. You would like to arrange the jars, say three of them, that you will be using often in a more accessible position and the remaining two jars in a less accessible position. In this situation the order of jars is important. In how many ways can you do it?

In another situation suppose you are painting your house. If a particular shade or colour is not available, you may be able to create it by mixing different colours and shades. While creating new colours this way, the order of mixing is not important. It is the combination or choice of colours that determine the new colours; but not the order of mixing.

To give another similar example, when you go for a journey, you may not take all your dresses with you. You may have 4 sets of shirts and trousers, but you may take only 2 sets. In such a case you are choosing 2 out of 4 sets and the order of choosing the sets doesn't matter. In these examples, we need to find out the number of choices in which it can be done.

In this lesson we shall consider simple counting methods and use them in solving such simple counting problems.

OBJECTIVES

After studying this lesson, you will be able to :

- find out the number of ways in which a given number of objects can be arranged;
- state the Fundamental Principle of Counting;
- define n! and evaluate it for defferent values of n;
- state that permutation is an arrangement and write the meaning of ${}^nP_r$;
- state that ${}^nP_r = \dfrac{n!}{(n-r)!}$ and apply this to solve problems;

- show that (i) $(n+1)\,{}^nP_n = {}^{n+1}P_n$ (ii) ${}^nP_{r+1} = (n-r)\,{}^nP_r$;
- state that a combination is a selection and write the meaning of ${}^nC_r$;
- distinguish between permutations and combinations;
- derive ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and apply the result to solve problems;
- derive the relation ${}^nP_r = r!\,{}^nC_r$;
- verify that ${}^nC_r = {}^nC_{n-r}$ and give its interpretation; and
- derive ${}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$ and apply the result to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Number Systems
- Four Fundamental Operations

11.1 FUNDAMENTAL PRINCIPLE OF COUNTING

Let us now solve the problem mentioned in the introduction. We will write t_1, t_2 to denote trains from Bangalore to Itarsi and T_1, T_2, T_3, for the trains from Itarsi to Allahabad. Suppose I take t_1 to travel from Bangalore to Itarsi. Then from Itarsi I can take T_1 or T_2 or T_3. So the possibilities are t_1T_1, t_1T_2 and t_1T_3 where t_1T_1 denotes travel from Bangalore to Itarsi by t_1 and travel from Itarsi to Allahabad by T_1. Similarly, if I take t_2 to travel from Bangalore to Itarsi, then the possibilities are t_2T_1, t_2T_2 and t_2T_3. Thus, in all there are $6\,(2\times 3)$ possible ways of travelling from Bangalore to Allahabad.

Here we had a small number of trains and thus could list all possibilities. Had there been 10 trains from Bangalore to Itarsi and 15 trains from Itarsi to Allahabad, the task would have been very tedious. Here the **Fundamental Principle of Counting** or simply the **Counting Principle** comes in use :

If any event can occur in *m* ways and after it happens in any one of these ways, a second event can occur in *n* ways, then both the events together can occur in $m \times n$ ways.

Example 11.1 How many multiples of 5 are there from 10 to 95 ?

Solution : As you know, multiples of 5 are integers having 0 or 5 in the digit to the extreme right (i.e. the unit's place).

The first digit from the right can be chosen in 2 ways.

The second digit can be any one of 1,2,3,4,5,6,7,8,9.

i.e. There are 9 choices for the second digit.

Thus, there are $2 \times 9 = 18$ multiples of 5 from 10 to 95.

Example 11.2 In a city, the bus route numbers consist of a natural number less than 100, followed by one of the letters *A,B,C,D,E* and *F*. How many different bus routes are possible?

Solution : The number can be any one of the natural numbers from 1 to 99.
There are 99 choices for the number.

The letter can be chosen in 6 ways.

$\therefore$ Number of possible bus routes are $99 \times 6 = 594$.

CHECK YOUR PROGRESS 11.1

1. (a) How many 3 digit numbers are multiples of 5?

   (b) A coin is tossed thrice. How many possible outcomes are there?

   (c) If you have 3 shirts and 4 trousers and any shirt can be worn with any of trousers, in how many ways can you wear your shirts and trousers?

2. (a) In how many ways can two vacancies be filled from among 4 men and 12 women if one vacancy is filled by a man and the other by a woman?

   (b) A tourist wants to go to another country by ship and return by air. She has a choice of 5 different ships to go by and 4 airlines to return by. In how many ways can she perform the journey?

So far, we have applied the counting principle for two events. But it can be extended to three or more, as you can see from the following examples :

Example 11.3 There are 3 questions in a question paper. If the questions have 4,3 and 2 solutions respectively, find the total number of solutions.

Solution : Here question 1 has 4 solutions, question 2 has 3 solutions and question 3 has 2 solutions.

$\therefore$ By the multiplication (counting) rule,

total number of solutions = $4 \times 3 \times 2$ = 24

Example 11.4 Consider the word ROTOR. Whichever way you read it, from left to right or from right to left, you get the same word. Such a word is known as *palindrome*. Find the maximum possible number of 5-letter palindromes.

Solution : The first letter from the right can be chosen in 26 ways because there are 26 alphabets. Having chosen this, the second letter can be chosen in 26 ways

$\therefore$ The first two letters can be chosen in $26 \times 26 = 676$ ways

Having chosen the first two letters, the third letter can be chosen in 26 ways.
$\therefore$ All the three letters can be chosen in $676 \times 26 = 17576$ ways.

It implies that the maximum possible number of five letter palindromes is 17576 because the fourth letter is the same as the second letter and the fifth letter is the same as the first letter.

Note : *In Example 11.4 we found the maximum possible number of five letter palindromes. There cannot be more than 17576. But this does not mean that there are 17576 palindromes. Because some of the choices like CCCCC may not be meaningful words in the English language.*

Example 11.5 How many 3-digit numbers can be formed with the digits 1,4,7,8 and 9 if the digits are not repeated.

Solution : Three digit number will have unit's, ten's and hundred's place.

Out of 5 given digits any one can take the unit's place.

This can be done in 5 ways. ... (*i*)

After filli ng the unit's place, any of the four remaining digits can take the ten's place.

This can be done in 4 ways. ... (*ii*)

After filling in ten's place, hundred's place can be filled from any of the three remaining digits.

This can be done in 3 ways. ... (*iii*)

$\therefore$ By counting principle, the number of 3 digit numbers = $5\times4\times3$ = 60

Let us now state the **General Counting Principle**

If there are *n* events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occured, the third event can occur in m_3 ways after the second event has ocurred, and so on, then all the *n* events can occur in $m_1\times m_2\times\ldots\times m_{n-1}\times m_n$ ways.

Example 11.6 Suppose you can travel from a place *A* to a place *B* by 3 buses, from place *B* to place *C* by 4 buses, from place *C* to place *D* by 2 buses and from place *D* to place *E* by 3 buses. In how many ways can you travel from *A* to *E*?

Solution : The bus from *A* to *B* can be selected in 3 ways.

The bus from *B* to *C* can be selected in 4 ways.

The bus from *C* to *D* can be selected in 2 ways.

The bus from *D* to *E* can be selected in 3 ways.

So, by the General Counting Principle, one can travel from *A* to *E* in $3\times4\times2\times3$ ways = 72 ways.

CHECK YOUR PROGRESS 11.2

1. (a) What is the maximum number of 6-letter palindromes?

   (b) What is the number of 6-digit palindromic numbers which do not have 0 in the first digit?

2. (a) In a school there are 5 English teachers, 7 Hindi teachers and 3 French teachers. A three member committee is to be formed with one teacher representing each language. In how many ways can this be done?

   (b) In a college students union election, 4 students are contesting for the post of President. 5 students are contesting for the post of Vice-president and 3 students are contesting for the post of Secretary. Find the number of possible results.

3. (a) How many three digit numbers greater than 600 can be formed using the digits 1,2,5,6,8 without repeating the digits?

   (b) A person wants to make a time table for 4 periods. He has to fix one period each for English, Mathematics, Economics and Commerce. How many different time tables can he make?

11.2 PERMUTATIONS

Suppose you want to arrange your books on a shelf. If you have only one book, there is only one way of arranging it. Suppose you have two books, one of History and one of Geography.

You can arrange the Geography and History books in two ways. Geography book first and the History book next, *GH* or History book first and Geography book next; *HG*. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say *GH*, you can put Mathematics book in one of the following ways: *MGH*, *GMH* or *GHM*. Similarly, corresponding to *HG*, you have three other ways of arranging the books. So, by the Counting Principle, you can arrange Mathematics, Geography and History books in 3×2 ways = 6 ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books.

In general, if you want to find the number of permutations of *n* objects $n \geq 1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3 \times 2 \times 1$ permutations of 3 objects. It may be that, there are $n \times (n-1) \times (n-2) \times ... \times 2 \times 1$ permutations of *n* objects. In fact, it is so, as you will see when we prove the following result.

Theorem 11.1 The total number of permutations of n objects is $n\,(n-1)$ ....2.1.

Proof : We have to find the number of possible arrangements of n different objects.

The first place in an arrangement can be filled in n different ways. Once it has been done, the second place can be filled by any of the remaining (n–1) objects and so this can be done in (n–1) ways. Similarly, once the first two places have been filled, the third can be filled in (n–2) ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object.

Using the counting principle, the total number of arrangements of n different objects is $n(n-1)(n-2)$ .... 2.1. **.....(11.1)**

The product $n\,(n-1)$ ... 2.1 occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by ***n*!** (or by $\underline{|n}$ read as ***n* factorial**).

$$\boldsymbol{n! = n\,(n-1)\,...\,3.2.1}$$

Here is an example to help you familiarise yourself with this notation.

Example 11.7 Evaluate (a) 3! (b) 2! + 4! (c) $2! \times 3!$

Solution : (a) $3! = 3 \times 2 \times 1 = 6$

(b) $2! = 2 \times 1 = 2$, $4! = 4 \times 3 \times 2 \times 1 = 24$

Therefore, $2! + 4! = 2 + 24 = 26$

(c) $2! \times 3! = 2 \times 6 = 12$

Notice that n! satisfies the relation, $n! = n \times (n-1)\,!$ **... (11.2)**

This is because, $n\,(n-1)! = n\,[(n-1)\,.\,(n-2)...\,2.1]$

$$= n\,.\,(n-1)\,.\,(n-2)\,...2.1 = n!$$

Of course, the above relation is valid only for $n \geq 2$ because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define

$$0! = 1$$

then the relation 11.2 holds for $n = 1$ also.

Example 11.8 Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution : We have to arrange 6 books.

The number of permutations of n objects is $n! = n.\,(n-1)\,.\,(n-2)\,...\,2.1$

Here $n = 6$ and therefore, number of permutations is 6.5.4.3.2.1 = 720

CHECK YOUR PROGRESS 11.3

1. (a) Evaluate : (i) 6! (ii) 7! (iii) 7! + 3! (iv) 6! $\times$ 4! (v) $\frac{5!}{3!.2!}$

   (b) Which of the following statements are true?

   (i) $2! \times 3! = 6!$ (ii) $2! + 4! = 6!$

   (iii) 3! divides 4! (iv) $4! - 2! = 2!$

2. (a) 5 students are staying in a dormitory. In how many ways can you allot 5 beds to them?

   (b) In how many ways can the letters of the word 'TRIANGLE' be arranged?

   (c) How many four digit numbers can be formed with digits 1, 2, 3 and 4 and with distinct digits?

11.3 PERMUTATION OF *r* OBJECTS OUT OF *n* OBJECTS

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give any one of the five books to Asha and after that you can give any one of the remaining four books to Akhtar. After that, you can give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in $5 \times 4 \times 3$ *ie.*60 ways.

More generally, suppose you have to arrange *r* objects out of *n* objects. In how many ways can you do it? Let us view this in the following way. Suppose you have *n* objects and you have to arrange *r* of these in *r* boxes, one object in each box.

n ways	$n-1$ ways	...	$n-r+1$ ways

r boxes

Fig. 11.1

Suppose there is one box. $r = 1$. You can put any of the *n* objects in it and this can be done in *n* ways. Suppose there are two boxes. $r = 2$. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining $n - 1$ objects. So, by the counting principle, the two boxes can be filled in $n(n-1)$ ways. Similarly, 3 boxes can be filled in $n(n-1)(n-2)$ ways.

In general, we have the following theorem.

Theorem 11.2 The number of permutations of *r* objects out of *n* objects is

$$n(n-1)\cdots(n-r+1).$$

The number of permutations of *r* objects out of *n* objects is usually denoted by ${}^nP_r$.

Thus, $${}^nP_r = n(n-1)(n-2)...(n-r+1) \qquad \textbf{... (11.3)}$$

Proof : Suppose we have to arrange r objects out of n different objects. In fact it is equivalent to filling r places, each with one of the objects out of the given n objects.

The first place can be filled in n different ways. Once this has been done, the second place can be filled by any one of the remaining (n–1) objects, in (n–1) ways. Similarly, the third place can be filled in (n – 2) ways and so on. The last place, the rth place can be filled in [n–(r–1)] i.e. (n–r+1) different ways. You may easily see, as to why this is so.

Using the Counting Principle, we get the required number of arrangements of r out of n objects is $n(n-1)(n-2)..........(n-r+1)$

Example 11.9 Evaluate : (a) ${}^4P_2$ (b) ${}^6P_3$ (c) $\dfrac{{}^4P_3}{{}^3P_2}$ (d) ${}^6P_3 \times {}^5P_2$

Solution : (a) ${}^4P_2 = 4(4-1) = 4 \times 3 = 12.$

(b) ${}^6P_3 = 6(6-1)\ (6-2) = 6 \times 5 \times 4 = 120.$

(c) $$\frac{{}^4P_3}{{}^3P_2} = \frac{4(4-1)(4-2)}{3(3-1)} = \frac{4 \times 3 \times 2}{3 \times 2} = 4$$

(d) ${}^6P_3 \times {}^5P_2 = 6\ (6-1)\ (6-2) \times 5\ (5-1),\ = 6 \times 5 \times 4 \times 5 \times 4\ =\ 2400$

Example 11.10 If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution : We have to find number of permutations of 4 objects out of 6 objects.

This number is ${}^6P_4 = 6(6-1)(6-2)(6-3) = 6.5.4.3 = 360$

Therefore, cards can be sent in 360 ways.

Consider the formula for ${}^nP_r$, namely, ${}^nP_r = n\,(n-1)\ ...\ (n-r+1)$. This can be obtained by removing the terms $n-r, n-r-1, ..., 2, 1$ from the product for n!. The product of these terms is $(n-r)\ (n-r-1)\ ...2.1$, i.e., $(n-r)!$.

$$\text{Now, } \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)(n-r)...2.1}{(n-r)\ (n-r-1)...2.1}$$

$$= n(n-1)(n-2)...(n-r+1) = {}^nP_r$$

So, using the factorial notation, this formula can be written as follows : $${}^nP_r = \frac{n!}{(n-r)!} \quad .....\textbf{(11.4)}$$

Example 11.11 Find the value of ${}^nP_0$.

Solution : Here $r = 0$. Using relation 11.4, we get ${}^{n}P_{0} = \dfrac{n!}{n!} = 1$

Example 11.12 Show that $(n + 1)\,{}^{n}P_{r} = {}^{n+1}P_{r+1}$

Solution : $(n + 1)\,{}^{n}P_{r} = (n+1)\dfrac{n!}{(n-r)!} = \dfrac{(n+1)\,n!}{(n-r)!}$

$$= \frac{(n+1)!}{[(n+1)-(r+1)]!} \quad \text{[writing } n - r \text{ as } [(n + 1) - (r + 1)]$$

$$= {}^{n+1}P_{r+1} \qquad \text{(By definition)}$$

CHECK YOUR PROGRESS 11.4

1. (a) Evaluate : (i) ${}^{4}P_{2}$ (ii) ${}^{6}P_{3}$ (iii) $\dfrac{{}^{4}P_{3}}{{}^{3}P_{2}}$ (iv) ${}^{6}P_{3} \times {}^{5}P_{2}$ (v) ${}^{n}P_{n}$

   (b) Verify each of the following statements :

   (i) $6 \times {}^{5}P_{2} = {}^{6}P_{2}$ (ii) $4 \times {}^{7}P_{3} = {}^{7}P_{4}$

   (iii) ${}^{3}P_{2} \times {}^{4}P_{2} = {}^{12}P_{4}$ (iv) ${}^{3}P_{2} + {}^{4}P_{2} = {}^{7}P_{4}$

2. (a) (i) What is the maximum possible number of 3- letter words in English that do not contain any vowel?

   (ii) What is the maximum possible number of 3- letter words in English which do not have any vowel other than 'a'?

   (b) Suppose you have 2 cots and 5 bedspreads in your house. In how many ways can you put the bedspreads on your cots?

   (c) You want to send Diwali Greetings to 4 friends and you have 7 greeting cards with you. In how many ways can you do it?

3. Show that ${}^{n}P_{n-1} = {}^{n}P_{n}$.

4. Show that $(n-r)\,{}^{n}P_{r} = {}^{n}P_{r+1}$.

11.4 PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 11.13 Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution : Let the beds be numbered 1 to 7.

Case 1 : Suppose Anju is allotted bed number 1.

Then, Parvin cannot be allotted bed number 2.

So Parvin can be allotted a bed in 5 ways.

After alloting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways.

So, in this case the beds can be allotted in $5 \times 5! ways = 600$ ways.

Case 2 : Anju is allotted bed number 7.

Then, Parvin cannot be allotted bed number 6

As in Case 1, the beds can be allotted in 600 ways.

Case 3 : Anju is allotted one of the beds numbered 2,3,4,5 or 6.

Parvin cannot be allotted the beds on the right hand side and left hand side of Anju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin.

Therefore, Parvin can be allotted a bed in 4 ways in all these cases.

After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways.

Therefore, in each of these cases, the beds can be allotted in $4 \times 5! = 480$ ways.

$\therefore$ The beds can be allotted in

$(2 \times 600 + 5 \times 480)\ ways = (1200 + 2400)\ ways = 3600$ ways.

Example 11.14 In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution : They have to be arranged in the following way :

L	T	L	T	L	T	L	T	L

The 5 lions should be arranged in the 5 places marked 'L'. This can be done in 5! ways.

The 4 tigers should be in the 4 places marked 'T'.This can be done in 4! ways.

Therefore, the lions and the tigers can be arranged in $5! \times 4!\ ways = 2880$ ways.

Example 11.15 There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairytales, novels and plays.

Solution : There are 4 books on fairy tales and they have to be put together.

They can be arranged in 4! ways. Similarly, there are 5 novels.

They can be arranged in 5! ways. And there are 3 plays.

They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in $4! \times 5! \times 3!$ ways = 17280 ways.

Example 11.16 Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 11.15. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution : First, we consider the books on fairy tales, novels and plays as single objects.

These three objects can be arranged in 3!ways = 6 ways.

Let us fix one of these 6 arrangements.

This may give us a specific order, say, novels $\rightarrow$ fairy tales $\rightarrow$ plays.

Given this order, the books on the same subject can be arranged as follows.

The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 5 novels can be arranged in 5! = 120 ways.

The 3 plays can be arranged in 3! = 6 ways.

For a given order, the books can be arranged in $24 \times 120 \times 6 = 17280$ ways.

Therefore, for all the 6 possible orders the books can be arranged in $6 \times 17280 = 103680$ ways.

Example 11.17 In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution : Let 4 girls be one unit and now there are 6 units in all.

They can be arranged in 6! ways.

In each of these arrangements 4 girls can be arranged in 4! ways.

$\therefore$ Total number of arrangements in which girls are always together

$= 6! \times 4! = 720 \times 24 = 17280$

Example 11.18 How many arrangements of the letters of the word 'BENGALI' can be made if the vowels are always together.

Solution : There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants. Considering vowels *a*, *e*, *i* as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.

$\therefore$ Total number of words $= 5! \times 3! = 120 \times 6 = 720$

CHECK YOUR PROGRESS 11.5

1. Mr. Gupta with Ms. Gupta and their four children are travelling by train. Two lower berths, two middle berths and 2 upper berths have been allotted to them. Mr. Gupta has

undergone a knee surgery and needs a lower berth while Ms. Gupta wants to rest during the journey and needs an upper berth. In how many ways can the berths be shared by the family?

2. Consider the word UNBIASED. How many words can be formed with the letters of the word in which no two vowels are together?

3. There are 4 books on Mathematics, 5 books on English and 6 books on Science. In how many ways can you arrange them so that books on the same subject are together and they are arranged in the order Mathematics $\rightarrow$ English $\rightarrow$ Science.

4. There are 3 Physics books, 4 Chemistry books, 5 Botany books and 3 Zoology books. In how many ways can you arrange them so that the books on the same subject are together?

5. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can this be done?

6. Find the number of permutations of the letters of the word 'TENDULKAR', in each of the following cases :

   (i) beginning with T and ending with R. (ii) vowels are always together.

   (iii) vowels are never together.

11.5 COMBINATIONS

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S_1, S_2, S_3, S_4. Then you can choose two pairs in the following ways :

1. $\{S_1, S_2\}$	2. $\{S_1, S_3\}$	3. $\{S_1, S_4\}$
4. $\{S_2, S_3\}$	5. $\{S_2, S_4\}$	6. $\{S_3, S_4\}$

[Observe that $\{S_1, S_2\}$ is the same as $\{S_2, S_1\}$]. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know from section 11.3, that it can be done in ${}^4P_2 = 12$ ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below :

1. $\{S_1, S_2\} \rightarrow S_1$ on Monday and S_2 on Tuesday or S_2 on Monday and S_1 on Tuesday
2. $\{S_1, S_3\} \rightarrow S_1$ on Monday and S_3 on Tuesday or S_3 on Monday and S_1 on Tuesday
3. $\{S_1, S_4\} \rightarrow S_1$ on Monday and S_4 on Tuesday or S_4 on Monday and S_1 on Tuesday

4. $\{S_2, S_3\} \rightarrow S_2$ on Monday and S_3 on Tuesday or S_3 on Monday and S_2 on Tuesday
5. $\{S_2, S_4\} \rightarrow S_2$ on Monday and S_4 on Tuesday or S_4 on Monday and S_2 on Tuesday
6. $\{S_3, S_4\} \rightarrow S_3$ on Monday and S_4 on Tuesday or S_4 on Monday and S_3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.

This argument holds good in general as we can see from the following theorem.

Theorem 11.3 Let $n \geq 1$ be an integer and $r \leq n$. Let us denote the number of ways of choosing *r* objects out of *n* objects by ${}^nC_r$. Then

$$^nC_r = \frac{^nP_r}{r!} \qquad \text{... (11.5)}$$

Proof : We can choose *r* objects out of *n* objects in ${}^nC_r$ ways. Each of the *r* objects chosen can be arranged in *r*! ways. The number of ways of arranging *r* objects is *r*!. Thus, by the counting principle, the number of ways of choosing *r* objects and arranging the *r* objects chosen can be done in ${}^nC_r r!$ ways. But, this is precisely ${}^nP_r$. In other words, we have

$$^nP_r = r!.{}^nC_r \qquad \text{... (11.6)}$$

Dividing both sides by *r*!, we get the result in the theorem.

Here is an example to help you to familiarise yourself with ${}^nC_r$.

Example 11.19 Evaluate each of the following :

(a) ${}^5C_2$ (b) ${}^5C_3$ (c) ${}^4C_3 + {}^4C_2$ (d) $\dfrac{^6C_3}{^4C_2}$

Solution : (a) ${}^5C_2 = \dfrac{^5P_2}{2!} = \dfrac{5.4}{1.2} = 10$. (b) ${}^5C_3 = \dfrac{^5P_3}{3!} = \dfrac{5.4.3}{1.2.3} = 10$.

(c) $${}^4C_3 + {}^4C_2 = \frac{^4P_3}{3!} + \frac{^4P_2}{2!} = \frac{4.3.2}{1.2.3} + \frac{4.3}{1.2} = 4 + 6 = 10$$

(d) $${}^6C_3 = \frac{^6P_3}{3!} = \frac{6.5.4}{1.2.3} = 20 \text{ and } {}^4C_2 = \frac{4.3}{1.2} = 6$$

$$\therefore \frac{^6C_3}{^4C_2} = \frac{20}{6} = \frac{10}{3}.$$

Example 11.20 Find the number of subsets of the set $\{1,2,3,4,5,6,7,8,9,10,11\}$ having 4 elements.

Solution : Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

By relation (11.5), this can be done in $^{11}C_4 = \dfrac{11.10.9.8}{1.2.3.4} = 330$ ways.

Example 11.21 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution : For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $^{12}C_4 = 495$. Therefore, we can draw 495 quadrilaterals.

Example 11.22 In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution : Number of ways of choosing 2 black pens from 5 black pens

$$= {}^5C_2 = \frac{^5P_2}{2!} = \frac{5.4}{1.2} = 10.$$

Number of ways of choosing 2 white pens from 3 white pens , $= {}^3C_2 = \dfrac{^3P_2}{2!} = \dfrac{3.2}{1.2} = 3$.

Number of ways of choosing 2 red pens from 4 red pens , $= {}^4C_2 = \dfrac{^4P_2}{2!} = \dfrac{4.3}{1.2} = 6$.

$\therefore$ By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in $10 \times 3 \times 6 = 180$ ways.

Example 11.23 A question paper consists of 10 questions divided into two parts *A* and *B*. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part *A* and at least 2 from part *B*. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution : The candidate has to select six questions in all of which at least two should be from Part *A* and two should be from Part *B*. He can select questions in any of the following ways :

	Part *A*	Part *B*
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is

$${}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$$

If the candidate follows choice (ii), the number of ways in which he can do so is

$${}^5C_3 \times {}^5C_3 = 10 \times 10 = 100.$$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is

$${}^5C_4 \times {}^5C_2 = 50.$$

Therefore, the candidate can select the question in 50 + 100 + 50 = 200 ways.

Example 11.24 A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when

(i) at least 2 women are included?

(ii) atmost 2 women are included?

Solution : (i) When at least 2 women are included.

The committee may consist of

2 women, 3 men : It can be done in ${}^4C_2 \times {}^6C_3$ ways.

or, 3 women, 2 men : It can be done in ${}^4C_3 \times {}^6C_2$ ways.

or, 4 women, 1 man : It can be done in ${}^4C_4 \times {}^6C_1$ ways.

$\therefore$ Total number of ways of forming the committee

$$= {}^4C_2 . {}^6C_3 + {}^4C_3 . {}^6C_2 + {}^4C_4 . {}^6C_1 = 6 \times 20 + 4 \times 15 + 1 \times 6 = 120 + 60 + 6 = 186$$

(ii) When atmost 2 women are included

The committee may consist of

2 women, 3 men : It can be done in ${}^4C_2 . {}^6C_3$ ways

or, 1 woman, 4 men : It can be done in ${}^4C_1 . {}^6C_4$ ways

or, 5 men : It can be done in ${}^6C_5$ ways

$\therefore$ Total number of ways of forming the committee

$$= {}^4C_2 . {}^6C_3 + {}^4C_1 . {}^6C_4 + {}^6C_5 = 6 \times 20 + 4 \times 15 + 6 = 120 + 60 + 6 = 186$$

Example 9.25 The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and atleast 4 bowlers?

Solution : We are to choose 11 players including 1 wicket keeper and 4 bowlers

or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players

$$= {}^2C_1.{}^5C_4.{}^9C_6$$

$$= 2\times\frac{5\times4\times3\times2.1}{4.3.2.1.}\times\frac{9\times8\times7\times6\times5\times4}{6\times5\times4\times3\times2\times1} = 2\times5\times\frac{9\times8\times7}{3\times2\times1} = 840$$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players

$$= {}^2C_1.{}^5C_5.{}^9C_5 \;= 2\times1\times\frac{9\times8\times7\times6\times5}{5\times4\times3\times2\times1} = 2\times1\times\frac{9\times8\times7\times6}{4\times3\times2\times1} = 252$$

$\therefore$ Total number of ways of selecting the team $= 840 + 252 = 1092$

CHECK YOUR PROGRESS 11.6

1. (a) Evaluate :

(i) ${}^{13}C_3$ (ii) ${}^9C_5$ (iii) ${}^8C_2 + {}^8C_3$ (iv) $\frac{{}^9C_3}{{}^6C_3}$

(b) Verify each of the following statement :

(i) ${}^5C_2 = {}^5C_3$ (ii) ${}^4C_3 \times {}^3C_2 = {}^{12}C_6$

(iii) ${}^4C_2 + {}^4C_3 = {}^8C_5$ (iv) ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3$

2. Find the number of subsets of the set {1, 3, 5, 7, 9, 11, 13, ..., 23}each having 3 elements.

3. There are 14 points lying on a circle. How many pentagons can be drawn using these points?

4. In a fruit basket there are 5 apples, 7 plums and 11 oranges. You have to pick 3 fruits of each type. In how many ways can you make your choice?

5. A question paper consists of 12 questions divided into two parts A and B, containing 5 and 7 questions repectively. A student is required to attempt 6 questions in all, selecting at least 2 from each part. In how many ways can a student select the questions?

6. Out of 5 men and 3 women, a committee of 3 persons is to be formed. In how many ways can it be formed selecting (i) exactly 1 woman. (ii) atleast 1 woman.

7. A cricket team consists of 17 players. It includes 2 wicket keepers and 4 bowlers. In how many ways can a playing eleven be selected if we have to select 1 wicket keeper and atleast 3 bowlers?

8. To fill up 5 vacancies, 25 applications were recieved. There were 7 S.C. and 8 O.B.C. candidates among the applicants. If 2 posts were reserved for S.C. and 1 for O.B.C. candidates, find the number of ways in which selection could be made?

11.6 SOME SIMPLE PROPERTIES OF ^nC_r

In this section we will prove some simple properties of ^nC_r which will make the computations of these coefficients simpler. Let us go back again to Theorem 11.3. Using relation 11.6 we can rewrite the formula for ^nC_r as: $^nC_r = \dfrac{n!}{r!(n-r)!}$ **....(11.7)**

Example 11.26 Find the value of $^nC_0$.

Solution : Here $r = 0$. Therefore, $^nC_0 = \dfrac{n!}{0!n!} = \dfrac{1}{0!} = 1$,

since we have defined 0! = 1.

The formula given in Theorem 11.3 was used in the previous section. As we will see shortly, the formula given in Equation 11.7 will be useful for proving certain properties of ^nC_r.

$$^nC_r = {^nC_{n-r}} \qquad \textbf{...(11.8)}$$

This means just that the number of ways of choosing r objects out of n objects is the same as the number of ways of not choosing $(n - r)$ objects out of n objects. In the example described in the introduction, it just means that the number of ways of selecting 2 sets of dresses is the same as the number of ways of rejecting 4 – 2 = 2 dresses. In Example 11.20, this means that the number of ways of choosing subsets with 4 elements is the same as the number of ways of rejecting 8 elements since choosing a particular subset of 4 elements is equivalent to rejecting its complement, which has 8 elements.

Let us now prove this relation using Equation 11.7. The denominator of the right hand side of this equation is $r!\,(n-r)!$. This does not change when we replace r by $n - r$.

$$(n-r)!.[n-(n-r)]! = (n-r)!.r!$$

The numerator is independent of r. Therefore, replacing r by $n - r$ in Equation 11.7 we get result.

How is the relation 11.8 useful? Using this formula, we get, for example, $^{100}C_{98}$ is the same as $^{100}C_2$. The second value is much more easier to calculate than the first one.

Example 11.27 Evaluate :

(a) $^7C_5$ (b) $^{11}C_9$ (c) $^{10}C_9$ (d) $^{12}C_9$

Solution : (a) From relation 11.8, we have

$$^7C_5 = {^7C_{7-5}} = {^7C_2} = \frac{7.6}{1.2} = 21$$

(b) Similarly $\quad {^{10}C_9} = {^{10}C_{10-9}} = {^{10}C_1} = 10$

(c) $\qquad {}^{11}C_9 = {}^{11}C_{11-9} = {}^{11}C_2 = \dfrac{11.10}{1.2} = 55$

(d) $\qquad {}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \dfrac{12.11}{1.2} = 66$

There is another relation satisfied by ${}^{n}C_r$ which is also useful. We have the following relation:

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^{n}C_r \qquad \textbf{....(11.9)} $$

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} $$

$$ = \frac{(n-1)!}{(n-r)(n-r-1)!(r-1)!} + \frac{(n-1)!}{r(n-r-1)!(r-1)!} $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right] $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{n}{(n-r)r}\right] $$

$$ = \frac{n(n-1)!}{(n-r)(n-r-1)!r(r-1)!} $$

$$ = \frac{n!}{(n-r)!r!} = {}^{n}C_r $$

Example 11.28 Evaluate :

(a) ${}^{6}C_2 + {}^{6}C_1$ $\qquad$ (b) ${}^{8}C_2 + {}^{8}C_1$ $\qquad$ (c) ${}^{5}C_3 + {}^{5}C_2$ $\qquad$ (d) ${}^{10}C_2 + {}^{10}C_3$

Solution : (a) Let us use relation (11.9) with n = 7, r = 2. So, ${}^{6}C_2 + {}^{6}C_1 = {}^{7}C_2 = 21$

(b) Here n = 9, r = 2. Therefore, ${}^{8}C_2 + {}^{8}C_1 = {}^{9}C_2 = 36$

(c) Here n = 6, r = 3. Therefore, ${}^{5}C_3 + {}^{5}C_2 = {}^{6}C_3 = 20$

(d) Here n = 11, r = 3. Therefore, ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3 = 165$

Example 11.29 If ${}^{n}C_{10} = {}^{n}C_{12}$ find n,

Solution : Using ${}^{n}C_r = {}^{n}C_{n-r}$ we get $n - 10 = 12$ or, $n = 12 + 10 = 22$

CHECK YOUR PROGRESS 11.7

1. (a) Find the value of ${}^{n}C_{n-1}$. Is ${}^{n}C_{n-1} = {}^{n}C_{n}$? (b) Show that ${}^{n}C_{n} = {}^{n}C_{0}$

2. Evaluate :

   (a) ${}^{9}C_{5}$ (b) ${}^{14}C_{10}$

   (c) ${}^{13}C_{9}$ (d) ${}^{15}C_{12}$

3. Evaluate :

   (a) ${}^{7}C_{3} + {}^{7}C_{2}$ (b) ${}^{8}C_{4} + {}^{8}C_{5}$

   (c) ${}^{9}C_{3} + {}^{9}C_{2}$ (d) ${}^{12}C_{3} + {}^{12}C_{2}$

4. If ${}^{10}C_{r} = {}^{10}C_{2r+1}$, find the value of r. 5. If ${}^{18}C_{r} = {}^{18}C_{r+2}$ find ${}^{r}C_{5}$

11.7 PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 11.30 There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf ?

Soluton : 4 novels can be selected out of 5 in ${}^{5}C_{4}$ ways. 2 biographies can be selected out of 4 in ${}^{4}C_{2}$ ways.

Number of ways of arranging novels and biographies = ${}^{5}C_{4} \times {}^{4}C_{2} = 5 \times 6 = 30$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6! = 720 ways.

By the Counting Principle, the total number of arrangements = $30 \times 720 = 21600$

Example 11.31 From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels ?

Solution : From 5 consonants, 3 consonants can be selected in ${}^{5}C_{3}$ ways.

From 4 vowels, 2 vowels can be selected in ${}^{4}C_{2}$ ways.

Now with every selection, number of ways of arranging 5 letters is ${}^{5}P_{5}$

$$\therefore \quad \text{Total number of words} = {}^{5}C_{3} \times {}^{4}C_{2} \times {}^{5}P_{5} = \frac{5\times 4}{2\times 1} \times \frac{4\times 3}{2\times 1} \times 5!$$

$$= 10 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7200$$

CHECK YOUR PROGRESS 11.8

1. There are 5 Mathematics, 4 Physics and 5 Chemistry books. In how many ways can you arrange 4 Mathematics, 3 Physics and 4 Chemistry books.

   (a) if the books on the same subjects are arranged together, but the order in which the books are arranged within a subject doesn't matter ?

   (b) if books on the same subjects are arranged together and the order in which books are arranged within subject matters ?

2. There are 9 consonants and 5 vowels. How many words of 7 letters can be formed using 4 consonents and 3 vowels ?

3. In how many ways can you invite at least one of your six friends to a dinner?

4. In an examination, an examinee is required to pass in four different subjects. In how many ways can he fail?

LET US SUM UP

- Fundamental principle of counting states.

  If there are n events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the n events can occur in

  $m_1 \times m_2 \times m_3 \times \ldots\ldots\ldots \times m_{n-1} \times m_n$ ways.

- The number of permutations of n objects taken all at a time is $n!$

- $${}^nP_r = \frac{n!}{(n-r)!}$$

- $${}^nP_n = n!$$

- The number of ways of selecting r objects out of n objects is ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

- $${}^nC_r = {}^nC_{n-r}$$

- $${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=XqQTXW7XfYA
http://www.youtube.com/watch?v=bCxMhncR7PU
www.mathsisfun.com/combinatorics/**combinations-permutations**.html

TERMINAL EXERCISE

1. There are 8 true - false questions in an examination. How many responses are possible ?
2. The six faces of a die are numbered 1,2,3,4,5 and 6. Two such dice are thrown simultaneously. In how many ways can they turn up ?
3. A restaurant has 3 vegetables, 2 salads and 2 types of bread. If a customer wants 1 vegetable, 1 salad and 1 bread, how many choices does he have ?
4. Suppose you want to paper your walls. Wall papers are available in 4 diffrent backgrounds colours with 7 different designs of 5 different colours on them. In how many ways can you select your wall paper ?
5. In how many ways can 7 students be seated in a row on 7 seats ?
6. Determine the number of 8 letter words that can be formed from the letters of the word *ALTRUISM*.
7. If you have 5 windows and 8 curtains in your house, in how many ways can you put the curtains on the windows ?
8. Determine the maximum number of 3- letter words that can be formed from the letters of the word *POLICY*.
9. There are 10 athletes participating in a race and there are three prizes, 1st, 2nd and 3rd to be awarded. In how many ways can these be awarded ?
10. In how many ways can you arrange the letters of the word *ATTAIN* so that the *T*s and As are together?
11. A group of 12 friends meet at a party. Each person shake hands once with all others. How many hand shakes will be there. ?
12. Suppose that you own a shop which sells televisions. You are selling 5 different kinds of television sets, but your show case has enough space for display of 3 televison sets only. In how many ways can you select the television sets for the display ?
13. A contractor needs 4 carpenters. Five equally qualified carpenters apply for the job. In how many ways can the contractor make the selection ?
14. In how many ways can a committe of 9 can be selected from a group of 13?
15. In how many ways can a committee of 3 men and 2 women be selected from a group of 15 men and 12 women ?
16. In how ways can 6 persons be selected from 4 grade 1 and 7 grade II officers, so as to include at least two officers from each category ?
17. Out of 6 boys and 4 girls, a committee of 5 has to be formed. In how many ways can this be done if we take :

    (a) 2 girls. (b) at least 2 girls.
18. The English alphabet has 5 vowels and 21 consonants. What is the maximum number of words, that can be formed from the alphabet with 2 different vowels and 2 different consonants?

19. From 5 consonants and 5 vowels, how many words can be formed using 3 consonants and 2 vowels?

20. In a school annual day function a variety programme was organised. It was planned that there would be 3 short plays, 6 recitals and 4 dance programmes. However, the chief guest invited for the function took much longer time than expected to finish his speech. To finish in time, it was decided that only 2 short plays, 4 recitals and 3 dance programmes would be perfomed, How many choices were available to them ?

    (a) if the programmes can be perfomed in any order ?

    (b) if the programmes of the same kind were perfomed at a stretch?

    (c) if the programmes of the same kind were perfomed at a strech and considering the order of performance of the programmes of the same kind ?

ANSWERS

CHECK YOUR PROGRESS 11.1

1.	(a) 180	(b) 8	(c) 12
2.	(a) 48	(b) 20	

CHECK YOUR PROGRESS 11.2

1.	(a)	17576	(b)	900
2.	(a)	105	(b)	60
3.	(a)	24	(b)	24

CHECK YOUR PROGRESS 11.3

1.	(a) (i) 720	(ii) 5040	(iii) 5046	(iv) 17280	(v) 10
	(b) (i) False	(ii) False	(iii) True	(iv) False	
2.	(a) 120	(b) 40320	(c) 24		

CHECK YOUR PROGRESS 11.4

1.	(a) (i) 12	(ii) 120	(iii) 4	(iv) 7200	(v) $n!$
	(b) (i) False	(ii) True	(iii) False	(iv) False	
2.	(a) (i) 7980	(ii) 9240	(b) 20	(c) 840	

CHECK YOUR PROGRESS 11.5

1.	96	2. 1152	3. 2073600	4. 2488320
5.	144	6. (i) 5040	(ii) 30240	(iii) 332640

CHECK YOUR PROGRESS 11.6

1.	(a)	(i) 286	(ii) 126	(iii) 84	(iv) $\frac{21}{5}$
	(b)	(i) True	(ii) False	(iii) False	(iv) True
2.	1771	3. 2002	4. 57750	5. 805	
6.	(i)	30	(ii)	46	
7.	3564	8. 7560			

CHECK YOUR PROGRESS 11.7

1. (a) n , No

2. (a) 126 (b) 1001 (c) 715 (d) 455

3. (a) 56 (b) 126 (c) 120 (d) 286

4. 3 5. 56

CHECK YOUR PROGRESS 11.8

1 (a) 600 (b) 207 3600

2. 6350400

3. 63 4. 15

TERMINAL EXERCISE

1.	256	2.	36	3.	12	4.	140
5.	5040	6.	40320	7.	6720	8.	120
9.	720	10.	24	11.	66	12.	10
13.	5	14.	715	15.	30030	16.	371

17. (a) 120 (b) 186

18. 50400 19. 12000

20. (a) 65318400 (b) 1080 (c) 311040

12

311en12

BINOMIAL THEOREM

Suppose you need to calculate the amount of interest you will get after 5 years on a sum of money that you have invested at the rate of 15% compound interest per year. Or suppose we need to find the size of the population of a country after 10 years if we know the annual growth rate. A result that will help in finding these quantities is the **binomial theorem.** This theorem, as you will see, helps us to calculate positive integral powers of any real binomial expression, that is, any expression involving two terms.

The binomial theorem, was known to Indian and Greek mathematicians in the 3rd century B.C. for some cases. The credit for the result for natural exponents goes to the Arab poet and mathematician Omar Khayyam (A.D. 1048-1122). Further generalisation to rational exponents was done by the British mathematician Newton (A.D. 1642-1727).

There was a reason for looking for further generalisation, apart from mathematical interest. The reason was its many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus, and in approximating numbers like $(1.02)^7$, etc. We shall discuss them in this lesson.

OBJECTIVES

After studying this lesson, you will be able to:

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x+y)^n$ for different values of x and y using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

12.1 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x + y)$. Now,

$(x+y)^1 = x+y$

$(x+y)^2 = (x+y)(x+y) = x^2+2xy+y^2$

$(x+y)^3 = (x+y)(x+y)^2 = x^3+3x^2y+3xy^2+y^3$

$(x+y)^4 = (x+y)(x+y)^3 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$

$(x+y)^5 = (x+y)(x+y)^4 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$ and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is x in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x+y)^4$, the number of terms is 5.

2. The exponent of x in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.

3. The exponent of y in the first term is zero (as $y^0=1$). The exponent of y in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.

4. The sum of the exponents of x and y in each term is equal to the exponent of the binomial. For example, in the expansion of $(x+y)^5$, the sum of the exponents of x and y in each term is 5.

   If we use the combinatorial co-efficients, we can write the expansion as

   $(x+y)^3 = {}^3C_0\, x^3 + {}^3C_1 x^2y + {}^3C_2 x\, y^2 + {}^3C_3 y^3$

   $(x+y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3y + {}^4C_2 x^2y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4$

   $(x+y)^5 = {}^5C_0 x^5 + {}^5C_1 x^4y + {}^5C_2 x^3y^2 + {}^5C_3 x^2y^3 + {}^5C_4 xy^4 + {}^5C_5 y^5$, and so on.

   More generally, we can write the binomial expansion of $(x+y)^n$, where n is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral**) **exponent**.

Theorem

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n \ldots\text{(A)}$$

where $n \in N$ and $x, y \in R$.

Proof : Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by $P(n)$, i.e.,

$$P(n):\ (x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \ldots$$

$$+ {}^nC_{n-1}\, xy^{\,n-1} + {}^nC_n\, y^{\,n} \qquad ...(i)$$

Let us examine whether $P(1)$ is true or not.

From (i), we have $P(1) : (x + y)^1 = {}^1C_0 x + {}^1C_1\, y = 1\times x + 1\times y$

i.e., $(x + y)^1 = x + y$ Thus, $P(1)$ holds.

Now, let us assume that $P(k)$ is true, i.e.,

$$P(k):\ (x + y)^k = {}^kC_0 x^{\,k} + {}^kC_1 x^{\,k-1} y + {}^kC_2 x^{\,k-2} y^{\,2} + {}^kC_3 x^{\,k-3} y^{\,3} + \ldots + {}^kC_{k-1} xy^{\,k-1} + {}^kC_k\, y^{\,k} \qquad ...(ii)$$

Assuming that $P(k)$ is true, if we prove that $P(k + 1)$ is true, then $P(n)$ holds, for all n. Now,

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)\left({}^kC_0 x^k + {}^kC_1 x^{k-1} y + {}^kC_2 x^{k-2} y^2 + \right.$$

$$\left. \ldots + {}^kC_{k-1} xy^{k-1} + {}^kC_k y^k\right)$$

$$= {}^kC_0 x^{k+1} + {}^kC_0 x^k y + {}^kC_1 x^k y + {}^kC_1 x^{k-1} y^2 + {}^kC_2 x^{k-1} y^2 + {}^kC_2 x^{k-2} y^3 +$$

$$\ldots + {}^kC_{k-1} x^2 y^{k-1} + {}^kC_{k-1} xy^k + {}^kC_k xy^k + {}^kC_k y^{k+1}$$

i.e. $\quad (x+y)^{k+1} = {}^kC_0 x^{k+1} + ({}^kC_0 + {}^kC_1) x^k y + ({}^kC_1 + {}^kC_2) x^{k-1} y^2 +$

$$\ldots + ({}^kC_{k-1} + {}^kC_k) xy^k + {}^kC_k y^{k+1} \qquad ...(iii)$$

From Lesson 11, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$, and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$ $\qquad ....(iv)$

Also, $\qquad {}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore, $\qquad {}^kC_0 + {}^kC_1 = {}^{k+1}C_1\ ,\ {}^kC_1 + {}^kC_2 = {}^{k+1}C_2\ ,\ {}^kC_2 + {}^kC_3 = {}^{k+1}C_3 \qquad .....(v)$

....

.... and so on

Using (iv) and (v), we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0 x^{k+1} + {}^{k+1}C_1 x^k y + {}^{k+1}C_2 x^{k-1} y^2 +\ ,\ \ldots + {}^{k+1}C_k xy^k + {}^{k+1}C_{k+1} y^{k+1}$$

which shows that P$(k+1)$ is true.

Thus, we have shown that (a) $P(1)$ is true, and (b) if $P(k)$ is true, then $P(k+1)$ is also true.

Therefore, by the principle of mathematical induction, $P(n)$ holds for any value of n. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

Example 12.1 Write the binomial expansion of $(x + 3y)^5$.

Solution : Here the first term in the binomial is x and the second term is $3y$. Using the binomial theorem, we have

$$(x+3y)^5 = {}^5C_0x^5 + {}^5C_1x^4(3y)^1 + {}^5C_2x^3(3y)^2 + {}^5C_3x^2(3y)^3 + {}^5C_4x(3y)^4 + {}^5C_5(3y)^5$$

$$= 1\times x^5 + 5x^4\times 3y + 10x^3\times(9y^2) + 10x^2\times(27y^3) + 5x\times(81y^4) + 1\times 243y^5$$

$$= x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Thus, $(x+3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

Example 12.2 Expand $(1+a)^n$ in terms of powers of a, where a is a real number.

Solution : Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1+a)^n = {}^nC_0(1)^n + {}^nC_1(1)^{n-1}a + {}^nC_2(1)^{n-2}a^2 + \ldots + {}^nC_{n-1}(1)a^{n-1} + {}^nC_na^n$$

i.e., $$(1+a)^n = 1 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_{n-1}a^{n-1} + {}^nC_na^n \quad \ldots \text{(B)}$$

(B) is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x+\frac{1}{x}\right)^5, \left(\frac{y}{x}+\frac{1}{y}\right)^5, \left(\frac{a}{4}+\frac{2}{a}\right)^5, \left(\frac{2t}{3}-\frac{3}{2t}\right)^6, \text{ etc.}$$

Let us illustrate it through an example.

Example 12.3 Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, where $x, y \neq 0$.

Solution : We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2 + {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}$$

Example 12.4 The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years ? Give the answer up to 2 decimal places.

Solution : Suppose the population is a at present. After 1 year it will be

$$a+\frac{3}{100}a=a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $\quad a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$= a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right) = a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places,

we get , $a\left(1+\frac{3}{100}\right)^5 \approx a\left[1+5(0.03)+10(0.03)^2\right] = a\times 1.159$

So, the increase is $0.159\times 100\% = \frac{159}{1000}\times 100\times\frac{1}{100}=15.9\%$ in 5 years.

Example 12.5 Using binomial theorem, evaluate , (*i*) 102^4 (*ii*) 97^3

Solution : (*i*) $102^4 = (100 + 2)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1(100)^3.2 + {}^4C_2(100)^2.2^2 + {}^4C_3\,(100).2^3 + {}^4C_4\,.2^4$

$= 100000000 + 8000000 + 240000 + 3200 + 16 = 108243216$

(*ii*) $(97)^3 = (100-3)^3 = {}^3C_0(100)^3 - {}^3C_1(100)^2.3 + {}^3C_2(100).3^2 - {}^3C_3.3^3$

$= 1000000 - 90000 + 2700 - 27 = 1002700 - 90027 = 912673$

CHECK YOUR PROGRESS 12.1

1. Write the expansion of each of the following :

   (a) $(2a + b)^3$ (b) $(x^2-3y)^6$ (c) $(4a-5b)^4$ (d) $(ax + by)^n$

2. Write the expansions of :

   (a) $(1 - x)^7$ (b) $\left(1+\frac{x}{y}\right)^7$ (c) $(1 + 2x)^5$

3. Write the expansions of :

(a) $\left(\frac{a}{3}+\frac{b}{2}\right)^5$ (b) $\left(3x-\frac{5}{x^2}\right)^7$ (c) $\left(x+\frac{1}{x}\right)^4$ (d) $\left(\frac{x}{y}+\frac{y}{x}\right)^5$

4. Suppose I invest Rs. 1 lakh at 18% per year compound interest. What sum will I get back after 10 years? Give your answer up to 2 decimal places.

5. The population of bacteria increases at the rate of 2% per hour. If the count of bacteria at 9 a.m. is 1.5×10^5, find the number at 1 p.m. on the same day.

6. Using binomial theorem, evaluate each of the following :

(*i*) $(101)^4$ (*ii*) $(99)^4$ (*iii*) $(1.02)^3$ (iv) $(0.98)^3$

12.2 GENERAL TERM IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x+y)^n$, i.e., in

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

We observe that , the first term is ${}^nC_0x^n$, i.e., ${}^nC_{1-1}x^ny^0$;

the second term is ${}^nC_1x^{n-1}y$, i.e., ${}^nC_{2-1}x^{n-1}y^1$;

the third term is ${}^nC_2x^{n-2}y^2$, i.e., ${}^nC_{3-1}x^{n-2}y^2$; and so on.

From the above, we can generalise that

the $(r+1)^{\text{th}}$ term is ${}^nC_{(r+1)-1}x^{n-r}y^r$, i.e., ${}^nC_rx^{n-r}y^r$.

If we denote this term by T_{r+1}, we have , $T_{r+1} = {}^nC_rx^{n-r}y^r$

T_{r+1} is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

Example 12.6 Find the $(r+1)^{\text{th}}$ term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$, where n is a natural number. Verify your answer for the first term of the expansion.

Solution : The general term of the expansion is given by :

$$T_{r+1} = {}^nC_r(x^2)^{(n-r)}\left(\frac{1}{x}\right)^r$$

$$= {}^nC_rx^{2n-2r}\frac{1}{x^r} = {}^nC_rx^{2n-3r} \qquad \ldots(i)$$

Hence, the $(r + 1)$th term in the expansion is ${}^nC_r x^{2n-3r}$.

On expanding $\left(x^2+\frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or x^{2n}.

Using (i), we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$ $\therefore$ $\qquad T_1 = {}^nC_0 x^{2n-0} = x^{2n}$

This verifies that the expression for T_{r+1} is correct for $r + 1 = 1$.

Example 12.7 Find the fifth term in the expansion of

$$\left(1-\frac{2}{3}x^3\right)^6$$

Solution : Using here $\mathrm{T}_{r+1} = \mathrm{T}_5$ which gives $r+1 = 5$, i.e., $r = 4$.

Also $n = 6$ and let $x = 1 and\ y = \frac{-2}{3}x^3$.

$$T_5 = {}^6C_4\left(-\frac{2}{3}x^3\right)^4 = {}^6C_2\left(\frac{16}{81}x^{12}\right) = \frac{6\times5}{2}\times\frac{16}{81}\times x^{12} = \frac{80}{27}x^{12}$$

Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$.

CHECK YOUR PROGRESS 12.2

1. For a natural number n, write the $(r + 1)^{\text{th}}$ term in the expansion of each of the following:

   (a) $(2x+y)^n$ (b) $(2a^2-1)^n$ (c) $(1-a)^n$ (d) $\left(3+\frac{1}{x^2}\right)^n$

2. Find the specified terms in each of the following expansions:

   (a) $(1+2y)^8$; 6th term (b) $(2x+3)^7$; 4th term (c) $(2a-b)^{11}$; 7th term

   (d) $\left(x+\frac{1}{x}\right)^6$; 4th term (e) $\left(x^3-\frac{1}{x^2}\right)^7$; 5th term

12.3 MIDDLE TERMS IN A BINOMIAL EXPANSION

Now you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even.

Thus, while finding the middle term in a binomial expansion, we come across two cases:

Case 1 : When n is even. To study such a situation, let us look at a particular value of n, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 12.1, you can see that there are three terms on either side of the fourth term.

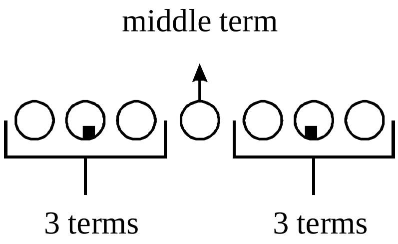


Fig. 12.1

In general, when the exponent n of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

Case 2: When n is odd , Let us take $n = 7$ as an example to see what happens in this case. The number of terms in the expansion will be 8. Looking at Fig. 12.2, do you find any one middle term in it? There is not. But we can partition the terms into two equal parts by a line as shown in the figure. We call the terms on either side of the partitioning line taken together, the middle terms. This is because there are an equal number of terms on either side of the two, taken together.

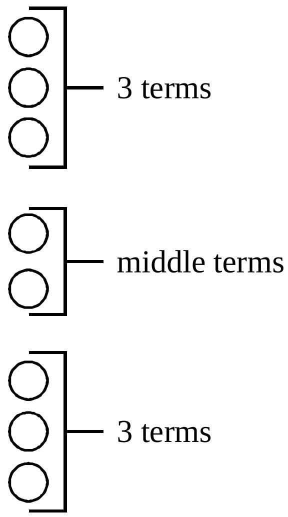


Fig. 12.2

Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if n = 13, then the $\left(\frac{13+1}{2}\right)$th and the $\left(\frac{13+3}{2}\right)$th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 12.3.

From the above, we conclude that

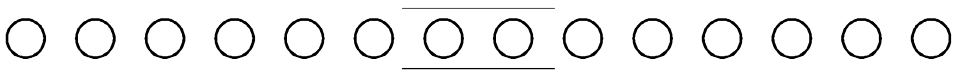


Fig. 12.3

When the exponent *n* of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are two middle terms in the corresponding binomial expansion.

Let us now consider some examples.

Example 12.8 Find the middle term in the expansion of $(x^2+y^2)^{8.}$

Solutuion : Here n = 8 (an even number).

Therefore, the $\left(\frac{8}{2}+1\right)$th, i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r(x^2)^{8-r}y^r$, $T_5 = {}^8C_4(x^2)^{8-4}(y^2)^4 = 70x^8y^8$

Example 12.9 Find the middle term(s) in the expansion of $\left(2x^2+\frac{1}{x}\right)^9$.

Solution : Here n = 9 (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)th$ and $\left(\frac{9+3}{2}\right)th$ are middle terms. i.e. T_5 and T_6 are middle terms.

For finding T_5 and T_6, putting r = 4 and r = 5 in the general term , $T_{r+1} = {}^9C_r(2x^2)^{9-r}\left(\frac{1}{x}\right)^6$,

$$T_5 = {}^9C_4(2x^2)^{9-4}\left(\frac{1}{x}\right)^4 \qquad = \frac{9\times8\times7\times6}{4\times3\times2}\times(32x^{10})\times\left(\frac{1}{x}\right)^4 = 4032\,x^6$$

and $T_6 = {}^9C_5\,(2x^2)^{9-5}\left(\frac{1}{x}\right)^5 = 2016\,x^3$

Thus, the two middle terms are $4032\,x^6$ and $2016\,x^3$.

CHECK YOUR PROGRESS 12.3

1. Find the middle term(s) in the expansion of each of the following :

   (a) $(2x+y)^{10}$ (b) $\left(1+\frac{2}{3}x^3\right)^8$ (c) $\left(x+\frac{1}{x}\right)^6$ (d) $\left(1-x^2\right)^{10}$

2. Find the middle term(s) in the expansion of each of the following :

   (a) $(a+b)^7$ (b) $(2a-b)^9$ (c) $\left(\frac{3x}{4}-\frac{4y}{3}\right)^7$ (d) $\left(x+\frac{1}{x^2}\right)^{11}$

LET US SUM UP

- For a natural number n,

  $$\left(x+y\right)^n = {}^nC_o\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \ldots + {}^nC_{n-1}\,xy^{n-1} + {}^nC_n\,y^n$$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1+a)^n = {}^nC_o + {}^nC_1\,a + {}^nC_2\,a^2 + \ldots. + {}^nC_{n-1}\,a^{n-1} + {}^nC_n\,a^n$
- The general term in the expansion of $(x+y)^n$ is ${}^nC_r\,x^{n-r}y^r$ and in the expansion of $(1+a)^n$ is ${}^nC_r a^r$, where n is a natural number and $0 \le r \le n$.
- If n is an even natural number, there is only one middle term in the expansion of $(x+y)^n$. If n is odd, there are two middle trems in the expansion.
- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=Cv4YhIMfbeM

http://www.youtube.com/watch?v=-fFWWt1m9k0

http://www.youtube.com/watch?v=xF_hJaXUNfE

TERMINAL EXERCISE

1. Write the expansion of each of the following :

   (a) $(3x + 2y)^5$ (b) $(p - q)^8$ (c) $(1- x)^8$

   (d) $\left(1+\frac{2}{3}x\right)^6$ (e) $\left(x+\frac{1}{2x}\right)^6$ (f) $\left(3x-y^2\right)^5$

   (g) $\left(\frac{x^2}{4}+\frac{2}{x}\right)^4$ (h) $\left(x^2-\frac{1}{x^3}\right)^7$ (i) $\left(x^3+\frac{1}{x^2}\right)^5$ (j) $\left(\frac{1}{x^2}-x^3\right)^4$

2. Write the (r + 1)th term in the expansion of each of the following, where $n \in N$:

   (a) $(3x - y^2)^n$ (b) $\left(x^3+\frac{1}{x}\right)^n$

3. Find the specified terms in the expansion of each of the following :

   (a) $(1 - 2x)^7$: 3rd term [Hint : Here $r = 2$] (b) $\left(x+\frac{1}{2x}\right)^6$: middle term (s)

   (c) $(3x - 4y)^6$: 4th term (d) $\left(y^2-\frac{1}{y}\right)^{11}$: middle term (s)

   (e) $(x^3 - y^3)^{12}$: 4th term (f) $(1 - 3x^2)^{10}$: middle term (s)

   (g) $(-3x-4y)^6$: 5th term (h) Write the rth term in the expansion of $(x - 2y)^6$.

4. If T_r, denotes the rth term in the expansion of $(1 + x)^n$ in ascending powers of x (n being a natural number) , prove that

   $r(r + 1)T_{r+2} = (n - r + 1)(n - r)\, x^2\, T_r$. [**Hint :** $T_r = {}^nC_{r-1}\, x^{r-1}$ and $T_{r+2} = {}^nC_{r+1}\, x^{r+1}$]

5. k_r is the coefficient of x^{r-1} in the expansion of $(1 + 2x)^{10}$ in ascending powers of x and $k_{r+2} = 4k_r$. Find the value of r. [**Hint :** $k_r = {}^{10}C_{r-1}\, 2^{r-1}$ and $k_{r+2} = {}^{10}C_{r+1} 2^{r+1}$]

6. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1+a)^n$ (n being a natural number) are in A.P. Find n. [**Hint :** ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$]

7. Expand $(1 + y + y^2)^4$. $\left[\textbf{Hint : } (1+y+y^2)^4 = \{(1+y)+y^2\}^4\right]$

MODULE-III
Algebra-I

Notes

ANSWERS

CHECK YOUR PROGRESS 12.1

1. (a) $8a^3+12a^2b+6ab^2+b^3$

(b) $x^{12}-18x^{10}y+135x^8y^2-540x^6y^3+1215x^4y^4-1458x^2y^5+729y^6$

(c) $256a^4-1280a^3b+2400a^2b^2-2000ab^3+625b^4$

(d) $a^nx^n+na^{n-1}x^{n-1}by+\dfrac{n(n-1)}{2!}a^{n-2}x^{n-2}b^2y^2+\ldots+b^ny^n$

2. (a) $1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7$

(b) $1+\dfrac{7x}{y}+\dfrac{21x^2}{y^2}+\dfrac{35x^3}{y^3}+\dfrac{35x^4}{y^4}+\dfrac{21x^5}{y^5}+\dfrac{7x^6}{y^6}+\dfrac{x^7}{y^7}$

(c) $1+10x+40x^2+80x^3+80x^4+32x^5$

3. (a) $\dfrac{a^5}{243}+\dfrac{5a^4b}{162}+\dfrac{5a^3b^2}{54}+\dfrac{5a^2b^3}{36}+\dfrac{5ab^4}{48}+\dfrac{b^5}{32}$

(b) $2187x^7-25515x^4+127575x-\dfrac{354375}{x^2}+\dfrac{590625}{x^5}-\dfrac{590625}{x^8}$

$+\dfrac{328125}{x^{11}}-\dfrac{78125}{x^{14}}$

(c) $x^4+4x^2+6+\dfrac{4}{x^2}+\dfrac{1}{x^4}$ (d) $\dfrac{x^5}{y^5}+5\dfrac{x^3}{y^3}+10\dfrac{x}{y}+10\dfrac{y}{x}+5\dfrac{y^3}{x^3}+\dfrac{y^5}{x^5}$

4. Rs 4.96 lakh 5. 162360

6. (i) 104060401 (ii) 96059601

(iii) 1.061208 (iv) 0.941192

CHECK YOUR PROGRESS 12.2

1. (a) ${}^nC_r\,2^{n-r}x^{n-r}y^r$ (b) ${}^nC_r\,2^{n-r}a^{2n-2r}(-1)^r$

(c) ${}^nC_r(-1)^ra^r$ (d) ${}^nC_r\,3^{n-r}.x^{-2r}$

2. (a) $1792y^5$ (b) $15120x^4$

(c) $14784a^5b^6$ (d) 20

(e) $35x$

CHECK YOUR PROGRESS 12.3

1. (a) $8064x^5y^5$ (b) $\dfrac{1120}{81}x^{12}$

(c) 20 (d) $-252x^{10}$

2. (a) $35a^4b^3,\ 35a^3b^4$

(b) $4032a^5b^4, -2016a^4b^5$

(c) $\dfrac{-105}{4}x^4y^3, \dfrac{140}{3}x^3y^4$

(d) $\dfrac{462}{x^4}, \dfrac{462}{x^7}$

TERMINAL EXERCISE

1. (a) $243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4+32y^5$

(b) $p^8-8p^7q+28p^6q^2-56p^5q^3+70p^4q^4-56p^3q^5+28p^2q^6-8pq^7+q^8$

(c) $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$

(d) $1+4x+\dfrac{20}{3}x^2+\dfrac{160}{27}x^3+\dfrac{80}{27}x^4+\dfrac{64}{81}x^5+\dfrac{64}{729}x^6$

(e) $x^6+3x^4+\dfrac{15}{4}x^2+\dfrac{5}{2}+\dfrac{15}{16x^2}+\dfrac{3}{16x^4}+\dfrac{1}{64x^6}$

(f) $243x^5-405x^4y^2+270x^3y^4-90x^2y^6+15xy^8-y^{10}$

(g) $\dfrac{x^8}{256}+\dfrac{x^5}{8}+\dfrac{3}{2}x^2+\dfrac{8}{x}+\dfrac{16}{x^4}$

(h) $x^{14}-7x^9+21x^4-\dfrac{35}{x}+\dfrac{35}{x^6}-\dfrac{21}{x^{11}}+\dfrac{7}{x^{16}}-\dfrac{1}{x^{21}}$

(i) $x^{15}+5x^{10}+10x^5+10+\dfrac{5}{x^5}+\dfrac{1}{x^{10}}$

(j) $\dfrac{1}{x^8}-\dfrac{4}{x^3}+6x^2-4x^7+x^{12}$

MODULE-III
Algebra-I

Notes

2. (a) $(-1)^r \, {}^nC_r 3^{n-r} x^{n-r} y^{2r}$

(b) ${}^nC_r x^{3n-4r}$

3. (a) $84x^2$

(b) $\frac{5}{2}$

(c) $-34560\ x^3y^3$

(d) $-462y^7,\ 462y^4$

(e) $-220x^{27}y^9$

(f) $-61236\ x^{10}$

(g) $34560x^2y^4$

(h) $(-2)^{r-1}\ {}^6C_{r-1}\ x^{7-r} y^{r-1}$

5. 5

6. 7, 14

7. $1 + 4y + 10y^2 + 16y^3 + 19y^4 + 16y^5 + 10y^6 + 4y^7 + y^8$

13

311en13

CARTESIAN SYSTEM OF RECTANGULAR CO-ORDINATES

You must have searched for your seat in a cinema hall, a stadium, or a train. For example, seat *H*-4 means the fourth seat in the H^{th} row. In other words, *H* and 4 are the coordinates of your seat. Thus, the geometrical concept of location is represented by numbers and alphabets (an algebraic concept).

Also a road map gives us the location of various houses (again numbered in a particular sequence), roads and parks in a colony, thus representing algebraic concepts by geometrical figures like straight lines, circles and polygons.

The study of that branch of Mathematics which deals with the interrelationship between geometrical and algebraic concepts is called Coordinate Geometry or Cartesian Geometry in honour of the famous French mathematician **Rene Descartes.**

In this lesson we shall study the basics of coordinate geometry and relationship between concept of straight line in geometry and its algebraic representation.

OBJECTIVES

After studying this lesson, you will be able to:

- define Cartesian System of Coordinates including the origin, coordinate axes, quadrants, etc;
- derive distance formula and section formula;
- derive the formula for area of a triangle with given vertices;
- verify the collinearity of three given points;
- state the meaning of the terms : inclination and slope of a line;
- find the formula for the slope of a line through two given points;
- state the condition for parallelism and perpendicularity of lines with given slopes;
- find the intercepts made by a line on coordinate axes;
- find the angle between two lines when their slopes are given;
- find the coordinates of a point when origin is shifted to some other point;
- find transformed equation of curve when oregin is shifted to another point.

MODULE-IV
Co-ordinate Geometry

Notes

EXPECTED BACKGROUND KNOWLEDGE

- Number system .
- Plotting of points in a coordinate plane.
- Drawing graphs of linear equations .
- Solving systems of linear equations .

13.1 RECTANGULAR COORDINATE AXES

Recall that in previous classes, you have learnt to fix the position of a point in a plane by drawing two mutually perpendicular lines. The fixed point O,where these lines intersect each other is called the **origin** O as shown in Fig. 13.1 These mutually perpencular lines are called the **coordinate axes**. The horizontal line XOX' is the **x-axis** or **axis** of x and the vertical line YOY' is the y- **axis** or **axis** of y.

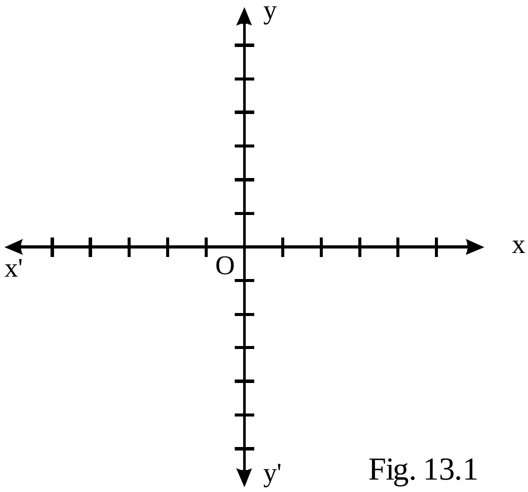


Fig. 13.1

9.1.1 CARTESIAN COORDINATES OF A POINT

To find the coordinates of a point we proceed as follows. Take X'OX and YOY' as coordinate axes. Let P be any point in this plane. From point P draw $PA \perp XOX'$ and $PB \perp YOY'$. Then the distance OA = x measured along x-axis and the distance OB = y measured along y-axis determine the position of the point P with reference to these axes. The distance OA measured along the axis of x is called the **abscissa** or x-coordinate and the distance OB (=PA) measured along $y-$axis is called the **ordinate** or y-coordinate of the point P. The abscissa and the ordinate taken together are called the **coordinates** of the point P. Thus, the coordinates of the point P are (x and y) which represent the position of the point P point in a plane. These two numbers are to form an **ordered pair** beacuse the order in which we write these numbers is important.

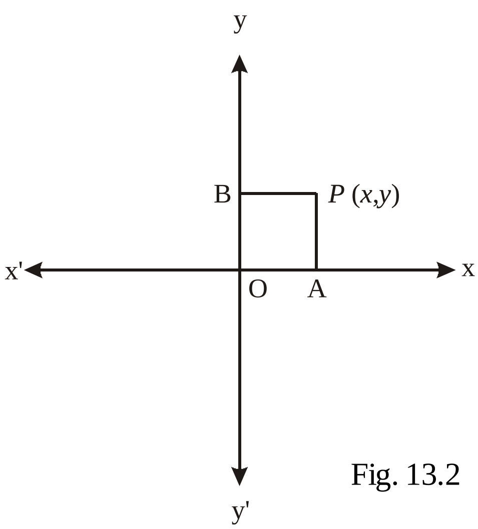


Fig. 13.2

In Fig. 13.3 you may note that the position of the ordered pair (3,2) is different from that of (2,3). Thus, we can say that (x,y) and (y,x) are two different ordered pairs representing two different points in a plane.

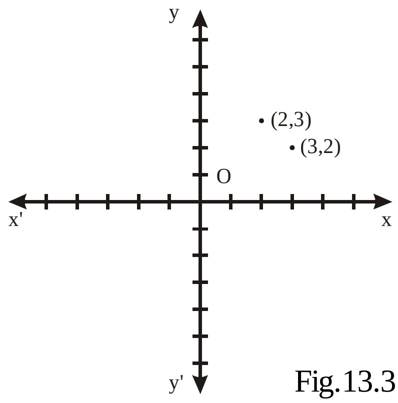


Fig.13.3

13.1.2 QUARDRANTS

We know that coordinate axes XOX' and YOY' divide the region of the plane into four regions. These regions are called the quardrants as shown in Fig. 13.4. In accordance with the convention of signs, for a point P (x,y) in different quadrants, we have

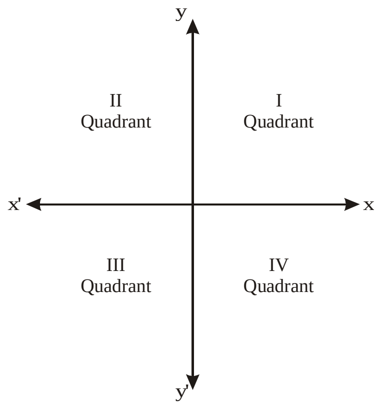


Fig.13.4

I	quadrant :	$x > 0, y > 0$
II	quadrant :	$x < 0, y > 0$
III	quadrant :	$x < 0, y < 0$
IV	quadrant :	$x > 0, y < 0$

13.2 DISTANCE BETWEEN TWO POINTS

Recall that you have derived the distance formula between two points P (x_1,y_1) and Q (x_2,y_2) in the following manner:

Let us draw a line $l \parallel XX'$ through *P*. Let *R* be the point of intersection of the perpendicular from Q to the line l. Then $\Delta\, PQR$ is a right-angled triangle.

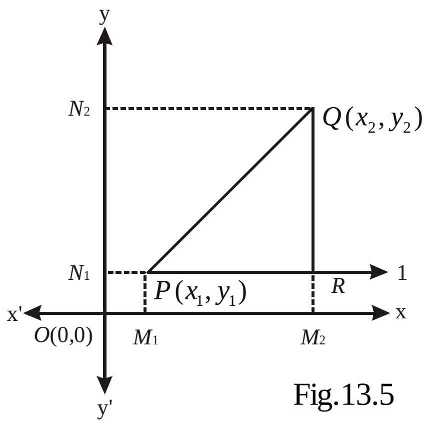


Fig.13.5

$$\text{Also } PR = M_1M_2$$

$$= OM_2 - OM_1$$

$$= x_2 - x_1$$

$$\text{and } QR = QM_2 - RM_2$$

$$= QM_2 - PM_1$$
$$= ON_2 - ON_1$$
$$= y_2 - y_1$$

Now $PQ^2 = PR^2 + QR^2$ (Pythagoras theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : This formula holds for points in all quadrants.

Also the distance of a point P(x,y) from the origin O(0,0)

is OP $= \sqrt{x^2 + y^2}$.

Let us illustrate the use of these formulae with some examples.

Example 13.1 Find the distance between the following pairs of points :

(*i*) $A(14,3)$ and $B(10,6)$ (*ii*) $M(-1,2)$ and $N(0,-6)$

Solution :

(*i*) Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $x_1 = 14$, $y_1 = 3$, $x_2 = 10$, $y_2 = 6$

$\therefore$ Distance between A and B $= \sqrt{(10-14)^2 + (6-3)^2}$

$$= \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Distance between A and B is 5 units.

(*ii*) Here $x_1 = -1$, $y_1 = 2$, $x_2 = 0$ and $y_2 = -6$

Distance between A and B $= \sqrt{(0-(-1))^2 + (-6-2)^2} = \sqrt{1+(-8)^2}$

$$= \sqrt{1+64} = \sqrt{65}$$

Distance between M and N $= \sqrt{65}$ units

Example 13.2 Show that the points P(–1, –1), Q(2, 3) and R (–2, 6) are the vertices of a right-angled triangle.

Solution: $PQ^2 = (2+1)^2 + (3+1)^2 = 3^2 + 4^2 = 9 + 16 = 25$

$QR^2 = (-4)^2 + (3)^2 = 16 + 9 = 25$

and $RP^2 = 1^2 + (-7)^2 = 1 + 49 = 50$

$\therefore$ $PQ^2 + QR^2 = 25+25=50= RP^2$

$\Rightarrow$ ΔPQR is a right-angled triangle (by converse of Pythagoras Theorem)

Example 13.3 Show that the points A(1, 2), B(4, 5) and C(–1, 0) lie on a straight line.

Solution: Here,

$$AB = \sqrt{(4-1)^2 + (5-2)^2} \text{ units} = \sqrt{18} \text{ units} = 3\sqrt{2} \text{ units}$$

$$BC = \sqrt{(-1-4)^2 + (0-5)^2} \text{ units} = \sqrt{50} \text{ units} = 5\sqrt{2} \text{ units}$$

and $$AC = \sqrt{(-1-1)^2 + (0-2)^2} \text{ units} = \sqrt{4+4} \text{ units} = 2\sqrt{2} \text{ units}$$

Now $$AB + AC = \left(3\sqrt{2} + 2\sqrt{2}\right) units = 5\sqrt{2}\, units = BC$$

i.e. BA + AC = BC

Hence, A, B, C lie on a straight line. In other words, A,B,C are collinear.

Example 13.4 Prove that the points (2a, 4a), (2a, 6a) and $\left(2a + \sqrt{3}a, 5a\right)$ are the vertices of an equilateral triangle whose side is 2a.

Solution: Let the points be A (2a, 4a), B (2a, 6a) and C $\left(2a + \sqrt{3}a, 5a\right)$

$$AB = \sqrt{0 + (2a)^2} = 2a \text{ units}$$

$$BC = \sqrt{(\sqrt{3}a)^2 + (-a)^2} \text{ units} = \sqrt{3a^2 + a^2} = 2a \text{ units}$$

and $$AC = \sqrt{(\sqrt{3}a)^2 + (+a)^2} = 2a \text{ units}$$

$\Rightarrow$ AB + BC > AC, BC + AC > AB and

AB + AC > BC and AB = BC = AC = 2a

$\Rightarrow$ A, B, C form the vertices of an equilateral triangle of side 2a.

CHECK YOUR PROGRESS 13.1

1. Find the distance between the following pairs of points.

   (a) (5, 4) and (2, –3) (b) (a, –a) and (b, b)

2. Prove that each of the following sets of points are the vertices of a right angled-trangle.

   (a) (4, 4), (3, 5), (–1,–1) (b) (2, 1), (0, 3), (–2, 1)

3. Show that the following sets of points form the vertices of a triangle:

   (a) (3, 3), (–3, 3) and (0, 0) (b) (0, a), (a, b) and (0, 0) (if ab = 0)

4. Show that the following sets of points are collinear :

   (a) (3, –6), (2, –4) and (–4, 8) (b) (0,3), (0, –4) and (0, 6)

5. (a) Show that the points (0, –1), (–2, 3), (6, 7) and (8, 3) are the vertices of a rectangle.

   (b) Show that the points (3, –2), (6, 1), (3, 4) and (0, 1) are the vertices of a square.

13.3 SECTION FORMULA

13.3.1 INTERNAL DIVISION

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points on a line l and $R(x, y)$ divide PQ internally in the ratio $m_1 : m_2$.

To find : The coordinates x and y of point R.

Construction : Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and L, M and N lie on XX'. Also draw $RT \perp QN$ and $PV \perp QN$.

Method : R divides PQ internally in the ratio $m_1 : m_2$.

$$\Rightarrow \quad R \text{ lies on } PQ \text{ and } \frac{PR}{RQ} = \frac{m_1}{m_2}$$

Also, in triangles, RPS and QRT,

$$\angle RPS = \angle QRT \qquad \text{(Corresponding angles as } PS \parallel RT\text{)}$$

and $\quad \angle RSP = \angle QTR = 90^0$

$\therefore \quad \Delta RPS \sim \Delta QRT \quad$ (AAA similarity)

$$\Rightarrow \quad \frac{PR}{RQ} = \frac{RS}{QT} = \frac{PS}{RT} \qquad \ldots (i)$$

Also, $PS = LM = OM - OL = x - x_1$

$RT = MN = ON - OM = x_2 - x$

$RS = RM - SM = y - y_1$

$QT = QN - TN = y_2 - y$.

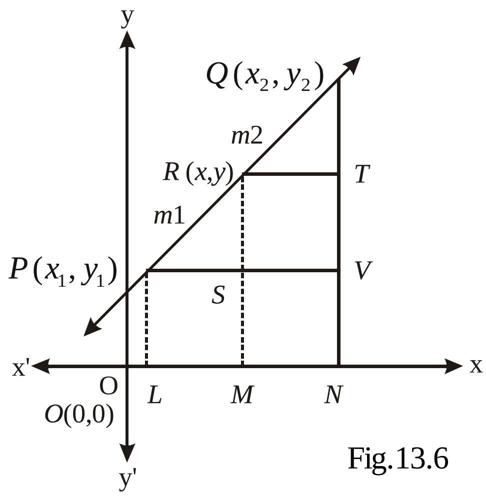


Fig.13.6

From (i), we have

$$\therefore \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow m_1(x_2 - x) = m_2(x - x_1)$$

and $\quad m_1(y_2 - y) = m_2(y - y_1)$

$$\Rightarrow \quad x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Thus, the coordinates of R are:

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

Coordinates of the mid-point of a line segment

If R is the mid point of PQ, then,

$m_1 = m_2 = 1$ (as R divides PQ in the ratio 1:1

Coordinates of the mid point are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

13.3.2 EXTERNAL DIVISION

Let R divide PQ externally in the ratio $m_1 : m_2$

To find : The coordinates of R.

Construction : Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and $PS \perp RM$ and $QT \perp RM$.

Clearly, $\Delta RPS \sim \Delta RQT$.

$$\therefore \frac{RP}{RQ} = \frac{PS}{QT} = \frac{RS}{RT}$$

or $\quad \dfrac{m_1}{m_2} = \dfrac{x - x_1}{x - x_2} = \dfrac{y - y_1}{y - y_2}$

$$\Rightarrow m_1(x - x_2) = m_2(x - x_1)$$

and $m_1(y - y_2) = m_2(y - y_1)$

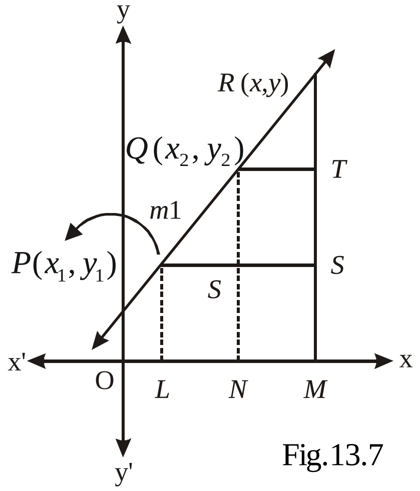


Fig.13.7

These give:

$$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2} \quad and \quad y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$$

Hence, the coordinates of the point of external division are

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \right)$$

Let us now take some examples.

Example 13.5 Find the coordinates of the point which divides the line segment joining the points (4, –2) and (–3, 5) internally and externally in the ratio 2:3.

Solution:

(i) Let $P(x, y)$ be the point of internal division.

$$\therefore\ x = \frac{2(-3)+3(4)}{2+3} = \frac{6}{5} \text{ and } y = \frac{2(5)+3(-2)}{2+3} = \frac{4}{5}$$

$\therefore$ P has coordinates $\left(\frac{6}{5}, \frac{4}{5}\right)$

If $Q(x', y')$ is the point of external division, then

$$x' = \frac{(2)(-3)-3(4)}{2-3} = 18 \text{ and } y' = \frac{(2)(5)-3(-2)}{2-3} = -16$$

Thus, the coordinates of the point of external division are (18, –16).

Example 13.6 In what ratio does the point (3, –2) divide the line segment joining the points (1,4) and (–3, 16) ?

Solution : Let the point $P(3, -2)$ divide the line segement in the ratio $k : 1$.

Then the coordinates of P are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1} \right)$

But the given coordinates of P are $(3, -2)$

$$\therefore \quad \frac{-3k+1}{k+1} = 3 \Rightarrow -3k+1 = 3k+3 \qquad \Rightarrow k = -\frac{1}{3}$$

$\Rightarrow P$ divides the line segement externally in the ratio 1:3.

Example 13.7 The vertices of a quadrilateral *ABCD* are respectively (1, 4), (–2,1), (0, –1) and (3, 2). If *E, F, G, H* are respectively the midpoints of *AB, BC, CD* and *DA*, prove that the quadrilateral *EFGH* is a parallelogram.

Solution : Since *E, F, G,* and *H,* are the midpoints of the sides *AB, BC, CD* and *DA,* therefore, the coordinates of *E, F, G,* and *H* respectively are :

$$\left(\frac{1-2}{2},\frac{4+1}{2}\right),\left(\frac{-2+0}{2},\frac{1-1}{2}\right),\left(\frac{0+3}{2},\frac{-1+2}{2}\right) \text{ and } \left(\frac{1+3}{2},\frac{4+2}{2}\right)$$

$$\Rightarrow\ E\left(\frac{-1}{2},\frac{5}{2}\right),\ F(-1,0), G\left(\frac{3}{2},\frac{1}{2}\right) \text{ and } H(2,\ 3) \text{ are the required points.}$$

Also, the mid point of diagonal *EG* has coordinates

$$\left(\frac{\frac{-1}{2}+\frac{3}{2}}{2},\frac{\frac{5}{2}+\frac{1}{2}}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$$

Coordinates of midpoint of *FH* are $\left(\frac{-1+2}{2},\frac{0+3}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$

Since, the midpoints of the diagonals are the same, therefore, the diagonals bisect each other.

Hence *EFGH* is a parallelogram.

CHECK YOUR PROGRESS 13.2

1. Find the midpoint of each of the line segements whose end points are given below:

   (a) (–2, 3) and (3, 5) (b) (6,0) and (–2,10)

2. Find the coordinates of the point dividing the line segment joining

   (–5, –2) and (3, 6) internally in the ratio 3:1.

3. (a) Three vertices of a parallelogram are (0,3), (0,6) and (2,9). Find the fourth vertex.

   (b) (4, 0), (– 4, 0), (0,– 4) and (0, 4) are the vertices of a square. Show that the quadrilateral formed by joining the midpoints of the sides is also a square.

4. The line segement joining (2, 3) and (5, –1) is trisected. Find the points of trisection.

5. Show that the figure formed by joining the midpoints of the sides of a rectangle is a rhombus.

13.4 AREA OF A TRIANGLE

Let us find the area of a triangle whose vertices are $A(x_1, y_1),\ B(x_2, y_2)\ and\ C(x_3, y_3)$

Draw AL, BM and CN perpendiculars to XX'.

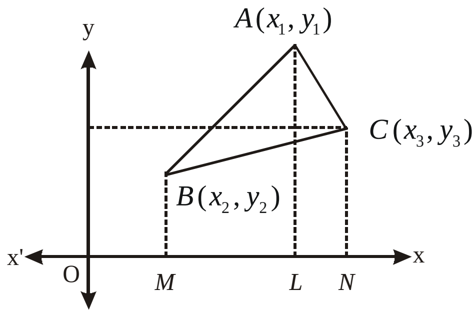


Fig.13.8

area of Δ ABC

= Area of trapzium. $BMLA$ + Area of trapzium. ALNC – Area of trapzium. BMNC

$$= \frac{1}{2}(BM + AL)ML + \frac{1}{2}(AL + CN)LN - \frac{1}{2}(BM + CN)MN$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_3y_3)\right]$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

This can be stated in the determinant form as follows :

$$\text{Area of } \Delta\,\text{ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example 13.8 Find the area of the triangle whose vertices are $A(3, 4)$, $B(6, -2)$ and $C(-4, -5)$.

Solution: The area of $\Delta\, ABC = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 6 & -2 & 1 \\ -4 & -5 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[3(-2+5) - 4(6+4) + 1(-30-8)\right] = \frac{1}{2}\left[9 - 40 - 38\right] = \frac{-69}{2}$$

As the area is to be positive

$\therefore$ Area of $\Delta\, ABC = \frac{69}{2}$ square units

Example 13.9 If the vertices of a triangle are (1, k), (4, –3) and (–9, 7) and its area is 15 square units, find the value(s) of k.

Solution : Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & k & 1 \\ 4 & -3 & 1 \\ -9 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[-3-7-k(4+9)+1(28-27)\right] = \frac{1}{2}\left[-10-13k+1\right] = \frac{1}{2}\left[-9-13k\right]$$

Since the area of the triangle is given to be15,

$\therefore \quad \frac{-9-13k}{2} = 15$ or, $-9-13k = 30$, $-13k = 39$, or, $k = -3$

CHECK YOUR PROGRESS 13.3

1. Find the area of each of the following triangles whose vertices are given below :

   (1) (0, 5), (5, –5), and (0,0) (b) (2, 3), (–2,–3) and (–2, 3)

   (c) (a, 0), (0, – a) and (0, 0)

2. The area of a triangle ABC, whose vertices are A (2, –3), B(3, –2) and C$\left(\frac{5}{2}, k\right)$ is $\frac{3}{2}$ sq unit. Find the value of k

3. Find the area of a rectangle whose vertices are (5, 4), (5,– 4), (–5, 4) and (–5, – 4)

4. Find the area of a quadrilateral whose vertices are (5, –2), (4, –7), (1, 1) and (3, 4)

13.5 CONDITION FOR COLLINEARITY OF THREE POINTS

The three points $A(x_1, y_1)$, $B(x_2, y_2)$ *and* $C(x_3, y_3)$ are collinear if and only if the area of the triangle ABC becomes zero.

i.e. $$\frac{1}{2}\left[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right] = 0$$

i.e. $$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$$

In short, we can write this result as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Let us illustrate this with the help of examples:

Example 13.10 Show that the points $A(a,\ b+c),\ B(b, c+a)\ and\ C(c, a+b)$ are collinear.

Solution : Area of triangle ABC $= \dfrac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2$)

$$= \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = \frac{1}{2}(a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Hence the points are collinear.

Example 13.11 For what value of k, are the points (1, 5) , (k, 1) and (4, 11) collinear ?

Solution : Area of the triangle formed by the given points is

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ k & 1 & 1 \\ 4 & 11 & 1 \end{vmatrix} = \frac{1}{2}[-10-5k+20+11k-4] = \frac{1}{2}[6k+6] = 3k+3$$

Since the given points are collinear, therefore

$3k + 3 = 0 \Rightarrow k = -1$

Hence, for $k = -1$, the given points are collinear.

CHECK YOUR PROGRESS 13.4

1. Show that the points (–1,–1), (5, 7) and (8, 11) are collinear.
2. Show that the points (3, 1), (5, 3) and (6, 4) are collinear.
3. Prove that the points $(a, 0)$, $(0, b)$ and (1, 1) are collinear if $\dfrac{1}{a} + \dfrac{1}{b} = 1$.
4. If the points $(a,\ b), (a_1, b_1)\ and\ (a-a_1,\ b-b_1)$ are collinear, show that $a_1 b = ab_1$

5. Find the value of k for which the points (5, 7), (k, 5) and (0, 2) are collinear.

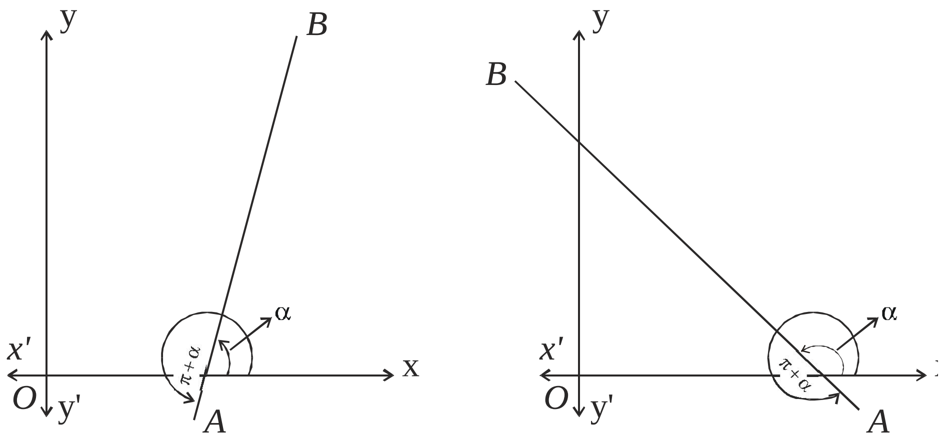
6. Find the values of k for which the point (k, 2–2k), (– k+1, 2k) and (– 4 – k, 6 – 2k) are collinear.

13.6 INCLINATION AND SLOPE OF A LINE

Look at the Fig. 13.9. The line AB makes an angle or $\pi+\alpha$ with the x-axis (measured in anticlockwise direction).

The *inclination* of the given line is represented by the measure of angle made by the line with the positive direction of x-axis (measured in anticlockwise direction)

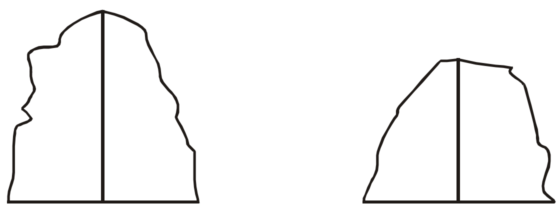
In a special case when the line is parallel to x-axis or it coincides with the x-*axis*, the inclination of the line is defined to be 0^0.



(a) (b)

Fig. 13.9

Again look at the pictures of two mountains given below. Here we notice that the mountain in Fig. 13.10 (a) is more steep compaired to mountain in Fig. 13.10 (b).



(a) Fig.13.10 (b)

How can we quantify this steepness ? Here we say that the angle of inclination of mountain (a) is more than the angle of inclination of mountain (b) with the ground.

Try to see the difference between the ratios of the maximum height from the ground to the base in each case.

Naturally, you will find that the ratio in case (a) is more as compaired to the ratio in case (b). That means we are concerned with height and base and their ratio is linked with tangent of an angle, so mathematically this ratio or the tangent of the inclination is termed as *slope*. We define the slope as tangent of an angle.

The *slope* of a line is the the tangent of the angle θ (say) which the line makes with the positive direction of x-axis. Generally, it is denoted by m $(= \tan\theta)$

Note : If a line makes an angle of 90^0 or 270^0 with the x–axis, the slope of the line can not be defined.

Example 13.12 In Fig. 13.9 find the slope of lines *AB* and *BA*.

Solution : Slope of line *AB* = $\tan\alpha$

Slope of line *BA* = $\tan(\pi+\alpha) = \tan\alpha$.

Note : From this example, we can observe that "slope is independent of the direction of the line segement".

Example 13.13 Find the slope of a line which makes an angle of 30^0 with the negative direction of *x*-axis.

Solution : Here $\theta = 180^0 - 30^0 = 150^0$

$\therefore$ m = slope of the line = $\tan(180^0 - 30^0)$

$= -\tan 30^0$

$= -\dfrac{1}{\sqrt{3}}$

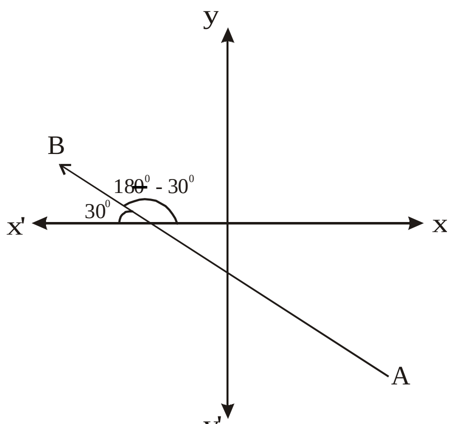


Fig. 13.11

Example 13.14 Find the slope of a line which makes an angle of 60^0 with the positive direction of *y*-axis.

Solution : Here $\theta = 90^0 + 60^0$

$\therefore$ m = slope of the line

$= \tan(90^0 + 60^0)$

$= -\cot 60^0$

$= -\tan 30^0 \quad = -\dfrac{1}{\sqrt{3}}$

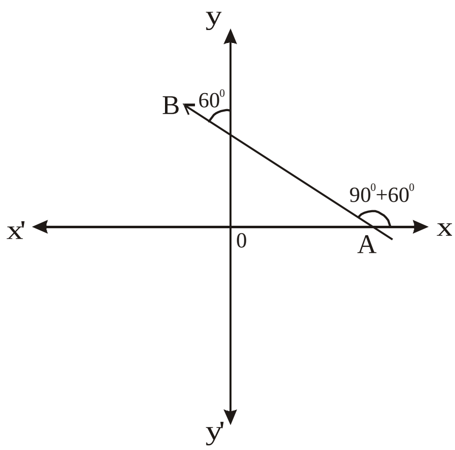


Fig. 13.12

Example 13.15 If a line is equally inclined to the axes, show that its slope is ± 1.

Solution : Let a line *AB* be equally inclined to the axes and meeting axes at points *A* and *B* as shown in the Fig. 13.13

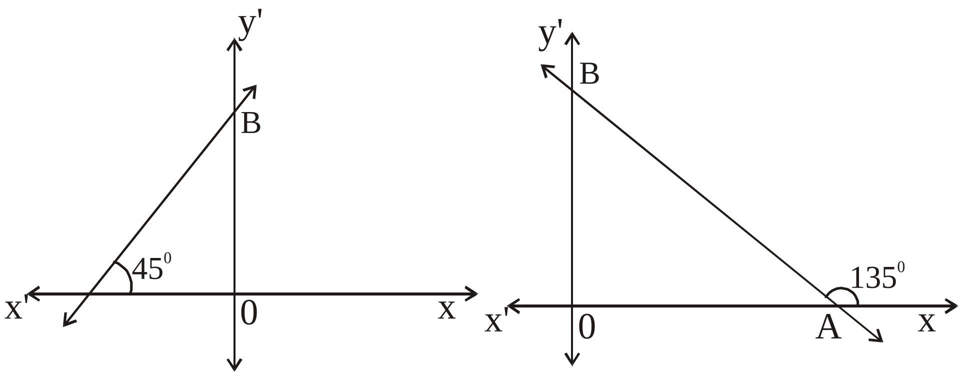


Fig. 13.13

In Fig 13.13(a), inclination of line $AB = \angle XAB = 45^0$

$\therefore$ Slope of the line $AB = \tan 45^0 = 1$

In Fig. 13.13 (b) inclination of line $AB = \angle XAB = 180^0 - 45^0 = 135^0$

$\therefore$ Slope of the line $AB = \tan 135^0 = \tan (180^0 – 45^0) = – \tan 45^0 = –1$

Thus, if a line is equally inclined to the axes, then the slope of the line will be ± 1.

CHECK YOUR PROGRESS 13.5

1. Find the Slope of a line which makes an angle of (i) 60^0, (ii) 150^0 with the positive direction of *x*-axis.
2. Find the slope of a line which makes an angle of 30^0 with the positive direction of *y*-axis.
3. Find the slope of a line which makes an angle of 60^0 with the negative direction of *x*-axis.

13.7 SLOPE OF A LINE JOINING TWO DISTINCT POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points. Draw a line through *A* and *B* and let the inclination of this line be θ. Let the point of intersection of a horizontal line through *A* and a vertical line through *B* be *M*, then the coordinates of *M* are as shown in the Fig. 13.14

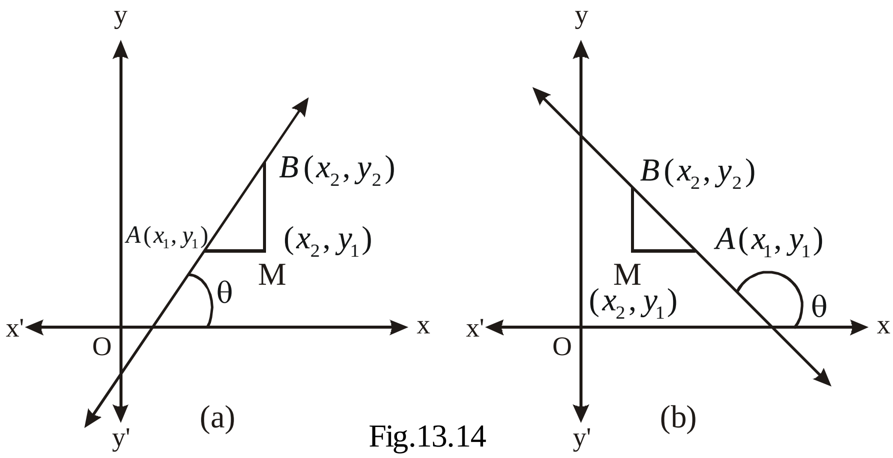


Fig.13.14

(A) In Fig 13.14 (a), angle of inclination *MAB* is equal to θ (acute). Consequently.

$$\tan\theta = \tan(\angle MAB) = \frac{MB}{AM} = \frac{y_2 - y_1}{x_2 - x_1}$$

(B) In Fig. 13.14 (b), angle of inclination θ is obtuse, and since θ and $\angle MAB$ are supplementary, consequently,

$$\tan\theta = -\tan(\angle MAB) = -\frac{MB}{MA} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence in both the cases, the slope *m* of a line through $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

***Note :** if $x_1 = x_2$, then m is not defined. In that case the line is parallel to y-axis.*

Is there a line whose slope is 1? Yes, when a line is inclined at 45^0 with the positive direction of *x*-axis.

Is there a line whose slope is $\sqrt{3}$? Yes, when a line is inclined at 60^0 with the positive direction of *x*-axis.

From the answers to these questions, you must have realised that given any real number *m*, there will be a line whose slope is *m* (because we can always find an angle α such that $\tan\alpha = m$).

Example 13.16 Find the slope of the line joining the points *A*(6, 3) and *B*(4, 10).

Solution : The slope of the line passing through the points (x_1, y_1) and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Here, $x_1 = 6,\ y_1 = 3;\ x_2 = 4,\ y_2 = 10.$

Now substituting these values, we have slope $= \dfrac{10-3}{4-6} = -\dfrac{7}{2}$

Example 13.17 Determine *x*, so that the slope of the line passing through the points (3, 6) and (*x*, 4) is 2.

Solution :

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-6}{x-3} = \frac{-2}{x-3}$$

$$\therefore \frac{-2}{x-3} = 2 \qquad \text{.............. (Given)}$$

$$\therefore \quad 2x - 6 = -2 \quad or \quad x = 2$$

CHECK YOUR PROGRESS 13.6

1. What is the slope of the line joining the points $A(6, 8)$ and $B(4, 14)$?
2. Determine x so that 4 is the slope of the line through the points $A(6,12)$ and $B(x, 8)$.
3. Determine y, if the slope of the line joining the points $A(-8, 11)$ and $B(2, y)$ is $-\dfrac{4}{3}$.
4. $A(2, 3)$ $B(0, 4)$ and $C(-5, 0)$ are the vertices of a triangle ABC. Find the slope of the line passing through *the point B* and the mid point of AC
5. $A(-2, 7)$, B(1, 0), C(4, 3) and D(1, 2) are the vertices of a quadrilateral $ABCD$. Show that

   (i) slope of AB = slope of CD (ii) slope of BC = slope of AD

13.8 CONDITIONS FOR PARALLELISM AND PERPENDI CULARITY OF LINES.

9.8.1 Slope of Parallel Lines

Let l_1, l_2, be two (non-vertical) lines with their slopes m_1 and m_2 respectively.

Let θ_1 and θ_2 be the angles of inclination of these lines respectively.

Case I : Let the lines l_1 *and* l_2 *be parallel*

Then $\theta_1 = \theta_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

Thus, if two lines are parallel then their slopes are equal.

Case II : Let the lines l_1, and l_2 have equal slopes.

i.e. $m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2 \; (0^\circ \le \theta \le 180^\circ)$

$\Rightarrow l_1 \parallel l_2$

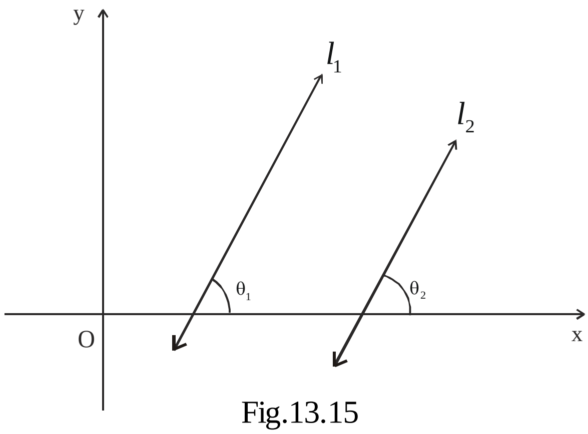
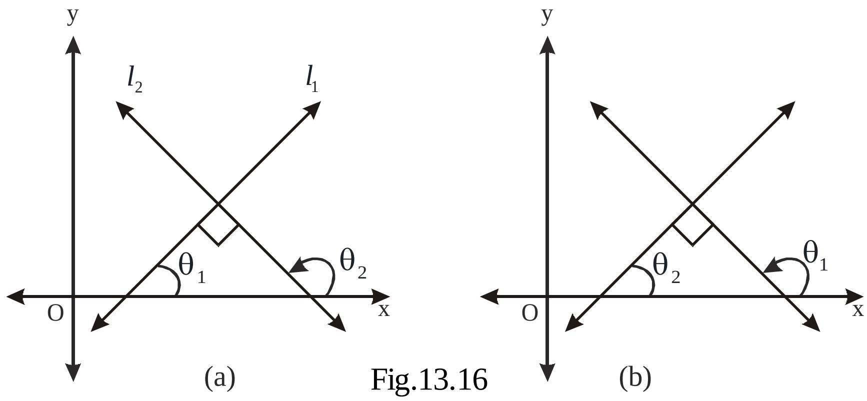


Fig.13.15

Hence, two (non-vertical) lines are parallel if and only if $m_1 = m_2$

13.8.2 SLOPES OF PERPENDICULAR LINES

Let l_1 and l_2 be two (non-vertical)lines with their slopes m_1 and m_2 respectively. Also let θ_1 and θ_2 be their inclinations respectively.



(a) Fig.13.16 (b)

Case-I : Let $l_1 \perp l_2$

$\Rightarrow \theta_2 = 90^0 + \theta_1$ or $\theta_1 = 90^0 + \theta_2$

$\Rightarrow \tan\theta_2 = \tan(90^0 + \theta_1)$ or $\tan\theta_1 = \tan(90^0 + \theta_2)$

$\Rightarrow \tan\theta_2 = -\cot(\theta_1)$ or $\tan\theta_1 = -\cot(\theta_2)$

$\Rightarrow \tan\theta_2 = -\dfrac{1}{\tan\theta_1}$ or $\Rightarrow \tan\theta_1 = -\dfrac{1}{\tan\theta_2}$

$\Rightarrow$ In both the cases, we have

$\tan\theta_1 \tan\theta_2 = -1$

or $m_1.m_2 = -1$

Thus, if two lines are perpendicular then the product of their slopes is equal to –1.

Case II : Let the two lines l_1 and l_2 be such that the product of their slopes is –1.

i.e. $m_1.m_2 = -1$

$$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$$

$$\Rightarrow \tan\theta_1 = -\frac{1}{\tan\theta_2} = -\cot\theta_2 = \tan\left(90^0 + \theta_2\right)$$

or

$$\tan\theta_2 = \frac{-1}{\tan\theta_1} = -\cot\theta_1 = \tan\left(90 + \theta_1\right)$$

$\Rightarrow$ Either $\theta_1 = 90^0 + \theta_2$ or $\theta_2 = 90^0 + \theta_1$ $\Rightarrow$ In both cases $l_1 \perp l_2$.

Hence, two (non-vertical) lines are perpendicular if and only if $m_1.m_2 = -1$.

Example 13.18 Show that the line passing through the points A(5,6) and B(2,3) is parallel to the line passing, through the points C(9,–2) and D(6,–5).

Solution : Slope of the line AB $= \dfrac{3-6}{2-5} = \dfrac{-3}{-3} = 1$

and slope of the line CD $= \dfrac{-5+2}{6-9} = \dfrac{-3}{-3} = 1$

As the slopes are equal $\therefore$ AB $\parallel$ CD.

Example 13.19 Show that the line passing through the points A(2,–5) and B(–2,5) is perpendicular to the line passing through the points L(6,3) and M(1,1).

Solution : Here

$$m_1 = \text{slope of the line AB} = \frac{5+5}{-2-2} = \frac{10}{-4} = \frac{-5}{2}$$

$$\text{and } m_2 = \text{slope of the line LM} = \frac{1-3}{1-6} = \frac{2}{5}$$

$$\text{Now } m_1.m_2 = \frac{-5}{2} \times \frac{2}{5} = -1$$

Hence, the lines are perpendicular to each other.

Example 13.20 Using the concept of slope, show that A(4,4), B(3,5) and C (-1,-1) are the vertices of a right triangle.

Solution : Slope of line AB $= m_1 = \dfrac{5-4}{3-4} = -1$

Slope of line BC $= m_2 = \dfrac{-1-5}{-1-3} = \dfrac{3}{2}$

and slope of line AC $= m_3 = \dfrac{-1-4}{-1-4} = 1$

Now $m_1 \times m_3 = -1 \quad \Rightarrow AB \perp AC$

$\Rightarrow \triangle ABC$ is a right-angled triangle.

Hence, A(4,4), B(3,5) and C(– 1, – 1) are the vertices of right triangle.

Example 13.21 What is the value of y so that the line passing through the points A(3,y) and B(2,7) is perpendicular to the line passing through the point C (– 1,4) and D (0,6)?

Solution : Slope of the line AB $= m_1 = \dfrac{7-y}{2-3} = y-7$

Slope of the line CD $= m_2 = \dfrac{6-4}{0+1} = 2$

Since the lines are perpendicular,

$\therefore m_1 \times m_2 = -1$ or $(y-7)\times 2 = -1$

or $2y - 14 = -1$ or $2y = 13$ or $y = \dfrac{13}{2}$

CHECK YOUR PROGRESS 13.7

1. Show that the line joining the points (2,–3) and (– 4,1) is

   (i) parallel to the line joining the points (7,–1) and (0,3).

   (ii) perpendicular to the line joining the points (4,5) and (0,–2).

2. Find the slope of a line parallel to the line joining the points (– 4,1) and (2,3).

3. The line joining the points (–5,7) and (0,–2) is perpendicular to the line joining the points (1,3) and (4,x). Find x.

4. A(–2,7), B(1,0), C(4,3) and D(1,2) are the vertices of quadrilateral ABCD. Show that the sides of ABCD are parallel.

5. Using the concept of the slope of a line, show that the points A(6, – 1), B(5,0) and C(2,3) are collinear.[Hint: slopes of AB, BC and CA must be equal.]

6. Find k so that line passing through the points (k,9) and (2,7) is parallel to the line passing through the points (2,–2) and (6,4).

7. Using the concept of slope of a line, show that the points (–4,–1), (–2–4), (4,0) and (2,3) taken in the given order are the vertices of a rectangle.

8. The vertices of a triangle ABC are A(–3,3), B(–1,– 4) and C(5,–2). M and N are the midpoints of AB and AC. Show that MN is parallel to BC and MN = $\frac{1}{2}$ BC.

13.9 INTERCEPTS MADE BY A LINE ON AXES

If a line l (not passing through the Origin) meets x-axis at A and y-axis at B as shown in Fig. 13.17, then

(i) OA is called the x-intercept or the intercept made by the line on x-axis.

(ii) OB is called y-intercept or the intercept made by the line on y-axis.

(iii) OA and OB taken together in this order are called the intercepts made by the line l on the axes.

(iv) AB is called the portion of the line intercepted between the axes.

(v) The coordinates of the point A on x-axis are $(a,0)$ and those of point B are $(0,b)$

To find the intercept of a line in a given plane on x-axis, we put $y = 0$ in the given equation of a line and the value of x so obtained is called the x intercept.

To find the intercept of a line on y-axis we put $x = 0$ and the value of y so obtained is called the y intercept.

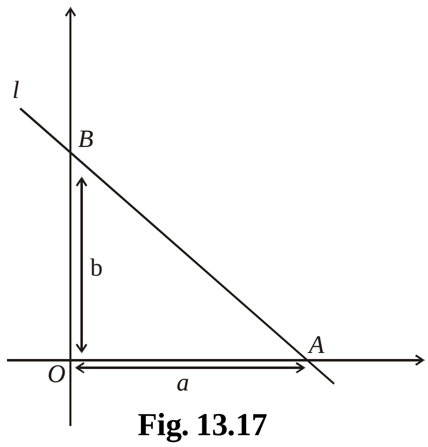


Fig. 13.17

Note: 1. A line which passes through origin makes no intercepts on axes.

2. A horizontal line has no x-intercept and vertical line has no y-intercept.

3. The intercepts on x- axis and y-axis are usually denoted by a and b respectively. But if only y-intercept is considered, then it is usually denoted by c.

Example 13.22 If a line is represented by $2x+3y=6$, find its x and y intercepts.

Solution : The given equation of the line is $2x+3y=6$ ... (i)

Putting $x = 0$ in (i), we get y = 2

Thus, y-intercept is 2.

Again putting $y = 0$ in (i), we get $2x = 6 \Rightarrow x = 3$

Thus, x-intercept is 3.

CHECK YOUR PROGRESS 13.8

1. Find x and y intercepts, if the equations of lines are :

(i) $x+3y=6$ (ii) $7x+3y=2$ (iii) $\frac{x}{2a}+\frac{y}{2b}=1$ (iv) $ax+by=c$

(v) $\frac{y}{2}-2x=8$ (vi) $\frac{y}{3}-\frac{2x}{3}=7$

13.10 ANGLE BETWEEN TWO LINES

Let l_1 and l_2 be two non vertical and non perpendicualr lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be the angles subtended by l_1 and l_2 respectively with the positive direction of x-axis. Then $m_1 = \tan \alpha_1$ and $m_2 = \tan \alpha_2$.

From figure 1, we have $\alpha_1 = \alpha_2 + \theta$

$\therefore \quad \theta = \alpha_1 - \alpha_2$

$\Rightarrow \quad \tan\theta = \tan(\alpha_1 - \alpha_2)$

i.e. $$\tan\theta = \frac{\tan\alpha_1 - \tan\alpha_2}{1+\tan\alpha_1 . \tan\alpha_2}$$

i.e. $$\tan\theta = \frac{m_1 - m_2}{1+m_1 m_2} \quad ...(1)$$

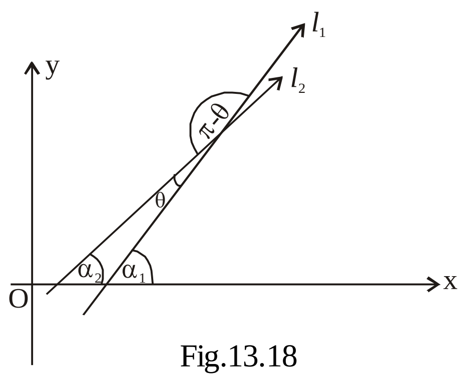


Fig.13.18

As it is clear from the figure that there are two angles θ and $\pi - \theta$ between the lines l_1 and l_2.

We know, $\tan(\pi-\theta) = -\tan\theta$

$$\therefore \quad \tan(\pi-\theta) = -\left(\frac{m_1-m_2}{1+m_1 m_2}\right)$$

Let $\pi - \theta = \phi$

$\therefore$ $$\tan\phi = -\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \quad \text{...(2)}$$

- If $\dfrac{m_1 - m_2}{1 + m_1 m_2}$ is positive then $\tan\theta$ is positive and $\tan\phi$ is negative i.e. θ is acute and ϕ is obtuse.

- If $\dfrac{m_1 - m_2}{1 + m_1 m_2}$ is negative then $\tan\theta$ is negative and $\tan\phi$ is positive i.e. θ is obtuse and ϕ is acute.

Thus the acute angle (say θ) between lines l_1 and l_2 with slopes m_1 and m_2 respectively is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{where } 1 + m_1 m_2 \neq 0.$$

The obtuse angle (say ϕ) can be found by using the formula $\phi = 180° - \theta$.

Example 13.23 Find the acute and obtuse angles between the lines whose slopes are $\dfrac{3}{4}$ and $\dfrac{-1}{7}$.

Solution : Let θ and ϕ be the acute and obtuse angle between the lines respectively.

$\therefore$ $$\tan\theta = \left|\frac{\frac{3}{4} + \frac{1}{7}}{1 + \left(\frac{3}{4}\right)\left(\frac{-1}{7}\right)}\right| = \left|\frac{21 + 4}{28 - 3}\right| = |1| = 1$$

$\Rightarrow$ $\theta = 45°$

$\therefore$ $\phi = 180° - 45° = 135°$.

Example 13.24 Find the angle (acute or obtuse) between x-axis and the line joining the points (3, –1) and (4, –2),

Solution : Slope of x-axis (say m_1) = 0

Slope of given line (say m_2) = $\dfrac{-2 + 1}{4 - 3} = -1$

$\therefore$ $$\tan\theta = \left|\frac{0 + 1}{1 + (0)(-1)}\right| = 1$$

$\Rightarrow$ $\theta = 45°$ as acute angle.

Example 13.25 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution : Here,
$$\tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \left(\frac{1}{2}\right)(m_2)}\right|$$

$$\Rightarrow \quad \left|\frac{1 - 2m_2}{2 + m_2}\right| = 1$$

$$\Rightarrow \quad \frac{1 - 2m_2}{2 + m_2} = 1 \text{ or } \frac{1 - 2m_2}{2 + m_2} = -1.$$

$$\Rightarrow \quad m_2 = -\frac{1}{3} \text{ or } m_2 = 3.$$

$\therefore$ Slope of other line is 3 or $-\frac{1}{3}$.

CHECK YOUR PROGRESS 13.9

1. Find the acute angle between the lines with slopes 5 and $\frac{2}{3}$.

2. Find the obtuse angle between the lines with slopes 2 and –3.

3. Find the acute angle between the lines l_1 and l_2 where l_1 is formed by joining the points (0, 0) and (2, 3) and l_2 by joining the points (2, –2) and (3, 5)

13.11 SHIFTING OF ORIGIN :

We know that by drawing x-axis and y-axis, any plane is divided into four quadrants and we represent any point in the plane as an ordered pair of real numbers which are the lengths of perpendicular distances of the point from the axes drawn. We also know that these axes can be chosen arbitrarily and therefore the position of these axes in the plane is not fixed. Position of the axes can be changed. When we change the position of axes, the coordinates of a point also get changed correspondingly. Consequently equations of curves also get changed.

The axes can be changed or transformed in the following ways :

(i) Translation of axes (ii) Rotation of axes (iii) Translation and rotation of axes. In the present section we shall discuss only one transformation i.e. translation of axes.

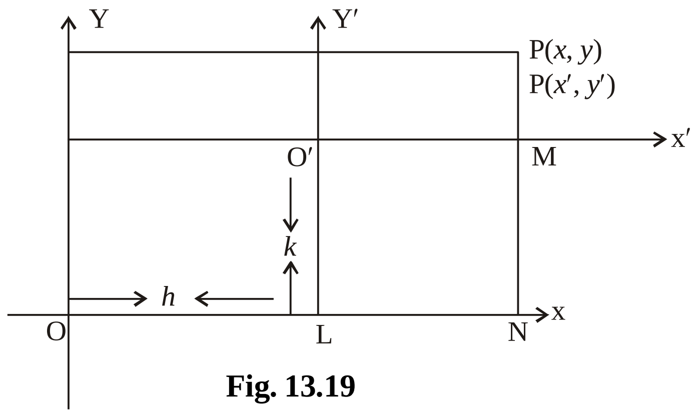


Fig. 13.19

The transformation obtained, by shifting the origin to a given point in the plane, without changing the directions of coordinate axes is called **translation of axes**.

Let us see how coordinates of a point in a plane change under a translation of axes. Let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be the given coordinate axes. Suppose the origin O is shifted to O′(h, k) by the translation of the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ be the new axes as shown in the above figure. Then with reference to $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ the point O' has coordinates (0, 0).

Let P be a point with coordinates (x, y) in the system $\overrightarrow{OX}$ and $\overrightarrow{OY}$ and with coordinates (x', y') in the system $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$. Then $O'L = K$ and $OL = h$.

Now $\quad x = \text{ON} = \text{OL} + \text{LN}$

$$= \text{OL} + \text{O}'\text{M}$$

$$= h + x'.$$

and $y = \text{PN} = \text{PM} + \text{MN} = \text{PM} + \text{O}'\text{L} = y' + k.$

Hence $x = x' + h;\ y = y' + k$

or $x' = x - h,\ y' = y - k$

- If the origin is shifted to (h, k) by translation of axes then coordinates of the point P(x, y) are transformed to P($x - h$, $y - k$) and the equation F(x, y) = 0 of the curve is transformed to F($x' + h$, $y' + k$) = 0.

- Translation formula always hold, irrespective of the quadrant in which the origin of the new system happens to lie.

Example 13.26 When the origin is shifted to (–3, 2) by translation of axes find the coordinates of the point (1, 2) with respect to new axes.

Solution : Here $(h, k) = (-3, 2)$, $(x, y) = (1, 2)$, $(x', y') = ?$

$x' = x - h = 1 + 3 = 4$

$y' = y - k = 2 - 2 = 0$

Therefore $(x', y') = (4, 0)$

Example 13.27 When the origin is shifted to the point (3, 4) by the translation of axes, find the transformed equation of the line $3x + 2y - 5 = 0$.

Solution : Here $(h, k) = (3, 4)$

$\therefore \quad x = x' + 3$ and $y = y' + 4$.

Substituting the values of x and y in the equation of line

we get $3(x' + 3) + 2(y' + 4) - 5 = 0$

i.e. $3x' + 2y' + 12 = 0$.

CHECK YOUR PROGRESS 13.10

1. (i) Does the length of a line segment change due to the translation of axes? Say yes or no.

   (ii) Are there fixed points with respect to translation of axes? Say yes or no.

   (iii) When the origin is shifted to the point (4, –5) by the translation of axes, the coordinates of the point (0, 3) are ...

   (iv) When the origin is shifted to (2, 3), the coordinates of a point P changes to (4, 5), coordinates of point P in original system are ...

   (v) If due to translation of axes the point (3, 0) changes to (2, –3), then the origin is shifted to the point ...

LET US SUM UP

- Distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- Coordinates of the point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$$

- Coordinates of the point dividing the line segment joining the the points (x_1, y_1) and (x_2, y_2) externally are in the ratio $m_1 : m_2$ are.

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

- Coordinates of the mid point of the line segment joining the points (x_1,y_1) and (x_2, y_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The area of a triangle with vertices (x_1, y_1) and (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

- Three points A, B, and C are collinear if the area of the triangle formed by them is zero.
- If θ is the angle which a line makes with the positive direction of *x*-axis, then the slope of the line is $m = \tan\theta$.
- Slope (m) of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- A line with the slope m_1 is parallel to the line with slope m_2 if $m_1 = m_2$.
- A line with the slope m_1 is perpendicular to the line with slope m_2 if $m_1 \times m_2 = -1$.
- If a line *l* (not passing through the origin) meets *x*- axis at A and y- axis at B then OA is called the *x*- intercept and OB is called the y- intercept.
- If θ be the angle between two lines with slopes m_1 and m_2, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

  where $1 + m_1m_2 \neq 0$

- If $\tan\theta$ is +ve, the angle (θ) between the lines is acute and if $\tan\theta$ is -ve then it is obtuse.
- When origin is shifted to (h,k) then transformed coordinates (x',y') (say) of a point P (x,y) are (x-h, y-k)

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=VhNkWdLGpmA
http://www.youtube.com/watch?v=5ctsUsvIp8w
http://www.youtube.com/watch?v=1op92ojA6q0

TERMINAL EXERCISE

1. Find the distance between the pairs of points:

   (a) (2, 0) and (1, cot θ) (b) (–sin A, cosA) and (sin B, cos B)

2. Which of the following sets of points form a triangle?

   (a) (3, 2), (–3, 2) and (0, 3) (b) (3, 2), (3, –2) and (3, 0)

3. Find the midpoint of the line segment joining the points (3. –5) and (–6, 8).

4. Find the area of the triangle whose vertices are:

   (a) (1, 2), (–2, 3), (–3, –4) (b)(c, a), (c + a, a), (c – a, –a)

5. Show that the following sets of points are collinear (by showing that area formed is 0).

   (a) (–2, 5) (2, –3) and (0, 1) (b) (a, b + c), (b, c + a) and (c, a + b)

6. If (–3, 12), (7, 6) and (x, a) are collinear, find x.

7. Find the area of the quadrilateral whose vertices are (4,3) (–5,6) (0,7) and (3,–6).

8. Find the slope of the line through the points

   (a) (1,2), (4,2) (b) (4, –6), (–2, –5)

9. What is the value of y so that the line pasing through the points (3, y) and (2,7) is parallel to the line passing through the points (–1, 4) and (0, 6) ?

10. Without using Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right-angled triangle.

11. Using the concept of slope, determine which of the following sets of points are collnear:

    (i) (–2, 3) , (8, –5) and (5, 4), (ii) (5, 1) , (1, –1) and (11, 4),

12. If A (2, –3) and B (3, 5) are two vertices of a rectangle ABCD, find the slope of

    (i) BC (ii) CD (iii) DA.

13. A quadrilateral has vertices at the points (7, 3), (3, 0), (0, –4) and (4, –1). Using slopes, show that the mid-points of the sides of the quadrilatral form a parallelogram.

14. Find the x-intercepts of the following lines:

    (*i*) $2x-3y=8$ (*ii*) $3x-7y+9=0$ (*iii*) $x-\dfrac{y}{2}=3$

15. When the origin is shifted to the point (3, 4) by translation of axes, find the transformed equation of $2x^2+4xy+5y^2=0$.

16. If the origin is shifted to the point (3, –4), the transformed equation of a curve is $(x^1)^2+(y^1)^2=4$, find the original equation of the curve.

17. If A(–2, 3), B(3, 8) and C(4, 1) are the vertices of a ΔABC. Find $\angle$ABC of the triangle.

18. Find the acute angle between the diagonals of a quadrilateral ABCD formed by the points A(9, 2), B(17, 11), C(5, – 3) and D(–3, –2) taken in order.

19. Find the acute angle between the lines AB and BC given that A = (5, –3), B = (–3, –2) and C = (9, 12).

ANSWERS

CHECK YOUR PROGRESS 13.1

(a) $\sqrt{58}$ (b) $\sqrt{2(a^2+b^2)}$

CHECK YOUR PROGRESS 13.2

1. (a) $\left(\frac{1}{2},4\right)$ (b) (2,5) 2. (1,4) 3. (a) (2,6)

4. $\left(3,\frac{5}{3}\right),\left(4,\frac{1}{3}\right)$

CHECK YOUR PROGRESS 13.3

1. (a) $\frac{25}{2}$ sq. units (b) 12 sq. units (c) $\frac{a^2}{2}$ sq. units

2. $k=\frac{5}{3}$ 3. 80 sq. units 4. $\frac{41}{2}$ sq. units

CHECK YOUR PROGRESS 13.4

5. $k = 3$ 6. $k = \frac{1}{2}, -1$

CHECK YOUR PROGRESS 13.5

1. (*i*) $\sqrt{3}$ (*ii*) $-\frac{1}{\sqrt{3}}$ 2. $-\sqrt{3}$ 3. $-\sqrt{3}$

CHECK YOUR PROGRESS 13.6

1. -3 2. 5 3. $-\frac{7}{3}$ 4. $\frac{5}{3}$

CHECK YOUR PROGRESS 13.7

2. $\frac{1}{3}$ 3. $\frac{14}{3}$. 6. $k=\frac{10}{3}$

CECK YOUR PROGRESS 13.8

1. (*i*) x-intercept = 6, y-intercept = 2

(*ii*) x-intercept = $\frac{2}{7}$, y-intercept = $\frac{2}{3}$

(*iii*) x-intercept = 2a, y-intercept = $2b$

(*iv*) x-intercept $= \frac{c}{a}$, y-intercept $= \frac{c}{b}$

(*v*) x-intercept $= -4$, y-intercept $= 16$

(*vi*) x-intercept $= \frac{-21}{2}$, y-intercept $= 21$

CHECK YOUR PROGRESS 13.9

1. $45°$ 2. $135°$ 3. $\tan = \frac{11}{23}$

CHECK YOUR PROGRESS 13.10

1. (i) No (ii) No (iii) (-4,8) (iv) (6,8) (v) (1,3)

TERMINAL EXERCISE

1. (a) cosec θ (b) $2 \sin \frac{A+B}{2}$

2. None of the given sets forms a triangle.

3. $\left(-\frac{3}{2}, \frac{3}{2}\right)$ 4. (a) 11 sq. unit (b) a^2 sq. unit.

6. $\frac{51-5a}{3}$ 7. 29 sq. unit.

8. (*a*) 0 (b) $-\frac{1}{6}$

9. $y = 3$ 11. Only (*ii*)

12. (*i*) $-\frac{1}{8}$ (*ii*) 8 (*iii*) $-\frac{1}{8}$

14. (*i*) 4 (*ii*) -3 (*iii*) 3

15. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

16. $x^2 + y^2 - 6x + 8y + 21 = 0$

17. $\tan^{-1}\left(\frac{4}{3}\right)$ 18. $\tan^{-1}\left(\frac{48}{145}\right)$ 19. $\tan^{-1}\left(\frac{62}{55}\right)$

14

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STRAIGHT LINES

We have read about lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions. We have also seen that graphs of linear equations, which came out to be straight lines

Interestingly, the reverse problem of the above is finding the equations of straight lines, under different conditions, in a plane. The analytical geometry, more commonly called coordinate geomatry, comes to our help in this regard. In this lesson. We shall find equations of a straight line in different forms and try to solve problems based on those.

OBJECTIVES

After studying this lesson, you will be able to :

- derive equations of a line parallel to either of the coordinate axes;
- derive equations of a line in different forms (slope-intercept, point-slope, two point, intercept, and perpendicular)
- find the equation of a line in the above forms under given conditions;
- state that the general equation of first degree represents a line;
- express the general equation of a line into

  (i) slope-intercept form (ii) intercept form and (iii) perpendicular form;
- derive an expression for finding the distance of a given point from a given line;
- calculate the distance of a given point from a given line;
- derive the equation of a line passing through a given point and parallel/perpendicular to a given line;
- find equation of family of lines passing through the point of intersection of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Congruence and similarity of traingles

14.1 STRAINGHT LINE PARALLEL TO AN AXIS

If you stand in a room with your arms stretched, we can have a line drawn on the floor parallel to one side. Another line perpendicular to this line can be drawn intersecting the first line between your legs.

In this situation the part of the line in front of you and going behind you is the y-axis and the one being parallel to your arms is the x-axis.

The direction part of the y-axis in front of you is positive and behind you is negative.

The direction of the part x-axis to your right is positive and to that to your left is negative.

Now, let the side facing you be at b units away from you, then the equation of this edge will be $y = b$ (parallel to x-axis)

where b is equal in absolute value to the distance from the x-axis to the opposite side.

If $b > 0$, then the line lies in front of you, i.e., above the x-axis.

If $b < 0$, then the line lies behind you, i.e., below the x-axis.

If $b = 0$, then the line passes through you and is the x-axis itself.

Again, let the side of the right of you is at c units apart from you, then the equation of this line will be $x = c$ (parallel to y - axis)

where c is equal in absolute value, to the distance from the y-axis on your right.

If $c > 0$, then the line lies on the right of you, i.e., to the right of y-axis.

If $c < 0$, then the line lies on the left of you, i.e., to the left of y-axis

If $c = 0$, then the line passes through you and is the y-axis.

Example 14.1 Find the equation of the line passing through (–2, –3) and

(i) parallel to x-axis (ii) parallel to y-axis

Solution :

(i) The equation of any line parallel to x-axis is $y = b$

Since it passes through (–2, –3), hence $-3 = b$

$\therefore$ The required equation of the line is $y = -3$

(i) The equation of any line parallel to y-axis is $x = c$

Since it passes through (–2, –3), hence $-2 = c$

$\therefore$ The required cquation of the line is $x = -2$

CHECK YOUR PROGRESS 14.1

1. Find the equation of the line passing through (–3, –4) and

   (a) parallel to x-axis. (b) parallel to y-axis.

2. Find the equation of a line passing through (5, –3) and perpendicular to x-axis.

3. Find the equation of the line passing through (–3, –7) and perpendicular to y-axis.

14.2 DERIVATION OF THE EQUATION OF STRAIGHT LINE IN VARIOUS STANDARD FORMS

So far we have studied about the inclination, slope of a line and the lines parallel to the axes. Now the questions is, can we find a relationship between x and y, where (x, y) is any arbitrary point on the line?

The relationship between x and y which is satisfied by the co-ordinates of arbitrary point on the line is called the equation of a straight line. The equation of the line can be found in various forms under the given conditions, such as

(a) When we are given the slope of the line and its intercept on y-axis.

(b) When we are given the slope of the line and it passes through a given point.

(c) When the line passes through two given points.

(d) When we are given the intercepts on the axes by the line.

(e) When we are given the length of perpendicular from origin on the line and the angle which the perpendicualr makes with the positive direction of x-axis.

We will discuss all the above cases one by one and try to find the equation of line in its standard forms.

(A) SLOPE-INTECEPT FORM

Let AB be a straight line making an angle θ with x-axis and cutting off an intercept $OD = c$ from OY.

As the line makes intercept OD = c on y-axis, it is called y-intercept.

Let AB intersect OX' at T.

Take any point $P(x, y)$ on AB. Draw $PM \perp OX$.

The $OM = x$, $MP = y$.

Draw $DN \perp MP$.

From the right-angled triangle DNP, we have

$$\tan\theta = \frac{NP}{DN} = \frac{MP - MN}{OM}$$

$$= \frac{y - OD}{OM}$$

$$= \frac{y - c}{x}$$

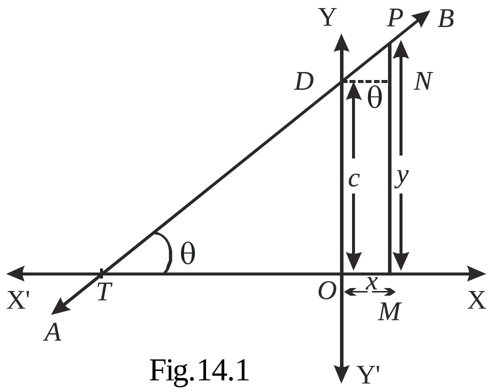

Fig.14.1

$\therefore \quad y = x\tan\theta + c$

$\tan\theta = m$ (slope)

$\therefore \quad y = mx + c$

Since, this equation is true for every point on AB, and clearly for no other point in the plane, hence it represents the equation of the line AB.

Note : (1) When $c = 0$ and $m \neq 0 \Rightarrow$ the line passes through the origin and its equation is $y = mx$

(2) When $c = 0$ and $m = 0 \Rightarrow$ the line coincides with x – axis and its equation is of the form $y = o$

(3) When $c \neq 0$ and $m = 0 \Rightarrow$ the line is parallel to x-axis and its equation is of the form $y = c$

Example 14.2 Find the equation of a line with slope 4 and y-intercept 0.

Solution : Putting $m = 4$ and $c = 0$ in the slope intercept form of the equation, we get $y = 4x$

This is the desired equation of the line.

Example 14.3 Determine the slope and the y-intercept of the line whose equation is

$8x + 3y = 5$.

Solution : The given equation of the line is

$$8x + 3y = 5 \quad \text{or,} \quad y = -\frac{8}{3}x + \frac{5}{3}$$

Comparing this equation with the equation $y = mx + c$ (Slope intercept form) we get

$$m = -\frac{8}{3} \text{ and } c = \frac{5}{3}$$

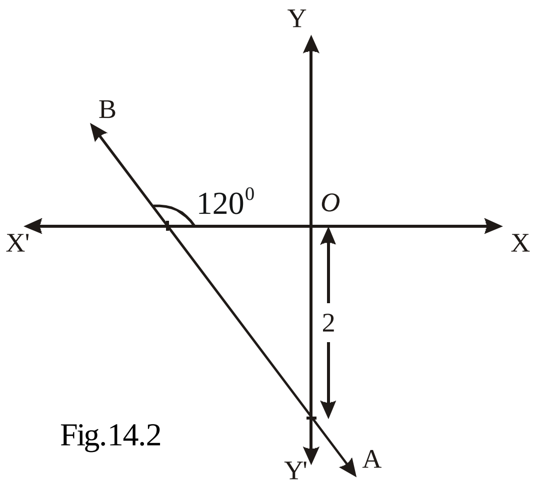

Fig.14.2

Therefore, slope of the line is $-\frac{8}{3}$ and its y-intercept is $\frac{5}{3}$.

Example 14.4 Find the equation of the line cutting off an intercept of length 2 from the negative direction of the axis of y and making an angle of 120^0 with the positive direction x-axis

Solution : From the slope intercept form of the line $\therefore \quad y = x \tan 120° + (-2)$

$$= -\sqrt{3}\, x - 2 \text{ or, } y + \sqrt{3}\, x + 2 = 0$$

Here $m = \tan 120°$, and $c = -2$, because the intercept is cut on the negative side of y-axis.

(b) POINT-SLOPE FORM

Here we will find the equation of a line passing through a given point $A(x_1, y_1)$ and having the slope m.

Let $P(x, y)$ be any point other than A on given the line. Slope $(\tan\theta)$ of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by

$$m = \tan\theta = \frac{y - y_1}{x - x_1}$$

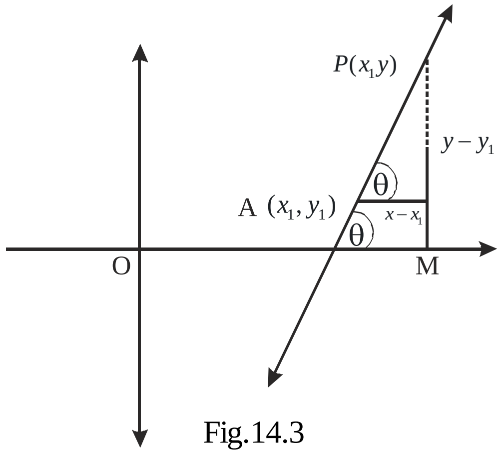

Fig.14.3

The slope of the line AP is given to be m.

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

$\therefore$ The equation of the required line is, $y - y_1 = m(x - x_1)$

***Note :** Since, the slope m is undefined for lines parallel to y-axis, the point-slope form of the equation will not give the equation of a line though A (x_1, y_1) parallel to y-axis. However, this presents no difficulty, since for any such line the abscissa of any point on the line is x_1. Therefore, the equation of such a line is $x = x_1$.*

Example 14.5 Determine the equation of the line passing through the point (2,– 1) and having slope $\frac{2}{3}$.

Solution : Putting $x_1 = 2, y_1 = -1$ and $m = \frac{2}{3}$ in the equation of the point-slope form of the line we get, $y - (-1) = \frac{2}{3}(x - 2)$

$$\Rightarrow \quad y+1=\frac{2}{3}(x-2) \Rightarrow \quad y=\frac{2}{3}x-\frac{7}{3}$$

which is the required equation of the line.

(c) TWO POINT FORM

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given distinct points.

Slope of the line passing through these points is given by

$$m=\frac{y_2-y_1}{x_2-x_1} \qquad (x_2 \neq x_1)$$

From the equation of line in point slope form, we get

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

which is the required equation of the line in two-point form.

Example 14.6 Find the equation of the line passing through (3, – 7) and (– 2,– 5).

Solution : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \cdots \text{(i)}$$

Since $x_1 = 3$, $y_1 = -7$ and $x_2 = -2$, and $y_2 = -5$, equation (i) becomes,

$$y+7=\frac{-5+7}{-2-3}(x-3)$$

or, $\quad y+7=\dfrac{2}{-5}(x-3)$ or, $\quad 2x+5y+29=0$

(d) INTERCEPT FORM

We want to find the equation of a line which cuts off given intercepts on both the co-ordinate axes.

Let PQ be a line meeting x-axis in A and y-axis in B. Let $OA = a$, $\quad OB = b$.

Then the co-ordinates of A and B are $(a,0)$ and (0, b,) respectively.

The equation of the line joining A and B is

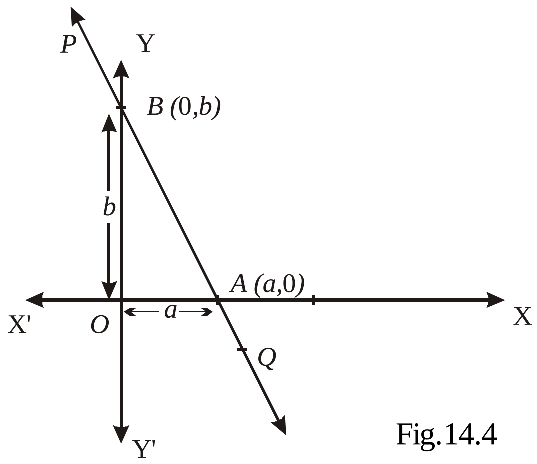

Fig.14.4

$$y - 0 = \frac{b-0}{0-a}(x-a) \text{ or, } \quad y = -\frac{b}{a}(x-a)$$

or, $$\frac{y}{b} = -\frac{x}{a} + 1 \text{ or, } \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the required equation of the line having intercepts a and b on the axes.

Example 14.7 Find the equation of a line which cuts off intercepts 5 and –3 on x and y axes respectively.

Solution : The intercepts are 5 and –3 on x and y axes respectively. i.e., $a = 5$, $b = -3$

The required equation of the line is

$$\frac{x}{5} + \frac{y}{-3} = 1, \; 3x - 5y - 15 = 0$$

Example 14.8 Find the equation of a line which passes through the point (3, 4) and makes intercepts on the axes equl in magnitude but opposite in sign.

Solution : Let the x-intercept and y-intercept be a and –a respectively

$\therefore$ The equation of the line is, $\dfrac{x}{a} + \dfrac{y}{-a} = 1, \; x - y = a$ $\quad\cdots$ (i)

Since (i) passes through (3, 4)

$\therefore$ $3 - 4 = a$ or $a = -1$

Thus, the required equation of the line is

$x - y = -1$ or $x - y + 1 = 0$

Example 14.9 Determine the equation of the line through the point (– 1,1) and parallel to x - axis.

Solution : Since the line is parallel to x-axis, so its slope ia zero. Therefore from the point slope form of the equation, we get, $y - 1 = 0\,[x - (-1)]$, $y - 1 = 0$

which is the required equation of the given line

Example 14.10 Find the intercepts made by the line

$3x - 2y + 12 = 0$ on the coordinate axes

Solution : Equation of the given line is, $3x - 2y = -12$.

Dividing by – 12, we get, $\dfrac{x}{-4} + \dfrac{y}{6} = 1$

Comparing it with the standard equation of the line in intercept form, we find $a = -4$ and $b =$

6. Hence the intercepts on the x-axis and y-axis repectively are –4. and 6.

Example 14.11 The segment of a line, intercepted between the coordinate axes is bisected at the point (x_1, y_1). Find the equation of the line

Solution : Let $P(x_1, y_1)$ be the middle point of the segment CD of the line AB intercepted between the axes. Draw $PM \perp OX$

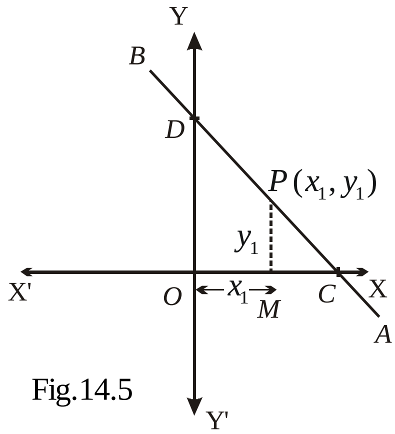

Fig.14.5

$\therefore$ $OM = x_1$ and $MP = y_1$

$\therefore$ $OC = 2x_1$ and $OD = 2y_1$

Now, from the intercept form of the line

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1 \text{ or, } \frac{x}{x_1} + \frac{y}{y_1} = 2$$

which is the required equation of the line.

(e) PERPENDICULAR FORM (NORMAL FORM)

We now derive the equation of a line when p be the length of perpendicular from the origin on the line and α, the angle which this perpendicular makes with the positive direction of x-a i x is are given.

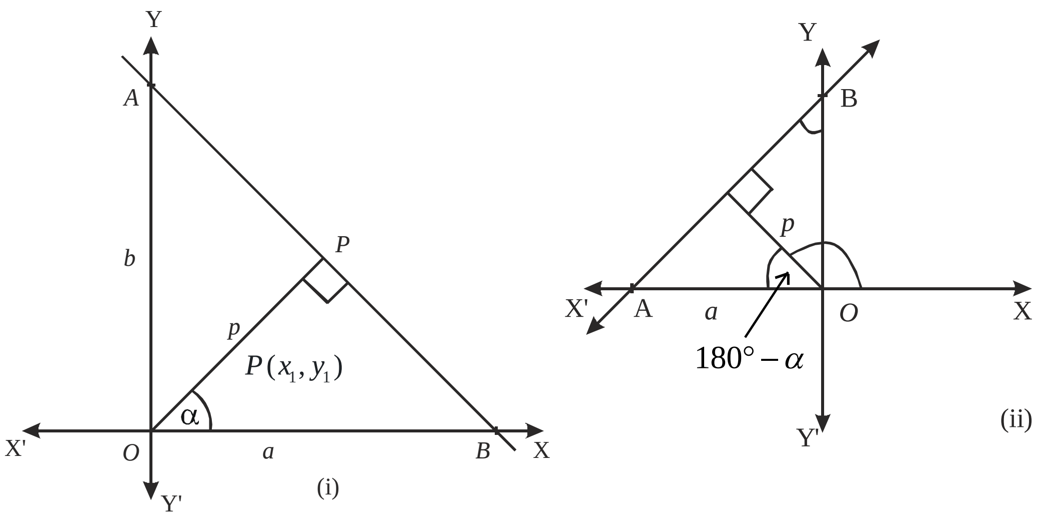

Fig. 14.6

(i) Let AB be the given line cutting off intercepts a and b on x–*axis* and y–*axis* respectively. Let OP be perpendicular from origin O on AB and $\angle POB = \alpha$ (See Fig. 14.6 (i))

$$\therefore \quad \frac{p}{a} = \cos\alpha \Rightarrow a = p\sec\alpha \quad \text{and,} \quad \frac{p}{b} = \sin\alpha \Rightarrow b = p\,\text{cosec}\,\alpha$$

$\therefore$ The equation of line AB is

$$\frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1$$

or, $\quad x\cos\alpha + y\sin\alpha = p$

(ii) $\quad \dfrac{p}{a} = \cos(180° - \alpha) = -\cos\alpha \qquad$ [From Fig. 14.6 (ii)]

$\Rightarrow \quad a = -p\sec\alpha$

similary, b = $p\operatorname{cosec}\alpha$

$\therefore$ The equation of the line AB is $\dfrac{x}{-a}+\dfrac{y}{b}=1$ *or* $x\cos\alpha + y\sin\alpha = p$

***Note :** 1. p is the length of perpendicular from the origin on the line and is always taken to be positive.*

2. α is the angle between positive direction of x-axis and the line perpendicular from the origin to the given line.

Example 14.12 Determine the equation of the line with $\alpha = 135°$ and perpendicular distance $p = \sqrt{2}$ from the origin.

Solution : From the standard equation of the line in normal form have

$$x\cos 135° + y\sin 135° = \sqrt{2}$$

or, $\quad -\dfrac{x}{\sqrt{2}}+\dfrac{y}{\sqrt{2}}=\sqrt{2}$ or, $\quad -x + y - 2 = 0$

or, $\quad x - y + 2 = 0$, which is the required equation of the straight line.

Example 14.13 Find the equation of the line whose perpendicular distance from the origin is 6 units and the perpendicular from the origin to line makes an angle of $30°$ with the positive direction of x-axis.

Solution : Here $\alpha = 30°$, $p = 6$ $\therefore$ The equation of the line is, $x\cos 30° + y\sin 30° = 6$

or, $\quad x\left(\dfrac{\sqrt{3}}{2}\right)+y\left(\dfrac{1}{2}\right)=6$ or, $\sqrt{3}\,x + y = 12$

CHECK YOUR PROGRESS 14.2

1. (a) Find the equation of a line with slope 2 and y – intercept equal to –2.

   (b) Determine the slope and the intercepts made by the line on the axes whose equation is $4x + 3y = 6$.

2. Find the equation of the line cutting off an interecept $\frac{1}{\sqrt{3}}$ on negative direction of axis of y and inclined at 120° to the positive direction of x-axis.

3. Find the slope and y-intercept of the line whose equation is $3x - 6y = 12$.

4. Determine the equation of the line passing through the point (–7, 4) and having the slope $-\frac{3}{7}$.

5. Determine the equation of the line passing through the point (1, 2) which makes equal angles with the two axes.

6. Find the equation of the line passing through (2, 3) and parallel to the line joining the points (2, –2) and (6, 4).

7. (a) Determine the equation of the line through (3, –4) and (–4, 3).

   (b) Find the equation of the diagonals of the rectangle ABCD whose vertices are A (3, 2), B (11, 8), C (8, 12) and D (0, 6).

8. Find the equation of the medians of a triangle whose vertices are (2, 0), (0, 2) and (4, 6).

9. Find the equation of the line which cuts off intercepts of length 3 units and 2 units on x-$ax$is and y-axis respectively.

10. Find the equation of a line such that the segment between the coordinate axes has its mid point at the point (1, 3)

11. Find the equation of a line which passes through the point (3, –2) and cuts off positive intercepts on x and y axes in the ratio of 4 : 3.

12. Determine the equation of the line whose perpendicular from the origin is of length 2 units and makes an angle of 45° with the positive direction of x-axis.

13. If p is the length of the perpendicular segment from the origin, on the line whose intercept on the axes are a and b, then show that, $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

14.3 GENERAL EQUATION OF FIRST DEGREE

You know that a linear equation in two variables x and y is given by $Ax + By + C = 0$ ...(1)

In order to understand its graphical representation, we need to take the following thres cases.

Case-1: (When both A and B are equal to zero)

In this case C is automaticaly zero and the equation does not exist.

Case-2: (When $A = 0$ and $B \neq 0$)

In this case the equation (1) becomes $By + C = 0$.

or $y = -\frac{C}{B}$ and is satisfied by all points lying on a line which is parallel to x-axis and the y-coordinate of every point on the line is $-\frac{C}{B}$. Hence this is the equation of a straight line. The case where B = 0 and A $\neq$ 0 can be treated similarly.

Case-3: (When A $\neq$ 0 and B $\neq$ 0)

We can solve the equation (1) for y and obtain., $y = -\frac{A}{B}x - \frac{C}{B}$

Clearly, this represents a straight line with slope $-\frac{A}{B}$ and y – intercept equal to $-\frac{C}{B}$.

14.3.1 CONVERSION OF GENERAL EQUATION OF A LINE INTO VARIOUS FORMS

If we are given the general equation of a line, in the form Ax + By + C = 0, we will see how this can be converted into various forms studied before.

14.3.2 CONVERSION INTO SLOPE-INTERCEPT FORM

We are given a first degree equation in x and y as $Ax + By + C = O$

Are you able to find slope and y-intercept ?

Yes, indeed, if we are able to put the general equation in slope-intercept form. For this purpose, let us re-arrange the given equation as.

$$Ax + By + C = 0 \text{ as, } By = -Ax - C$$

or $\quad y = -\frac{A}{B}x - \frac{C}{B}$ (Provided $B \neq 0$)

which is the required form. Hence, the slope = $-\frac{A}{B}$, y – intercept = $-\frac{C}{B}$.

Example 14.14 Reduce the equation $x + 7y - 4 = 0$ to the slope – intercept form.

Here find its slope and y intercept.

Solution : The given equation is, $x + 7y - 4 = 0$

or $\quad 7y = -x + 4$, or $y = -\frac{1}{7}x + \frac{4}{7}$

Here slope $= -1/7$ and y interecept $= 4/7$

14.3.3 CONVERSION INTO INTERCEPT FORM

Suppose the given first degree equation in x and y is $\quad Ax + By + C = 0. \quad$ ...(i)

In order to convert (i) in intercept form, we re arrange it as $Ax + By = -C$ or $\frac{Ax}{-C} + \frac{By}{-C} = 1$

or $\quad \frac{x}{(-\frac{C}{A})} + \frac{y}{(-\frac{C}{B})} = 1$ (Provided $A \neq 0$ and $B \neq 0$)

which is the requied converted form. It may be noted that intercept on x – axis $= \frac{-C}{A}$ and

intercept on y – axis $= \frac{-C}{B}$

Example 14.15 Reduce $3x + 5y = 7$ into the intercept form and find its intercepts on the axes.

Solution : The given equation is, $3x + 5y = 7$

or, $\quad \frac{3}{7}x + \frac{5}{7}y = 1$ or, $\quad \frac{x}{\frac{7}{3}} + \frac{y}{\frac{7}{5}} = 1$

$\therefore$ The x– intercept $= \frac{7}{3}$ and, y – intercept $= \frac{7}{5}$

14.3.4 CONVERSION INTO PERPENDICULAR FORM

Let the general first degree equation in x and y be, $Ax + By + C = 0$ ... (i)

We will convert this general equation in perpendicular form. For this purpose let us re-write the given equation (i) as $Ax + By = -C$

Multiplying both sides of the above equation by λ, we have

$$\lambda Ax + \lambda By = -\lambda C \qquad \text{... (ii)}$$

Let us choose λ such that $(\lambda A)^2 + (\lambda B)^2 = 1$

or $\quad \lambda = \frac{1}{\sqrt{(A^2 + B^2)}}$ (Taking positive sign)

Substituting this value of λ in (ii), we have

$$\frac{Ax}{\sqrt{(A^2 + B^2)}} + \frac{By}{\sqrt{(A^2 + B^2)}} = -\frac{C}{\sqrt{(A^2 + B^2)}} \qquad \text{... (iii)}$$

This is required conversion of (i) in perpendicular form. Two cases arise according as C is negative or positive.

(i) If $C < 0$, the equation (ii) is the required form.

(ii) If $C > 0$, the R. H. S. of the equation of (iii) is negative.

$\therefore$ We shall multiply both sides of the equation of (iii) by –1.

$\therefore$ The required form will be $-\dfrac{Ax}{\sqrt{(A^2+B^2)}} - \dfrac{By}{\sqrt{(A^2+B^2)}} = \dfrac{C}{\sqrt{(A^2+B^2)}}$

Thus, length of perpendicular from the origin = $\dfrac{|C|}{\sqrt{(A^2+B^2)}}$

Inclination of the perpendicular with the positwe direction of x-ax is

is given by $\cos\theta = \mp\dfrac{A}{\sqrt{A^2+B^2}}$

or $\sin\theta = \left(\mp\dfrac{B}{\sqrt{(A^2B^2)}}\right)$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$. If $C = 0$, the line passes through the origin and there is no perpendicular from the origin on the line.

With the help of the above three cases, we are able to say that

"The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously."

Is the converse of the above statement true? ***The converse of the above statement is that every straight line can be expressed as a general equation of first degree in x and y.***

In this lesson we have studied about the various forms of equation of straight line. For example, let us take some of them as $y = mx + c$, $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $x\cos\alpha + y\sin\alpha = p$. Obviously, all are linear equations in x and y. We can re-arrange them as $y - mx - c = 0$, $bx + ay - ab = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ respectively. Clearly, these equations are nothing but a different arrangement of general equation of first degree in x and y. Thus, we have established that

"Every straight line can be expressed as a general equation of first degree in x and y".

Example 14.16 Reduce the equation $x + \sqrt{3}\,y + 7 = 0$ into perpendicular form.

Solution : The equation of given line is $x + \sqrt{3}y + 7 = 0$ ... (i)

Comparing (i) with general equation of straight line, we have, $A = 1$ and $B = \sqrt{3}$

$$\therefore \quad \sqrt{A^2 + B^2} = 2$$

Dividing equation (i) by 2, we have, $\frac{x}{2} + \frac{\sqrt{3}}{2} y + \frac{7}{2} = 0$

or $\quad \left(-\frac{1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)y - \frac{7}{2} = 0$ or $x \cos \frac{4\pi}{3} + y \sin \frac{4\pi}{3} = \frac{7}{2}$

(cos θ and sin θ being both negative in the third quadrant, value of θ will lie in the third quadrant).

This is the representation of the given line in perpendicular form.

Example 14.17 Find the perpendicular distance from the origin on the line $\sqrt{3}\,x - y + 2 = 0$. Also, find the inclination of the perpendicular from the origin.

Solution : The given equation is $\sqrt{3}\,x - y + 2 = 0$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, we have

$$\frac{\sqrt{3}}{2} x - \frac{1}{2} y + 1 = 0 \text{ or, } \quad \frac{\sqrt{3}}{2} x - \frac{1}{2} y = -1$$

Multiplying both sides by -1, we have, $-\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 1$

or, $\quad x \cos \frac{5\pi}{6} + y \sin \frac{5\pi}{6} = 1$ (cos θ is –ve in second quadrant and sin θ is +ve in second quadrant, so value of θ lies in the second quadrant).

Thus, inclination of the perpendicular from the origin is 150° and its length is equal to 1.

Example 14.18 Find the equation of a line which passess through the point (3,1) and bisects the portion of the line $3x + 4y = 12$ intercepted between coordinate axes.

Solution : First we find the intercepts on coordinate axes cut off by the line whose equation is

$$3x + 4y = 12 \text{ or } \frac{3x}{12} + \frac{4y}{12} = 1 \text{ or } \qquad \frac{x}{4} + \frac{y}{3} = 1$$

Hence, intercepts on x-axis and y-axis are 4 and 3 respectively.

Thus, the coordinates of the points where the line meets the coordinate axes are A (4, 0) and B (0, 3).

$\therefore$ Mid – point of AB is $\left(2, \frac{3}{2}\right)$ is

Hence the equation of the line through (3,1) and

is and $\left(2, \frac{3}{2}\right)$ *is*, $y - 1 = \dfrac{\frac{3}{2} - 1}{2 - 3}(x - 3)$

or $\quad y - 1 = -\frac{1}{2}(x - 3)$

or $\quad 2(y - 1) + (x - 3) = 0$

or $\quad 2y - 2 + x - 3 = 0$, or $\quad x + 2y - 5 = 0$

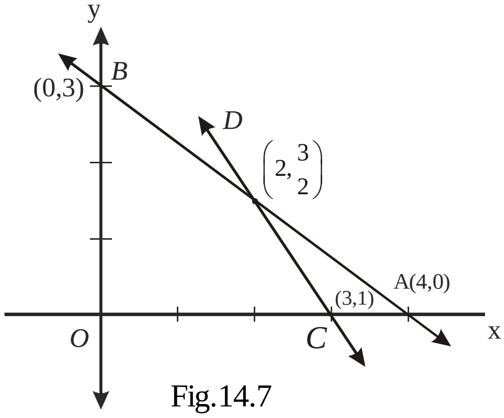

Fig.14.7

Example 14.19 Prove that the line through (8, 7) and (6, 9) cuts off equal intercepts on coordinate axes.

Solution : The equation of the line passing through (8, 7) and (6, 9) is, $y - 7 = \dfrac{9 - 7}{6 - 8}(x - 8)$

or $\quad y - 7 = -(x - 8)$, or $\quad x + y = 15$

or $\quad \dfrac{x}{15} + \dfrac{y}{15} = 1$

Hence, intercepts on both axes are 15 each.

Example 14.20 Find the ratio in which the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8).

Solution : The equation of the line joining C (–5, 1) and D (1, –3) is

$$y - 1 = \frac{-3 - 1}{1 + 5}(x + 5), \text{ or}$$

$$y - 1 = -\frac{4}{6}(x + 5)$$

or $\quad 3y - 3 = -2x - 10$, or $\quad 2x + 3y + 7 = 0$ ... (i)

Let line (i) divide the join of A (3, 4) and B (7, 8) at the point P.

If the required ratio is $\lambda : 1$ in which line (i) divides the join of A (3, 4) and B (7, 8), then the coordinates of P are

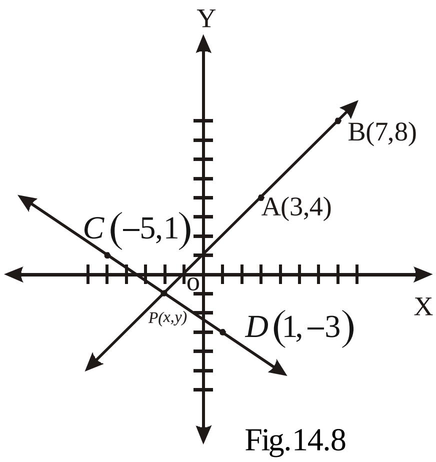

Fig.14.8

$$\left(\frac{7\lambda+3}{\lambda+1}, \frac{8\lambda+4}{\lambda+1}\right)$$

Since *P* lies on the line (i), we have

$$2\left(\frac{7\lambda+3}{\lambda+1}\right)+3\left(\frac{8\lambda+4}{\lambda+1}\right)+7=0$$

$$\Rightarrow \quad 14\lambda + 6 + 24\lambda + 12 + 7\lambda + 7 = 0, \Rightarrow \quad 45\lambda + 25 = 0 \Rightarrow \lambda = -\frac{5}{9}$$

Hence, the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8) externally in the ratio 5 : 9.

CHECK YOUR PROGRESS 14.3

1. Under what condition, the general equation Ax + By + C = 0 of first degree in x and y represents a line?
2. Reduce the equation 2x + 5y + 3 = 0 to the slope intercept form.
3. Find the x and y intercepts for the following lines :

   (a) y = mx + c (b) 3y = 3x + 8 (c) 3x – 2y + 12 = 0
4. Find the length of the line segment AB intercepted by the straight line 3x – 2y + 12 = 0 between the two axes.
5. Reduce the equation x cos α + y sin α = p to the intercept form of the equation and also find the intercepts on the axes.
6. Reduce the following equations into normal form.

   (a) 3x – 4y + 10 = 0 (b) 3x – 4y = 0
7. Which of the lines 2x – y + 3 = 0 and x – 4y – 7 = 0 is nearer from the origin?

14.4 DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

In this section, we shall discuss the concept of finding the distance of a given point from a given line or lines.

Let $P(x_1, y_1)$ be the given point and *l* be the line $Ax + By + C = 0$.

Let the line *l* intersect *x* axis and *y* axis *R* and *Q* respectively.

Draw $PM \perp l$ and let $PM = d$.

Let the coordinates of *M* be (x_2, y_2)

$$d = \sqrt{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}} \qquad \text{...(i)}$$

$\because$ M lies on l, $\therefore$ $Ax_2 + By_2 + C = 0$ or $C = -(Ax_2 + By_2)$ ...(ii)

The coordinates of R and Q are $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ respectively.

The slope of $QR = \dfrac{0+\frac{C}{B}}{-\frac{C}{A}-0} = -\dfrac{A}{B}$ and,

the slope of $PM = \dfrac{y_2 - y_1}{x_2 - x_1}$

As $PM \perp QR \Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} \times \left(-\dfrac{A}{B}\right) = -1.$ or $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{B}{A}$ ...(iii)

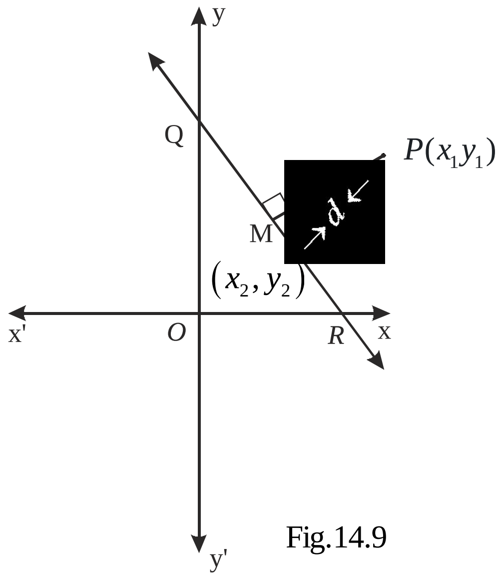

Fig.14.9

From (iii) $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{\sqrt{\left\{(x_1 - x_2) + (y_1 - y_2)^2\right\}}}{\sqrt{(A^2 + B^2)}}$ ...(iv)

(Using properties of Ratio and Proportion)

Also $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{A(x_1 - x_2) + B(y_1 - y_2)}{A^2 + B^2}$ ...(v)

From (iv) and (v), we get

$$\frac{\sqrt{\left\{(x_1-x_2)^2(y_1-y_2)^2\right\}}}{\sqrt{(A^2+B^2)}}=\frac{A(x_1-x_2)+B(y_1-y_2)}{A^2+B^2}$$

or $$\frac{d}{\sqrt{A^2+B^2}} + \frac{Ax_1+By_1-(Ax_2+By_2)}{A^2+B^2} \qquad \text{[Using (i)]}$$

or $$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}} \qquad \text{[Using (ii)]}$$

Since the distance is always positive, we can write

$$d=\left|\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}\right|$$

> ***Note :*** *The perpendicular distance of the origin* (0, *0) from $Ax + By + C = 0$ is*
>
> $$\frac{A(0)+B(0)+C}{\sqrt{(A^2+b^2)}}=\frac{C}{\sqrt{(A^2+B^2)}}$$

Example 14.21 Find the points on the x-axis whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a.

Solution : Let $(x_1, 0)$ be any point on x-axis.

Equation of the given line is $bx + ay - ab = 0$. The perpendicular distance of the point $(x_1, 0)$ from the given line is, $a=\pm\frac{bx_1+a.0-ab}{\sqrt{(a^2+b^2)}}$ $\quad\therefore\quad x_1=\frac{a}{b}\left\{b\pm\sqrt{(a^2+b^2)}\right\}$

Thus, the point on x-axis is $x_1=\left(\frac{a}{b}b\pm\sqrt{(a^2+b^2)},0\right)$

CHECK YOUR PROGRESS 14.4

1. Find the perpendicular distance of the point (2, 3) from $3x + 2y + 4 = 0$.
2. Find the points on the axis of y whose perpendicular distance from the straight line

$\frac{x}{a} + \frac{y}{b} = 1$ is b.

3. Find the points on the axis of y whose perpendicular distance from the straight line 4x + 3y = 12 is 4.

4. Find the perpendicular distance of the origin from 3x + 7y + 14 = 0

14.6 EQUATION OF PARALLEL (OR PERPENDICULAR) LINES

Till now, we have developed methods to find out whether the given lines are prallel or perpendicular. In this section, we shall try to find, the equation of a line which is parallel or perpendicular to a given line.

14.6.1 EQUATION OF A STRAIGHT LINE PARALLEL TO THE GIVEN LINE

$$\boldsymbol{Ax + By + c = 0}$$

Let $\quad A_1x + B_1y + C_1 = 0 \qquad$ ...(i)

be any line parallel to the given line, $Ax + By + C = 0$

The condition for parallelism of (i) and (ii) is ...(ii)

$$\frac{A_1}{A} = \frac{B_1}{B} = K_1 \quad \text{(say)} \Rightarrow \qquad A_1 = AK_1,\ B_1 = \mathrm{B}K_1$$

with these values of A_1 and B_1, (i) gives

$$AK_1x + BK_1\,y + C_1 = 0 \quad \text{or} \qquad Ax + By + \frac{C_1}{K_1} = 0$$

or $\quad Ax + By + K = 0, \qquad$ where $K = \frac{C_1}{K_1} \qquad$ ... (iii)

This is a line parallel to the given line. From equations (ii) and (iii) we observe that

(i) coefficients of x and y are same

(ii) constants are different, and are to evaluated from given conditions.

Example 14.22 Find equation of the straight line, which passes through the point (1, 2) and which is parallel to the straight line $2x + 3y + 6 = 0$.

Solution : Equation of any straight line parallel to the given equation can be written if we put

(i) the coefficients of x and y as same as in the given equation.

(ii) constant to be different from the given equation, which is to be evaluated under given condition.

Thus, the required equation of the line will be, $2x + 3y + K = 0$ for some constant K

Since it passes through the point (1, 2) hence, $2 \times 1 + 3 \times 2 + K = 0$

or $\quad K = -8$

$\therefore$ Required equation of the line is $2x + 3y = 8$.

14.7 STRAIGHT LINE PERPENDICULAR TO THE GIVEN LINE

$$\boldsymbol{Ax + By + C = 0}$$

Let $\quad A_1 x + B_1 y + C_1 = 0 \quad$ ... (i), be any line perpendicular to the given line

$$Ax + By + C = 0$$

Condition for perpendicularity of lines (i) and (ii) is ... (ii)

$$AA_1 + BB_1 = 0 \;\Rightarrow\; \frac{A_1}{B} = -\frac{B_1}{A} = K_1 \qquad \text{(say)}$$

$\Rightarrow \quad A_1 = BK_1$ and $B_1 = -AK_1$

With these values of A_1 and B_1, (i) gives, $Bx - Ay + \frac{C_1}{K_1} = 0 = 0$

or $\quad Bx - Ay + K = 0$ where $K = \dfrac{C_1}{K_1}$ ... (iii)

Hence, the line (iii) is perpendicular to the given line (ii)

We observe that in order to get a line perpendicular to the given line we have to follow the following procedure : (i) Interchange the coefficients of x and y

(ii) Change the sign of one of them.

(iii) Change the Constant term to a new constant K (say), and evaluate it from given condition.

Example 14.23 Find the equation of the line which passes through the point (1, 2) and is perpendicular to the line $2x + 3y + 6 = 0$.

Solution : Following the procedure given above, we get the equation of line perpendicular to the given equation as $3x - 2y + K = 0$ ...(i)

(i) passes through the point (1, 2), hence

$3 \times 1 - 2 \times 2 + K = 0$ or $K = 1$

$\therefore$ Required equation of the straight line is $3x - 2y + 1 = 0$.

Example 14.24 Find the equation of the line which passes through the point (x_2, y_2) and is perpendicular to the straight line $y y_1 = 2a\,(x + x_1)$.

Solution : The given straight line is $yy_1 - 2ax - 2ax_1 = 0$ ...(i)

Any straight line perpendicular to (i) is $2ay + xy_1 + C = 0$

This passes through the point (x_2, y_2) $\therefore$ $2ay_2 + x_2 y_1 + C = 0$

$\Rightarrow$ $C = -2ay_2 - x_2y_1$

$\therefore$ Required equation of the spraight line is, $2a\,(y - y_2) + y_1\,(x - x_2) = 0$

CHECK YOUR PROGRESS 14.5

1. Find the equation of the straight line which passes through the point (0, –2) and is parallel to the straight line 3x + y = 2.

2. Find the equation of the straight line which passes through the point (–1, 0) and is parallel to the straight line y = 2x + 3.

3. Find the equation of the straight line which passes through the point (0, –3) and is perpendicular to the straight line x + y + 1 = 0.

4. Find the equation of the line which passes through the point (0, 0) and is perpendicular to the straight line x + y = 3.

5. Find the equation of the straight line which passes through the point (2, –3) and is perpendicular to the given straight line 2a (x + 2) + 3y = 0.

6. Find the equation of the line which has x - intercept –8 and is perpendicular to the line 3x + 4y – 17 = 0.

7. Find the equation of the line whose y-intercept is 2 and is parallel to the line 2x – 3y + 7 = 0.

8. Prove that the equation of a straight line passing throngh ($a\cos^3\theta$, $a\sin^3\theta$) and perpendicular to the sine x sec θ + y cosec θ = a is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

14.8 EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES :

Let $l_1 : a_1x + b_1y + c_1 = 0$ ...(i)

and $l_2 : a_2x + b_2y + c_2 = 0$, be two intersecting lines.

Let P(*h*, *k*) be the point of intersection of l_1 and l_2, then

$$a_1h + b_1k + c_1 = 0 \qquad ...(iii)$$

and $$a_2h + b_2k + c_2 = 0 \qquad ...(iv)$$

Now consider the equation

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2) = 0 \qquad ...(v)$$

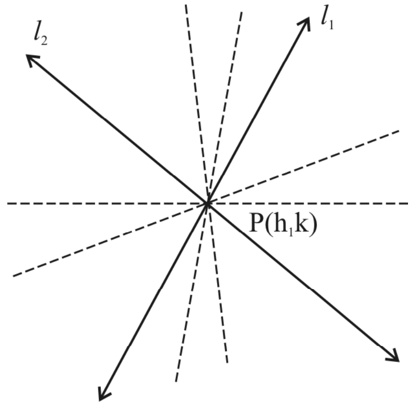

Fig.14.10

It is a first degree equation in x and y. So it will represent different lines for different values of λ. If we replace x by h and y by k we get

$$(a_1h+b_1k+c_1)+\lambda(a_2h+b_2k+c_2) = 0 \qquad ...(vi)$$

using (iii) and (iv) in (vi) we get

$$0 + \lambda 0 = 0 \text{i.e.} \quad 0 = 0 \text{ which is true.}$$

So equation (v) represents a family of lines passing throught the point (h, k) i.e. the point of intersection of the given lines l_1 and l_2.

- A particular member of the family is obtained by giving a particular value to λ. This value of λ can be obtained from other given conditions.

Example 14.25 Find the equation of the line passing through the point of intersection of the lines $x + y + 1 = 0$ and $2x - y + 7 = 0$ and containing the point (1, 2).

Solution : Equation of family of lines passing through the intersection of given lines is $(x + y + 1) + \lambda\ (2x - y + 7) = 0$

This line will contain the point (1, 2) if

$$(1 + 2 + 1) + \lambda\ (2 \times 1 - 1 \times 2 + 7) = 0$$

i.e. $\qquad 4 + 7\lambda = 0 \Rightarrow \lambda = -\dfrac{4}{7}.$

Therefore the equation of required line is, $(x+y+1)-\dfrac{4}{7}(2x-y+7) = 0$

i.e. $\qquad 7(x+y+1)-4(2x-y+7) = 0$ i.e. $-x + 11y - 21 = 0$

or $\qquad x - 11y + 21 = 0$

Example 14.26 Find the equation of the line passing through the intersection of lines $3x + y - 9 = 0$ and $4x + 3y - 7 = 0$ and parallel to y-axis.

Solution : Equation of family of lines passing through the intersection of given lines is

$$(3x+y-9)+\lambda(4x+3y-7) = 0, \text{ i.e. } (3+4\lambda)x+(1+3\lambda)y-(9+7\lambda) = 0 \quad ...(i)$$

We know that if a line is parallel to y-axis then co-efficient of y in its equation must be zero.

$\therefore\ 1 + 3\lambda = 0 \Rightarrow \lambda = -1/3.$

Hence, equation of the required line is, $\left\{3+4\left(-\frac{1}{3}\right)\right\}x+0y-\left\{9+7\left(\frac{-1}{3}\right)\right\} = 0$

i.e. $\qquad x = 4$

CHECK YOUR PROGRESS 14.6

1. Find the equation of the line passing through the intersection of the lines $x + y = 5$ and $2x - y - 7 = 0$ and parallel to x-axis.

2. Find the equation of the line passing though the intersection of the lines $x + y + 1 = 0$ and $x - y - 1 = 0$ and containing the point $(-3, 1)$

LET US SUM UP

- The equation of a line parallel to y-axis is $x = a$ and parallel to x-axis is $y = b$.
- The equation of the line which cuts off intercept c on y-axis and having slope m is $y = mx + c$
- The equation of the line passing through $A(x_1, y_1)$ and having the slope m is $y - y_1 = m(x - x_1)$
- The equation of the line passing through two points $A(x, y_1)$ and $B(x_2, y_2)$ is

  $$y - \mathrm{y}_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- The equation of the line which cuts off intercepts a and b on x – axis and y – axis respectively is $\frac{x}{a}+\frac{y}{b} = 1$
- The equation of the line in normal or perpendicular form is $x\cos\alpha + y\sin\alpha = p$

  where p is the length of perpendicular from the origin to the line and α is the angle which this perpendicular makes with the positive direction of the x-axis.

- The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously.
- From general equation $Ax + By + C = 0$ we can evaluate the following :

(i) Slope of the line $= -\dfrac{A}{B}$ (ii) x-intercept $= -\dfrac{C}{A}$ (iii) y-intercept $= -\dfrac{C}{B}$

(iv) Length of perpendicular from the origin to the line $= \dfrac{|C|}{\sqrt{(A^2 + B^2)}}$

(v) Inclination of the perpendicular from the origin is given by

$$\cos\alpha = \frac{\mp A}{\sqrt{(A^2 + B^2)}} \; ; \; \sin\alpha = \frac{\mp B}{\sqrt{(A^2 + B^2)}}$$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$; but if $C = 0$ then either only the upper sign or only the lower sign are taken.

- Distance of a given point (x_1, y_1) from a given line $Ax + By + C = 0$ is $d = \left|\dfrac{Ax_1 + By_1 + C}{\sqrt{(A^2 + B^2)}}\right|$
- Equation of a line parallel to the line $Ax + By + C = 0$ is $Ax + By + k = 0$
- Equation of a line perpendicular to the line $Ax + By + C = 0$ is $Bx - Ay + k = 0$
- Equation of a line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1) + k\,(a_2x + b_2y + c_2) = 0$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Straight_lines
http://mathworld.wolfram.com/Straight_lines

TERMINAL EXERCISE

1. Find the equation of the straight line whose y-intercept is –3 and which is:

   (a) parallel to the line joining the points (–2,3) and (4, –5).

   (b) perpendicular to the line joining the points (0, –5) and (–1, 3).

2. Find the equation of the line passing through the point (4, –5) and

   (a) parallel to the line joining the points (3, 7) and (–2, 4).

   (b) perpendicular to the line joining the points (–1, 2) and (4, 6).

3. Show that the points $(a, 0)$, $(0, b)$ and $(3a, -2b)$ are collinear. Also find the equation of the line containing them.

4. A (1, 4), B (2, –3) and C (–1, –2) are the vertices of triangle ABC. Find

   (a) the equation of the median through A.

   (b) the equation of the altitude througle A.

   (c) the right bisector of the side BC.

5. A straight line is drawn through point A (2, 1) making an angle of $\frac{\pi}{6}$ with the positive direction of x-axis. Find the equation of the line.

6. A straight line passes through the point (2, 3) and is parallel to the line $2x + 3y + 7 = 0$. Find its equation.

7. Find the equation of the line having a and b as x-intercept and y-intercepts respectively.

8. Find the angle between the lines $y = (2 - \sqrt{3})\, x + 5$ and $y = (2 + \sqrt{3})\, x - d$.

9. Find the angle between the lines $2x + 3y = 4$ and $3x - 2y = 7$

10. Find the length of the per pendicular drawn from the point (3, 4) on the straight line $12\,(x + 6) = 5\,(y - 2)$.

11. Find the length of the perpendicula from (0, 1) on $3x + 4\,y + 5 = 0$.

12. Find the distance between the lines $2x + 3y = 4$ and $4x + 6y = 20$

13. Find the length of the perpendicular drawn from the point (–3, – 4) on the line $4x - 3y = 7$.

14. Show that the product of the perpendiculars drawn from the points on the straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

15. Prove that the equation of the straight line which passes through the point $(a\cos^3\theta, b\sin^3\theta)$ and is perpendicular to $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ is $x\cos\theta - y\,\mathrm{cosec}\,\theta = a\cos 2\theta$

16. Find the equation of a straight line passing through the point of intersection of the lines $3x + y - 9 = 0$ and $4x + 4y - 7 = 0$ and perpendicular to the line $5x - 4y + 1 = 0$.

17. Find the equation of a straight line passing thought the point of intersection of the lines $2x + 3y - 2 = 0$ and $x - 2y + 1 = 0$ and having x-intercept equal to 3.

18. Find the equation of a line through the point of intersection of the lines $3x + 4y - 7 = 0$ and $x - y + 2 = 0$ and with slope 5.

19. Find the equation of a line through the point of intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$ and perpendicular to the line $5x - 3y - 1 = 0$.

20. Find the equation of a line through the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ which cut off equal intercepts on the axes.

ANSWERS

CHECK YOUR PROGRESS 14.1

1. (a) $y = -4$ (b) $x = -3$
2. $x = 5$
3. $y + 7 = 0$

CHECK YOUR PROGRESS 14.2

1. (a) $y = 2x - 2$ (b) Slope $= \frac{-4}{3}$, y – intercept = 2
2. $\sqrt{3}y = -3x - 1$
3. Slope $= \frac{1}{2}$, y-intercept = –2
4. $3x + 7y = 7$
5. $y = x + 1$; $x + y - 3 = 0$
6. $3x - 2y = 0$
7. (a) $x + y = -1$ (b) Equation of the diagonal AC $= 2x - y - 4 = 0$

   Equation of the diagonal BD $= 2x - 11y + 66 = 0$
8. $x - 2 = 0$, $x - 3y + 6 = 9$ and $5x - 3y - 2 = 0$
9. $2x + 3y = 6$
10. $3x + y = 6$
11. $3x + 4y = 1$
12. $x + y = 2\sqrt{2}$

CHECK YOUR PROGRESS 14.3

1. A and B are not both simltaneously zero
2. $y = \frac{-2}{5}x - \frac{3}{5}$
3. (a) x–intercept $= \frac{-c}{m}$; y-intercept = c

(b) x-intercept $= \frac{-8}{3}$; y-intercept $= \frac{8}{3}$

(c) x-intercept $= -4$; y-intercept $= 6$

4. $2\sqrt{13}$ units

5. $$\frac{x}{p\sec\alpha} + \frac{x}{p\,\mathrm{cosec}\,\alpha} = 1$$

6. (a) $\frac{-3}{5}x + \frac{4}{5}y - 2 = 0$ (b) $\frac{-3}{5}x + \frac{4}{5}y = 0$

7. The first line is nearer from the origin.

CHECK YOUR PROGRESS 14.4

1. $d = \frac{16}{\sqrt{13}}$

2. $\left(0, \frac{b}{a}(a \pm \sqrt{a^2 + b^2})\right)$

3. $\left(0, \frac{32}{3}\right)$

4. $\frac{14}{\sqrt{58}}$

CHECK YOUR PROGRESS 14.5

1. $3x + y + 2 = 0$
2. $y = 2x + 2$
3. $x - y = 3$
4. $y = x$
5. $3x - 2ay = 6(a - 1)$
6. $4x - 3y + 32 = 0$
7. $2x - 3y + 6 = 0$

CHECK YOUR PROGRESS 14.6

1. $y=1$
2. $2x+3y+3=0$

TERMINAL EXERCISE

1. (a) $4x + 3y + 9 = 0$ (b) $x - 8y - 24 = 0$

2. (a) $3x - 5y - 37 = 0$ (b) $5x - 8y - 60 = 0$

4. (a) $13x - y - 9 = 0$ (b) $3x - y + 1 = 0$

 (c) $3x - y - 4 = 0$

5. $x - \sqrt{3y} = 2 - \sqrt{3}$

6. $2x + 3y + 13 = 0$

7. $bx + ay = ab$

8. $\frac{\pi}{2}$

9. $\frac{\pi}{2}$

10. $\frac{98}{13}$

11. $\frac{9}{5}$

12. $\frac{6}{\sqrt{13}}$

13. $\frac{7}{5}$

16. $32x + 40y - 41 = 0$

17. $x + 5y - 3 = 0$

18. $35x - 7y + 18 = 0$

19. $63x + 105y - 781 = 0$

20. $23x + 23y - 11 = 0$

15

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CIRCLES

Notice the path in which the tip of the hand of a watch moves. (see Fig. 15.1)

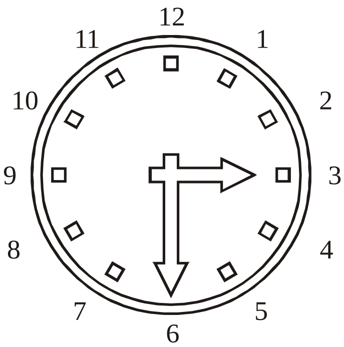

Fig.15.1

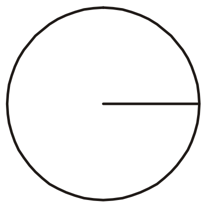

Fig.15.2

Again, notice the curve traced out when a nail is fixed at a point and a thread of certain length is tied to it in such a way that it can rotate about it, and on the other end of the thread a pencil is tied. Then move the pencil around the fixed nail keeping the thread in a stretched position (See Fig 15.2)

Certainly, the curves traced out in the above examples are of the same shape and this type of curve is known as a ***circle***.

The distance between the tip of the pencil and the point, where the nail is fixed is known as the ***radius*** of the circle.

We shall discuss about the curve traced out in the above examples in more details.

OBJECTIVES

After studying this lesson, you will be able to :

- derive and find the equation of a circle with a given centre and radius;
- state the conditions under which the general equation of second degree in two variables represents a circle;
- derive and find the centre and radius of a circle whose equation is given in general form;
- find the equation of a circle passing through :

  (*i*) three non-collinear points (*ii*) two given points and touching any of the axes;

EXPECTED BACKGROUND KNOWLEDGE

- Terms and concepts connected with circle.
- Distance between two points with given coordinates.
- Equation of a straight line in different forms.

15.1 DEFINITION OF THE CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the same plane remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

15.2 EQUATION OF A CIRCLE

Can we find a mathematical expression for a given circle?

Let us try to find the equation of a circle under various given conditions.

15.2.1 WHEN COORDINATES OF THE CENTRE AND RADIUS ARE GIVEN

Let C be the centre and a be they radius of the circle. Coordinates of the centre are given to be (h, k), say.

Take any point $P(x, y)$ on the circle and draw perpendiculars CM and PN on OX. Again, draw CL perpendicular to PN.

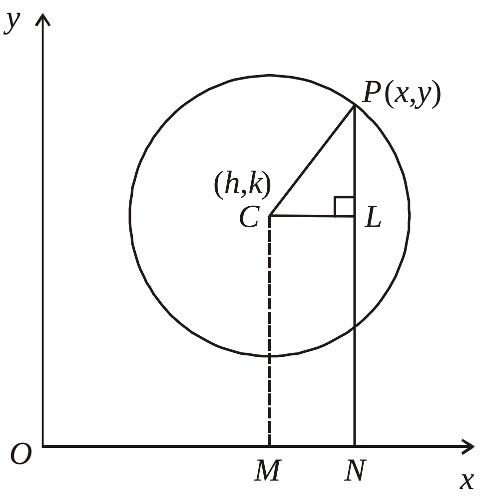

Fig.15.3

We have

$$CL = MN = ON - OM = x - h$$

and $\quad PL = PN - LN = PN - CM = y - k$

In the right angled triangle CLP, $CL^2 + PL^2 = CP^2$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2 \qquad ...(1)$$

This is the required equation of the circle under given conditions. This form of the circle is known as ***standard form*** of the circle.

Conversely, if (x, y) is any point in the plane satisfying (1), then it is at a distance 'a' from (h, k). So it is on the circle.

What happens when the

(*i*) circle passes through the origin?

(ii) circle does not pass through origin and the centre lies on the x-axis?

(iii) circle passes through origin and the x-axis is a diameter?

(iv) centre of the circle is origin?

(v) circle touches the x-axis?

(vi) circle touches the y-axis?

(vii) circle touches both the axes?

We shall try to find the answer of the above questions one by one.

(i) In this case, since (0, 0) satisfies (1), we get

$$h^2 + k^2 = a^2$$

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ky = 0$ ...(2)

(ii) In this case $k = 0$

Hence the equation (1) reduces to $(x-h)^2 + y^2 = a^2$ ...(3)

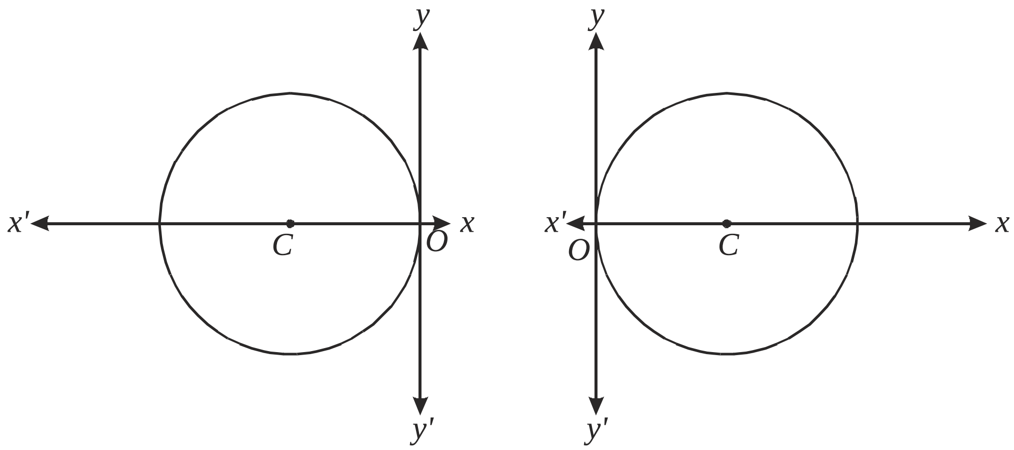

Fig.15.4

(iii) In this case $k = 0$ and $h = \pm a$ (see Fig. 15.4)

Hence the equation (1) reduces to $x^2 + y^2 \pm 2ax = 0$ ...(4)

(iv) In this case $h = 0 = k$, Hence the equation (1) reduces to $x^2 + y^2 = a^2$ ...(5)

(v) In this case $k = a$ (see Fig. 15.5)

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ ...(6)

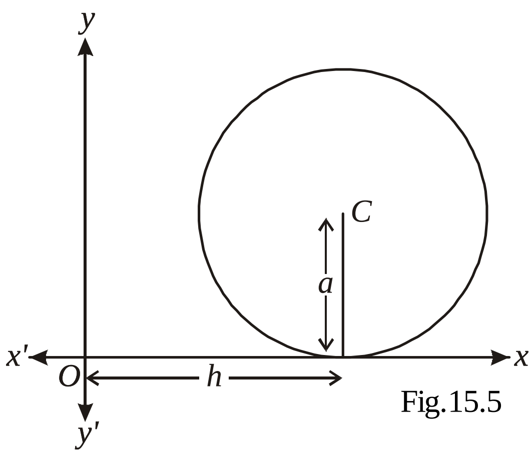

Fig.15.5

(*vi*) In this case $h = a$

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ky + k^2 = 0$ ...(7)

(*vii*) In this case $h = k = a$. (See Fig. 15.6)

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(8)

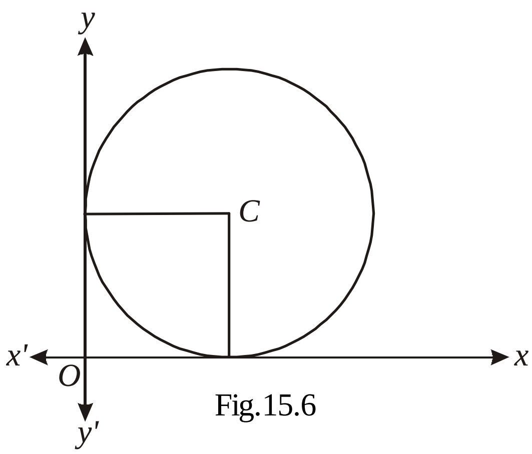

Fig.15.6

Example 15.1 Find the equation of the circle whose centre is (3, –4) and radius is 6.

Solution : Comparing the terms given in equation (1), we have

$h = 3$, $k = -4$ and $a = 6$.

$\therefore$ $(x-3)^2 + (y+4)^2 = 6^2$ or $x^2 + y^2 - 6x + 8y - 11 = 0$

Example 15.2 Find the centre and radius of the circle given by $(x+1)^2 + (y-1)^2 = 4$.

Solution: Comparing the given equation with $(x-h)^2 + (y-k)^2 = a^2$ we find that

$-h = 1, -k = -1, a^2 = 4$

$\therefore \quad h = -1, k = 1, a = 2.$

So the given circle has its centre (–1,1) and radius 2.

15.3 GENERAL EQUATION OF THE CIRCLE IN SECOND DEGREE IN TWO VARIABLES

The standard equation of a circle with centre (h, k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{... (1)}$$

or $$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \qquad \text{... (2)}$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0.$

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (3)}$$

$$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$\Rightarrow \quad [x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \qquad \text{... (4)}$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$$

where $h = -g, \quad k = -f, \quad r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

15.3.1 CONDITONS UNDER WHICH THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES REPRESENTS A CIRCLE

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) It is a second degree equation in x, y in which coefficients of the terms involving x^2 and y^2 are equal.

(ii) It contains no term involving xy

Note : *In solving problems, we keep the coefficients of x^2 and y^2 unity.*

Example 15.3 Find the centre and radius of the circle

$$45x^2 + 45y^2 - 60x + 36y + 19 = 0$$

Solution : Given equation can be written on dividing by 45 as

$$x^2 + y^2 - \frac{4}{3}x + \frac{4}{5}y + \frac{19}{45} = 0$$

Comparing it with the equation, $x^2 + y^2 + 2gx + 2fy + c = 0$ we get

$$g = -\frac{2}{3},\ f = \frac{2}{5} \text{ and } c = \frac{19}{45}$$

Thus, the centre is $\left(\frac{2}{3}, -\frac{2}{5}\right)$ and radius is $\sqrt{g^2 + f^2 - c} = \frac{\sqrt{41}}{15}$

Example 15.4 Find the equation of the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

Solution: Let the equation of the circle be, $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Since the circle passes through three given points so they will satisfy the equation (1). Hence

$$1 + 2g + c = 0 \qquad \cdots(2)$$

and

$$36 - 12f + c = 0 \qquad \cdots(3)$$

$$25 + 6g + 8f + c = 0 \qquad \cdots(4)$$

Subtracting (2) from (3) and (3) from (4), we have $2g + 12f = 35$ and $6g + 20f = 11$

Solving these equations for *g* and *f*, we get $g = -\frac{71}{4},\quad f = \frac{47}{8}$

Substituting *g* in (2), we get $c = \frac{69}{2}$

and substituting *g*, *f* and *c* in (1), the required equation of the circle is

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

Exmaple 15. 5 Find the equation of the circles which touches the axis of *x* and passes through the points (1,–2) and (3, –4).

Solution : Since the circle touches the *x*-axis, put *k* = *a* in the standard form (See result 6) of the equation of the circle, we have, $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ ... (1)

This circle passes through the point (1, –2) $\therefore$ $h^2 - 2h + 4a + 5 = 0$ ... (2)

Also, the circle passes through the point (3, – 4) $\therefore$ $h^2 - 6h + 8a + 25 = 0$ ...(3)

Eliminationg '*a*' from (2) and (3), we get $\Rightarrow$

$$h^2 + 2h - 15 = 0$$

$$h = 3 \ or \ h = -5.$$

From (3) the corresponding values of a are – 2 and –10 respectively. On substituting the values of h and a in (1) we get , $x^2 + y^2 - 6x + 4y + 9 = 0$ ... (4)

and $x^2 + y^2 + 10x + 20y + 25 = 0$ ... (5)

(4) and (5) represent the required equations.

CHECK YOUR PROGRESS 15.1

1. Find the equation of the circle whose

   (a) centre is (0, 0) and radius is 3. (b) centre is (– 2,3) and radius is 4.

2. Find the centre and radius of the circle

   (a) $x^2 + y^2 + 3x - y = 6$ (b) $4x^2 + 4y^2 - 2x + 3y - 6 = 0$

3. Find the equation of the circle which passes through the points (0, 2) (2, 0) and (0, 0).

4. Find the equation of the circle which touches the y-axis and passes through the points (–1,2) and (–2,1)

LET US SUM UP

- **Standard form of the circle**

  $(x-h)^2 + (y-k)^2 = a^2$, Centre is (h, k) and radius is a

- The general form of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

  Its centre: $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c}$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=6r1GQCxyMKI

www.purplemath.com/modules/**circle**2.htm

www.purplemath.com/modules/**circle**.htm

TERMINAL EXERCISE

1. Find the equation of a circle with centre (4, – 6) and radius 7.

2. Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$

3. Find the equation of the circle passes through the point (1,0), (–1,0) and (0,1)

MODULE-IV
Co-ordinate Geometry

Notes

ANSWERS

CHECK YOUR PROGRESS 15.1

1. (a) $x^2 + y^2 = 9$

   (b) $x^2 + y^2 + 4x - 6y - 3 = 0$

2. (a) $\left(-\frac{3}{2}, 1\right); \frac{\sqrt{37}}{2}$

   (b) $\left(\frac{1}{4}, -\frac{3}{8}\right); \frac{\sqrt{109}}{8}$

3. $x^2 + y^2 - 2x - 2y = 0$

4. $x^2 + y^2 + 2x - 2y + 1 = 0$

TERMINAL EXERCISE

1. $x^2 + y^2 - 8x + 12y + 3 = 0$

2. Centre $(-2, 3)$; Radius = $\sqrt{13}$

3. $x^2 + y^2 = 1$.

16

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CONIC SECTIONS

While cutting a carrot you might have noticed different shapes shown by the edges of the cut. Analytically you may cut it in three different ways, namely

(*i*) Cut is parallel to the base (see Fig.16.1)

(*ii*) Cut is slanting but does not pass through the base (see Fig.16.2)

(*iii*) Cut is slanting and passes through the base (see Fig.16.3)

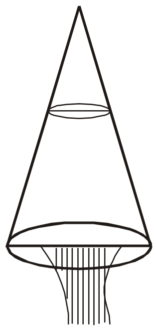


Fig.16.1

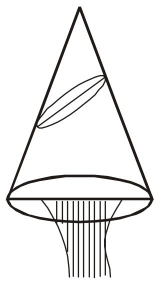


Fig.16.2

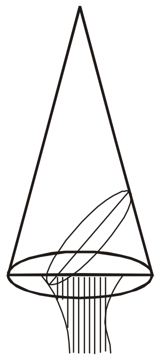


Fig.16.3

The different ways of cutting, give us slices of different shapes.

In the first case, the slice cut represent a circle which we have studied in previous lesson.

In the second and third cases the slices cut represent different geometrical curves, which we shall study in this lesson.

OBJECTIVES

After studying this leson, you will be able to :

- recognise a circle, parabola, ellipse and hyperbola as sections of a cone;
- recognise the parabola, ellipse and hyperbola as certain loci;
- identify the concept of eccentricity, directrix, focus and vertex of a conic section;
- identify the standard equations of parabola, ellipse and hyperbola;
- find the equation of a parabola, ellipse and hyperbola given its directrix and focus.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of coordinate Geometry
- Various forms of equation of a straight line
- Equation of a circle in various forms

16.1 CONIC SECTION

In the introduction we have noticed the various shapes of the slice of the carrot. Since the carrot is conical in shape so the section formed are sections of a cone. They are therefore called conic sections.

Mathematically, a conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed line.

The fixed point is called the ***focus*** and is usually denoted by S.

The fixed straight line is called the ***Directrix.***

The straight line passing through the focus and perpendicular to the directrix is called the ***axis.***

The constant ratio is called the ***eccentricity*** and is denoted by e.

What happens when

(*i*) $e<1$ (*ii*) $e=1$ (*iii*) $e>1$

In these cases the conic section obtained are known as ellipse, parabola and hyperbola respectively.

In this lesson we shall study about ellipse, parabola, and hyperbola.

16.2 ELLIPSE

Recall the cutting of slices of a carrot. When we cut it obliquely, slanting without letting the knife pass through the base, what do we observe?

You might have come across such shapes when you cut a boiled egg vertically.

The slice thus obtained represents an ellipse. Let us define the ellipse mathematically as follows:

> ***"An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity".***

16.2.1 STANDARD EQUATION OF AN ELLIPSE

Let S be the focus, ZK be the directrix and P be a moving point. Draw SK perpendicular from S on the directrix. Let e be the eccentricity.

Divide SK internally and externally at A and A' (on KS produced) repectively in the ratio $e:1$, as $e<1$.

$SA = e.AK$ $\quad\cdots$ (1)

and $SA' = e.A'K$ $\quad\cdots$ (2)

Since A and A' are points such that their distances from the focus bears a constant ratio e $(e<1)$ to their respective distances from the directrix and so they lie on the ellipse. These points are called vertices of the ellipse.

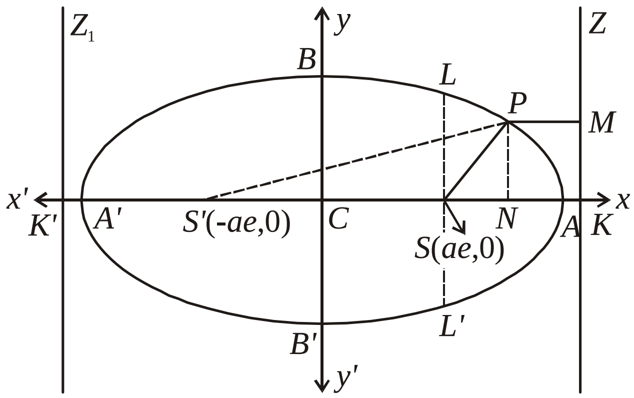


Fig.16.4

Let AA' be equal to $2a$ and C be its mid point, i.e., $CA = CA' = a$

The point C is called the centre of the ellipse.

Adding (1) and (2), we have

$SA + SA' = e.AK + e.A'K$ $\qquad A$

or $\quad AA' = e(CK - CA + A'C + CK)$ or $\quad 2a = e.2CK$ or $\quad CK = \dfrac{a}{e}$ $\quad\cdots$ (3)

Subtracting (1) from (2), we have

$SA' - SA = e(A'K - AK)$

or $\quad (SC + CA') - (CA - CS) = e.A'A$

or $\quad 2\ CS = e.2a$ or $\quad CS = ae$ $\quad\cdots$ (4)

Let us choose C as origin, CAX as x-axis and CY, a line perpendicular to CX as y-axis.

$\therefore$ Coordinates of S are then $(ae, 0)$ and equation of the directrix is $x = \dfrac{a}{e}$

Let the coordinates of the moving point P be (x, y). Join SP, draw $PM \perp ZK$.

By definition $SP = e.PM$ or $\quad SP^2 = e^2 .\ PM^2$

or $\quad SN^2 + NP^2 = e^2.(NK)^2$ or $\quad (CN - CS)^2 + NP^2 = e^2.(CK - CN)^2$

or $\quad (x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2$ or $\quad x^2(1-e^2)+y^2=a^2(1-e^2)$

or $\quad \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \qquad$ [On dividing by $a^2(1-e^2)$]

Putting $a^2(1-e^2)=b^2$, we have the standard form of the ellipse as, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Major axis : The line joining the two vertices A' and A, i.e., $A'A$ is called the major axis and its length is $2a$.

Minor axis : The line passing through the centre perpendicular to the major axis, i.e., BB' is called the minor axis and its length is $2b$.

Principal axis : The two axes together (major and minor) are called the principal axes of the ellipse.

Latus rectum : The length of the line segment LL' is called the latus rectum and it is given by

$$\frac{2b^2}{a}$$

Equation of the directrix : $x=\pm\frac{a}{e}$

Eccentricity : e is given by $e^2=1-\frac{b^2}{a^2}$

Example 16.1 Find the equation of the ellipse whose focus is (1,–1), eccentricity $e=\frac{1}{2}$ and the directrix is $x-y=3$.

Solution : Let $P\,(h,k)$ be any point on the ellipse then by the definition, its distance from the focus = e. Its distance from directrix or $SP^2=e^2.PM^2$

(M is the foot of the perpendicular drawn from P to the directrix).

or $\quad (h-1)^2+(k+1)^2=\frac{1}{4}\left(\frac{h-k-3}{\sqrt{1+1}}\right)^2$

or $\quad 7(h^2+k^2)+2hk-10h+10k+7=0$

$\therefore \quad$ The locus of P is, $7(x^2+y^2)+2xy-10x+10y+7=0$

which is the required equation of the ellipse.

Example 16.2 Find the eccentricity, coordinates of the foci and the length of the axes of the ellipse $3x^2 + 4y^2 = 12$

Solution : The equation of the ellipse can be written in the following form, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

On comparing this equation with that of the standard equation of the ellipse, we have $a^2 = 4$ and $b^2 = 3$, then

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

(ii) coordinates of the foci are (1,0) and (–1,0)

[$\because$ The coordinate are ($\pm$ ae, 0)]

(*iii*) Length of the major axes $2a = 2 \times 2 = 4$ and

length of the minor axis = $2b = 2 \times \sqrt{3} = 2\sqrt{3}.$

CHECK YOUR PROGRESS 16.1

1. Find the equation of the ellipse referred to its centre

   (a) whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$

   (b) whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

   (c) whose foci are the points (4,0) and (– 4,0) and whose eccentricity is $\frac{1}{3}$.

2. Find the eccentricity of the ellipse, if its latus rectum be equal to one half its minor axis.

16.3 PARABOLA

Recall the cutting of slice of a carrot. When we cut obliquely and letting the knife pass through the base, what do we observe?

Also when a batsman hits the ball in air, have you ever noticed the path of the ball?

Is there any property common to the edge of the slice of the carrot and the path traced out by the ball in the example cited above?

Yes, the edge of such a slice and path of the ball have the same shape which is known as a parabola. Let us define parabola mathematically.

"A parabola is the locus of a point which moves in a plane so that its distance

from a fixed point in the plane is equal to its distance from a fixed line in the plane."

16.3.1 STANDARD EQUATION OF A PARABOLA

Let S be the fixed point and ZZ' be the directrix of the parabola. Draw SK perpendicular to ZZ'. Bisect SK at A.

Since $SA = AK$, by the definition of the parabola A lies on the parabola. A is called the vertex of the parabola.

Take A as origin, AX as the x-axis and AY perpendicular to AX through A as the y-axis.

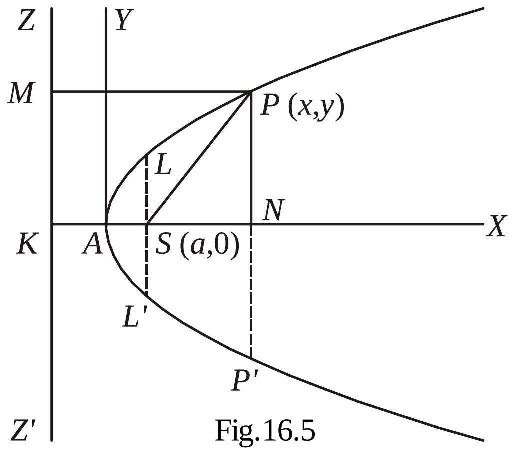


Fig.16.5

Let $\quad KS = 2a \qquad \therefore\ AS = AK = a$

$\therefore$ The coordinates of A and S are $(0,0)$ and $(a,0)$ respectively.

Let $P(x,y)$ be any point on the parabola. Draw $PN \perp AS$ produced

$\therefore\quad AN = x$ and $NP = y$

Join SP and draw $PM \perp ZZ'$

$\therefore$ By definition of the parabola

$SP = PM$ or $\qquad SP^2 = PM^2$

or $\quad (x-a)^2 + (y-0)^2 = (x+a)^2 \qquad [\because PM = NK = NA + AK = x + a]$

or $\quad (x-a)^2 - (x+a)^2 = -y^2$ or $\qquad y^2 = 4ax$

which is the ***standard equation*** of the parabola.

Note : *In this equation of the parabola*

(i) Vertex is $(0,0)$

(ii) Focus is $(a,0)$

(iii) Equation of the axis is $y = 0$

(iv) Equation of the directrix is $x + a = 0$

(v) Latus rectum $= 4a$

16.3.2 OTHER FORMS OF THE PARABOLA

What will be the equation of the parabola when

(*i*) focus is $(-a, 0)$ and directrix is $x - a = 0$

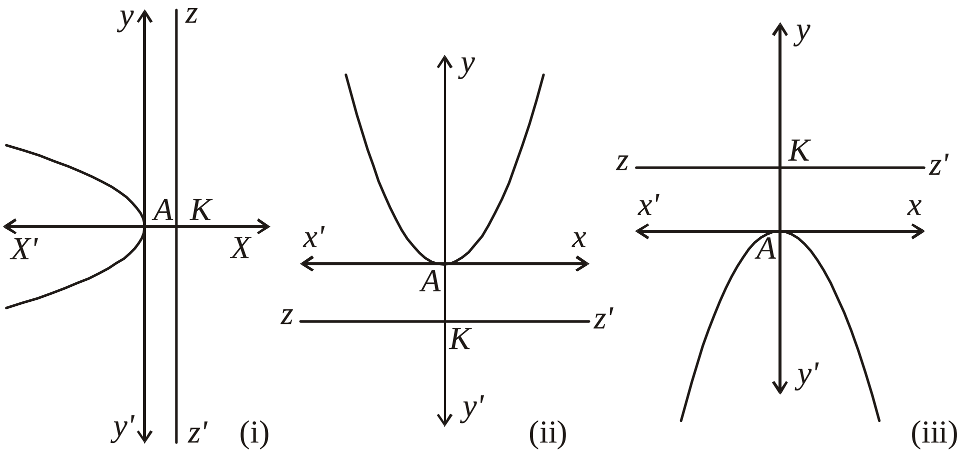
(*ii*) focus is $(0, a)$ and directrix is $y + a = 0$,

(*iii*) focus is $(0, -a)$ and directrix is $y - a = 0$?

It can easily be shown that the equation of the parabola with above conditions takes the following forms:

(*i*) $y^2 = -4ax$ (*ii*) $x^2 = 4ay$ (*iii*) $x^2 = -4ay$

The figures are given below for the above equations of the parabolas.



$y^2 = -4ax$ $x^2 = 4ay$ $x^2 = -4ay$

Fig.16.6

Corresponding results of above forms of parabolas are as follows:

Forms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0,0)	(0,0)	(0,0)	(0,0)
Coordinates of focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Coordinates of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Coordinates of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
length of Latus rectum	$4a$	$4a$	$4a$	$4a$

Example 16.3 Find the equation of the parabola whose focus is the origin and whose directrix is the line $2x + y - 1 = 0$.

Solution : Let S (0,0) be the focus and ZZ' be the directrix whose equation is $2x + y - 1 = 0$

Let $P(x, y)$ be any point on the parabola.

Let PM be perpendicular to the directrix (See Fig. 16.5)

$\therefore$ By definition $SP = PM$ or $SP^2 = PM^2$

or $$x^2 + y^2 = \frac{(2x + y - 1)^2}{\left(\sqrt{2^2 + 1}\right)^2}$$

or $5x^2 + 5y^2 = 4x^2 + y^2 + 1 + 4xy - 2y - 4x$ or $x^2 + 4y^2 - 4xy + 2y + 4x - 1 = 0.$

Example 16.4 Find the equation of the parabola, whose focus is the point (2, 3) and whose directrix is the line $x - 4y + 3 = 0$.

Solution : Given focus is $S(2,3)$; and the equation of the directrix is $x - 4y + 3 = 0$.

$\therefore$ As in the above example, $(x-2)^2 + (y-3)^2 = \left\{\dfrac{x - 4y + 3}{\sqrt{1^2 + 4^2}}\right\}^2$

$\Rightarrow$ $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$

CHECK YOUR PROGRESS 16.2

1. Find the equation of the parabola whose focus is (a, b) and whose directrix is $\frac{x}{a} + \frac{y}{b} = 1$.
2. Find the equation of the parabola whose focus is (2,3) and whose directrix is $3x + 4y = 1$.

16.4 HYPERBOLA

Hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its distance from a fixed straight line in the same plane is greater than one. In other words hyperbola is the conic in which eccentricity is greater than unity. The fixed point is called focus and the fixed straight line is called directrix.

Equation of Hyperbola in Standard from :

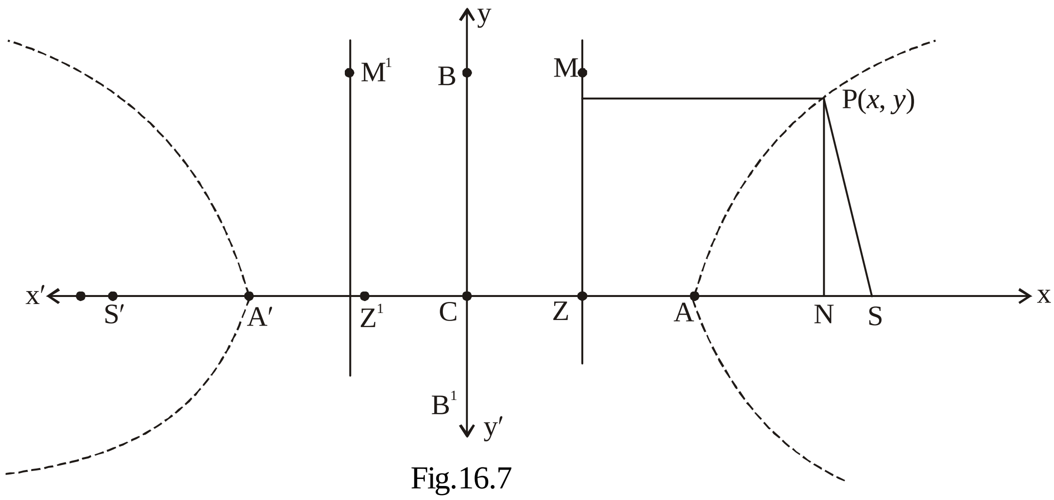


Fig.16.7

Let S be the focus and ZM be the directrix. Draw SZ perpendicular from S on directix we can divide SZ both internally and externally in the ratio $e : 1$ ($e > 1$). Let the points of division be A and A′ as shown in the above figure. Let C be the mid point of AA′. Now take CZ as the x-axis and the perpendicular at C as y-axis.

Let $\quad AA' = 2a$

Now $\quad \dfrac{SA}{AZ} = e \ (e > 1)$ and $\dfrac{SA'}{A'Z} = e \ (e > 1)$.

i.e. $\quad SA = eAZ \qquad$ ...(i)

i.e. $\quad SA' = eA'Z \qquad$ ...(ii)

Adding (i) and (ii) we get

$$SA + SA' = e(AZ + A'Z)$$

$$(CS - CA) + (CS + CA') = eAA'$$

$$\Rightarrow \quad 2CS = e.2a \ (\because \ CA = CA')$$

$$\Rightarrow \quad CS = ae$$

Hence focus point is $(ae, 0)$.

Subtracting (i) from (ii) we get

$$SA' - SA = e(A'Z - AZ)$$

i.e. $\quad AA' = e\left[(CZ + CA') - (CA - CZ)\right]$

i.e. $\quad AA' = e[2CZ] \quad (\because \ CA' = CA)$

i.e. $\quad 2a = e(2CZ)$

$$\Rightarrow \quad CZ = \frac{a}{e}$$

$\therefore$ Equation of directrix is $x = \dfrac{a}{e}$.

Let P(x, y) be any point on the hyperbola, PM and PN be the perpendiculars from P on the directrix and x-axis respectively.

Thus, $\dfrac{SP}{PM} = e \quad \Rightarrow \text{SP} = e\text{PM}$

$\Rightarrow \quad (\text{SP})^2 = e^2(\text{PM})^2$

i.e. $\quad (x-ae)^2+(y-0)^2 = e^2\left(x-\dfrac{a}{e}\right)^2$

i.e. $\quad x^2+a^2e^2-2aex+y^2 = e^2\left(\dfrac{e^2x^2+a^2-2aex}{e^2}\right)$

i.e. $\quad x^2+a^2e^2+y^2 = e^2x^2+a^2$

i.e. $\quad (e^2-1)x^2-y^2 = a^2(e^2-1)$

i.e. $\quad \dfrac{x^2}{a^2}-\dfrac{y^2}{a^2(e^2-1)} = 1$

Let $\quad a^2(e^2-1) = b^2$

$$\therefore \quad \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1}$$

Which is the equation of hyperbola in standard from.

- Now let S′ be the image of S and Z′M′ be the image of ZM w.r.t y-axis. Taking S′ as focus and Z′M′ as directrix, it can be seen that the corresponding equation of hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$ Hence for every hyperbola, there are two foci and two directrices.
- We have $b^2 = a^2(e^2-1)$ and $e > 1$

  $\Rightarrow \quad e = \sqrt{\dfrac{a^2+b^2}{a^2}}$
- If we put $y = 0$ in the equation of hyperbola we get

  $x^2 = a^2 \Rightarrow x = \pm a$

  $\therefore$ Hyperbola cuts x-axis at A(a, 0) and A′($-a$, 0).
- If we put $x = 0$ in the equation of hyperbola we get

$y^2 = -b^2 \Rightarrow y = \pm\sqrt{-1}.b = \pm ib$

Which does not exist in the cartesian plane.

$\therefore$ Hyperbola does not interesct y-axis.

- AA′ = 2a, along the x-axis is called **transverse axis** of the hyperbola and BB′ = 2b, along y-axis is called **conjugate axis** of the hyperbola. Notice that hyperbola does not meet its conjugate axis.
- As in case of ellipse, hyperbola has two foci

  S(ae, 0), S′(–ae, 0) and two directrices $x = \pm\frac{a}{e}$.
- C is called the centre of hyperbola.
- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola. As in ellipse, it can be proved that the length of the latus rectum of hyperbola is $\frac{2b^2}{a}$.
- Hyperbola is symmetric about both the axes.
- Foci of hyperbola are always on transverse axis. It is the positive term whose denominator gives the transverse axis. For example $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has transverse axis along x-axis and length of transverse axis is 6 units. While $\frac{y^2}{25} - \frac{x^2}{16} = 1$ has transverse axis along y-axis of length 10 unit.
- The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of given hyperbola, is called the conjugate hyperbola of the given hyperbola. This equation is of the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

In this case : Transverse axis is along y-axis and conjugate axis is along x-axis.

- Length of transverse axis = 2b.
- Length of conjugate axis = 2a
- Length of latus rectum = $\frac{2a^2}{b}$.
- Equations of directrices $y = \pm\frac{b}{e}$.
- Vertices (0, ± b)
- Foci (0, ± be)
- Centre (0, 0)
- Eccentricity (e) = $\sqrt{\frac{b^2 + a^2}{b^2}}$.

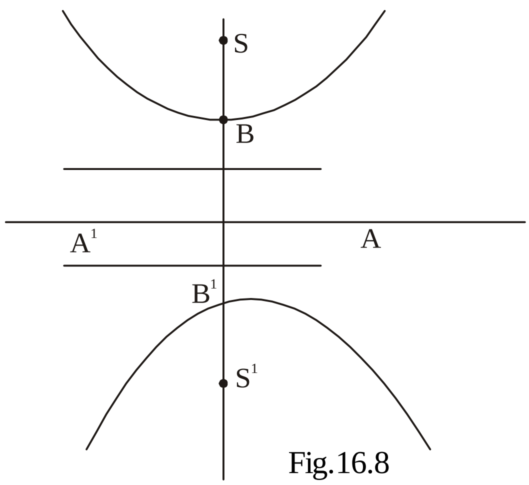


Fig.16.8

16.4.1 RECTANGULAR HYPERBOLA

If in a hyperbola the length of the transverse axis is equal to the length of the conjugate axis, then the hyperbola is called a rectangular hyperbola.

Its equation is $x^2 - y^2 = a^2$ or $y^2 - x^2 = b^2$ ($\because a = b$)

In this case $e = \sqrt{\dfrac{a^2+a^2}{a^2}}$ or $\sqrt{\dfrac{b^2+b^2}{b^2}} = \sqrt{2}$

i.e. the eccentricity of rectangualr hyperbola is $\sqrt{2}$.

Example 16.5 For the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, find the following (i) Eccentricity (ii) Foci (iii) Vertices (iv) Directrices (v) Length of transverse axis (vi) Length of conjugate axis (vii) Length of latus rectum (viii) Centre.

Solution : Here $a^2 = 16$ and $b^2 = 9$, $\Rightarrow a = 4$ and $b = 3$.

(i) Eccentricity (e) = $\sqrt{\dfrac{a^2+b^2}{a^2}} = \sqrt{\dfrac{16+9}{16}} = \dfrac{5}{4}$

(ii) Foci = $(\pm\ ae, 0) = \left(\pm\dfrac{4.5}{4}, 0\right) = (\pm 5, 0)$

(iii) Vertices = $(\pm\ a,\ 0) = (\pm\ 4,\ 0)$

(iv) Directrices $x = \pm\dfrac{a}{e} \Rightarrow x = \pm\dfrac{4}{5/4} \Rightarrow x = \pm\dfrac{16}{5}$.

(v) Length of transverse axis = $2a = 2 \times 4 = 8$.

(vi) Length of conjugate axis = $2a = 2 \times 3 = 6$

(vii) Length of latus rectum = $\dfrac{2b^2}{a} = \dfrac{2\times 9}{4} = \dfrac{9}{2}$.

(viii) Centre = (0, 0)

Example 16.6 Find the equation of hyperbola with vertices (± 2, 0) and foci (± 3, 0)

Solution : Here $a = 2$ and $ae = 3$.

$\therefore$ $e = 3/2$.

We know that $b^2 = a^2(e^2 - 1)$

$\Rightarrow$ $b^2 = 4\left(\dfrac{9}{4} - 1\right) = 5$

$\therefore$ Equation of hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

Example 16.7 For hyperbola $\frac{y^2}{9} - \frac{x^2}{27} = 1,$ find the following :

(i) Eccentricity (ii) Centre (iii) Foci (iv) Vertices (v) Directrices (vi) Length of transverse axis (vii) Length of conjugate axis (viii) Latus rectum.

Solution : Here $b^2 = 9$ and $a^2 = 27 \Rightarrow b = 3$ and $a = 3\sqrt{3}.$

(i) $e = \sqrt{\frac{27+9}{9}} = \sqrt{4} = 2.$ (ii) Centre = (0, 0)

(iii) Foci = $(0, \pm be) = (0, \pm 3.2) = (0, \pm 6).$

(iv) Vertices = $(0, \pm b) = (0, \pm 3).$

(v) Directrices, $y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{3}{2}.$

(vi) Length of transverse axis = $2b = 2 \times 3 = 6$

(vii) Length of conjugate axis = $2a = 2 \times 3\sqrt{3} = 6\sqrt{3}$

(viii) Length of latus rectum = $\frac{2a^2}{b} = \frac{2 \times 27}{3} = 18.$

CHECK YOUR PROGRESS 16.3

1. (i) Transverse axis of the hyperbola $\frac{y^2}{25} - \frac{x^2}{16}$ is along ....

   (ii) Eccentricity of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is ...

   (iii) Eccentricity of rectangular hyperbola is ...

   (iv) Length of latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is ...

   (v) Foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is at ...

   (vi) Equation of directrices of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is ...

   (vii) Vertices of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at ...

2. For the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, complete the following.

   (i) Eccentricity (e) = ...

   (ii) Centre = ...

(iii) Foci = ...

(iv) Vertices = ...

(v) Equations of directrices, y = ...

(vi) Length of latus rectum = ...

(vii) Length of transverse axis = ...

(viii) Length of conjugate axis = ...

(ix) Transverse axis is along ...

(x) Conjugate axis is along ...

LET US SUM UP

- **Conic Section**

"A conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line".

(*i*) **Focus :** The fixed point is called the focus.

(*ii*) **Directrix :** The fixed straight line is called the directrix.

(*iii*) **Axis :** The straight line passing through the focus and pependicular to the directrix is called the axis.

(*iv*) **Eccentricity :** The constant ratio is called the eccentricity.

- (*v*) **Latus Rectum :** The double ordinate passing through the focus and parallel to the directrix is known as latus rectum. (In Fig.16.5 LSL' is the latus rectum).

- **Standard Equation of the Ellipse is :** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(*i*) Major axis = $2a$ (*ii*) Minor axis = $2b$

(*iii*) Equation of directrix is $x=\pm\frac{a}{e}$ (*iv*) Foci : $(\pm ae, 0)$

(*v*) Eccentricity, i.e., e is given by $e^2=1-\frac{b^2}{a^2}$ vi Latus Reotam $=\frac{2b^2}{a}$

- **Standard Equation of the Parabola is :** $y^2=4ax$

(*i*) Vertex is (0,0) (*ii*) Focus is $(a,0)$

(*iii*) Axis of the parabola is $y=0$ (*iv*) Directrix of the parabola is $x+a=0$

(*v*) Latus rectum = $4a$.

- **OTHER FORMS OF THE PARABOLA ARE**

  (*i*) $y^2 = -4ax$ (concave to the left).

  (*ii*) $x^2 = 4ay$ (concave upwards).

  (*iii*) $x^2 = -4ay$ (concave downwards).

- Equation of hyperbola having transverse axis along x-axis and conjugate axis along y-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

  For this hyperbola (i) $e = \sqrt{\frac{a^2 + b^2}{a^2}}$.

  (ii) Centre = (0, 0) (iii) Foci = ($\pm$ *ae*, 0)

  (iv) Vetrices = ($\pm$ *a*, 0)(v) Length of latus rectum = $\frac{2b^2}{a}$

  (vi) Length of transverse axis = 2*a*

  (vii) Length of conjugate axis = 2*b*

  (viii) Equations of directrixes are given by $x = \pm \frac{a}{e}$.

- Equations of hyperbola having transverse axis along y-axis and conjugate axis along x-axis is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$

  For this hyperbola :

  (i) Vertices = (0, $\pm$ *b*)(ii) Centre = (0, 0)

  (iii) Foci = (0, $\pm$ *be*) (iv) $e = \sqrt{\frac{a^2 + b^2}{b^2}}$

  (v) Length of latus rectum = $\frac{2a^2}{b}$.

  (vi) Length of transverse axis = 2*b*.

  (vii) Length of conjugate axis = 2*a*.

  (viii) Equations of directrixes are given by $y = \pm \frac{b}{e}$.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=0A7RR0oy2ho

http://www.youtube.com/watch?v=lvAYFUIEpFI

http://www.youtube.com/watch?v=QR2vxfwiHAU

http://www.youtube.com/watch?v=pzSyOTkAsY4

http://www.youtube.com/watch?v=hl58vTCqVIY

http://www.youtube.com/watch?v=lGQw-W1PxBE

http://www.youtube.com/watch?v=S0Fd2Tg2v7M

TERMINAL EXERCISE

1. Find the equation of the ellipse in each of the following cases, when

   (a) focus is (0, 1), directrix is $x + y = 0$ and $e = \frac{1}{2}$.

   (b) focus is (–1,, 1), directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.

2. Find the coordinates of the foci and the eccentricity of each of the following ellipses:

   (a) $4x^2 + 9y^2 = 1$ (b) $25x^2 + 4y^2 = 100$

3. Find the equation of the parabola whose focus is (–8, –2) and directrix is $y - 2x + 9 = 0$.

4. Find the equation of the hyperbola whose foci are (± 5, 0) and the length of the transverse axis is 8 units.

5. Find the equation of the hyperbola with vertices at (0, ± 6) and $e = \frac{5}{3}$.

6. Find the eccentricity, length of transverse axis, length of conjugate axis, vertices, foci, equations of directrices, and length of latus rectum of the hyperbola (i) $25x^2 - 9y^2 = 225$ (ii) $16y^2 - 4x^2 = 1$.

7. Find the equation of the hyperbola with foci $(0, \pm\sqrt{10})$, and passing through the point (2, 3).

8. Find the equation of the hyperbola with foci (± 4, 0) and length of latus rectum 12.

ANSWERS

CHECK YOUR PROGRESS 16.1

1. (a) $20x^2 + 36y^2 = 405$

   (b) $x^2 + 2y^2 = 100$

   (c) $8x^2 + 9y^2 = 1152$

2. $\dfrac{\sqrt{3}}{2}$

CHECK YOUR PROGRESS 16.2

1. $(ax-by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0.$

2. $16x^2 + 9y^2 - 94x - 142y - 24xy + 324 = 0$

CHECK YOUR PROGRESS 16.3

1. (i) y-axis (ii) $\dfrac{5}{3}$

   (iii) $\sqrt{2}$ (iv) $\dfrac{2b^2}{a}$

   (v) $(\pm\ ae,\ 0)$ (vi) $x = \pm\dfrac{a}{e}$

   (vii) $(\pm\ a,\ 0)$

2. (i) $\sqrt{\dfrac{b^2+a^2}{b^2}}$ (ii) $(0,\ 0)$ (iii) $(0,\ \pm\ be)$

   (iv) $(0,\ \pm\ \mathrm{b})$ (v) $\dfrac{\pm b}{e}$ (vi) $\dfrac{2a^2}{b}$

   (vii) $2b$ (viii) $2a$ (ix) y-axis (x) x-axis

TERMINAL EXERCISE

1. (a) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$

   (b) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

2. (a) $\left(\pm\dfrac{\sqrt{5}}{6}, 0\right);\ \dfrac{\sqrt{5}}{3}$

   (b) $\left(0\ ,\ \pm\sqrt{21}\right);\ \dfrac{\sqrt{21}}{5}$

3. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

4. $9x^2 - 16y^2 = 144$

5. $16y^2 - 9x^2 = 576$

6. (i) Eccentricity = $\dfrac{\sqrt{34}}{3}$, length of transverse axis = 6, length of conjugate axis = 10, vertices (± 3, 0), Foci $(\pm\sqrt{34}, 0)$, equations of directrices $x = \pm\dfrac{1}{\sqrt{34}}$, latus rectum = $\dfrac{50}{3}$.

   (ii) Eccentricity = $\sqrt{5}$, length of transverse axis = $\dfrac{1}{2}$, length of conjugate axis = 1, vertices $\left(0, \pm\dfrac{1}{4}\right)$, Foci $\left(0, \pm\dfrac{\sqrt{5}}{4}\right)$, equations of directrices, $y = \dfrac{1}{4\sqrt{5}}$, latus rectum = 2.

7. $y^2 - x^2 = 5$

8. $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$

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17

MEASURES OF DISPERSION

You have learnt various measures of central tendency. Measures of central tendency help us to represent the entire mass of the data by a single value.

Can the **central tendency** describe the data fully and adequately?

In order to understand it, let us consider an example.

The daily income of the workers in two factories are :

Factory A	:	35	45	50	65	70	90	100
Factory B	:	60	65	65	65	65	65	70

Here we observe that in both the groups the mean of the data is the same, namely, 65

(i) In group A, the observations are much more scattered from the mean.

(ii) In group B, almost all the observations are concentrated around the mean.

Certainly, the two groups differ even though they have the same mean.

Thus, there arises a need to differentiate between the groups. We need some other measures which concern with the measure of scatteredness (or spread).

To do this, we study what is known as **measures of dispersion.**

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of dispersion through examples;
- define various measures of dispersion – range, mean deviation, variance and standard deviation;
- calculate mean deviation from the mean of raw and grouped data;
- calculate mean deviation from the median of raw and grouped data.
- calculate variance and standard deviation of raw and grouped data; and
- illustrate the properties of variance and standard deviation.
- Analyses the frequencys distributions with equal means.

EXPECTED BACKGROUND KNOWLEDGE

- Mean of grouped data
- Median of ungrouped data

17.1 MEANING OF DISPERSION

To explain the meaning of dispersion, let us consider an example.

Two sections of 10 students each in class X in a certain school were given a common test in Mathematics (maximum marks 40). The scores of the students are given below :

Section A :	6	9	11	13	15	21	23	28	29	35
Section B:	15	16	16	17	18	19	20	21	23	25

The average score in section A is 19.

The average score in section B is 19.

Let us construct a dot diagram, on the same scale for section A and section B (see Fig. 17.1)

The position of mean is marked by an arrow in the dot diagram.

Section A

Section B

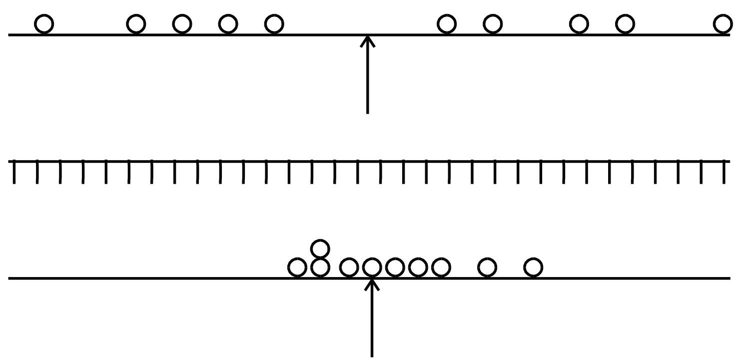

Fig. 17.1

Clearly, the extent of spread or dispersion of the data is different in section A from that of B. The measurement of the scatter of the given data about the average is said to be a measure of dispersion or scatter.

In this lesson, you will read about the following measures of dispersion :

(a) Range

(b) Mean deviation from mean

(c) Mean deviation from median

(d) Variance

(e) Standard deviation

17.2 DEFINITION OF VARIOUS MEASURES OF DISPERSION

(a) **Range :** In the above cited example, we observe that

(i) the scores of all the students in section A are ranging from 6 to 35;

(ii) the scores of the students in section B are ranging from 15 to 25.

The difference between the largest and the smallest scores in section A is 29 (35 – 6)

The difference between the largest and smallest scores in section B is 10 (25 – 15).

Thus, the difference between the largest and the smallest value of a data, is termed as the range of the distribution.

(b) **Mean Deviation from Mean :** In Fig. 17.1, we note that the scores in section B cluster around the mean while in section A the scores are spread away from the mean. Let us take the deviation of each observation from the mean and add all such deviations. If the sum is 'large', the dispersion is 'large'. If, however, the sum is 'small' the dispersion is small.

Let us find the sum of deviations from the mean, i.e., 19 for scores in section A.

Observations (x_i)	Deviations from mean ($x_i - \overline{x}$)
6	–13
9	–10
11	–8
13	–6
15	–4
21	+2
23	+4
28	+9
29	+10
35	16
190	0

Here, the sum is zero. It is neither 'large' nor 'small'. Is it a coincidence ?

Let us now find the sum of deviations from the mean, i.e., 19 for scores in section B.

Observations (x_i)	Deviations from mean ($x_i - \overline{x}$)
15	–4
16	–3
16	–3
17	–2
18	–1
19	0
20	1
21	2
23	4
25	6
190	0

Again, the sum is zero. Certainly it is not a coincidence. In fact, we have proved earlier that **the sum of the deviations taken from the mean is always zero for any set of data.** Why is the sum always zero ?

On close examination, we find that the signs of some deviations are positive and of some other deviations are negative. Perhaps, this is what makes their sum always zero. In both the cases, we get sum of deviations to be zero, so, we cannot draw any conclusion from the sum of deviations. But this can be avoided if we take only the **absolute value of the deviations** and then take their sum.

If we follow this method, we will obtain a measure (descriptor) called the mean deviation from the mean.

The mean deviation is the sum of the absolute values of the deviations from the mean divided by the number of items, (i.e., the sum of the frequencies).

(c) **Variance :** In the above case, we took the absolute value of the deviations taken from mean to get rid of the negative sign of the deviations. Another method is to square the deviations. Let us, therefore, square the deviations from the mean and then take their sum. If we divide this sum by the number of observations (i.e., the sum of the frequencies), we obtain the average of deviations, which is called variance. **Variance is usually denoted by** σ^2.

(d) **Standard Deviation :** If we take the positive square root of the variance, we obtain the root mean square deviation or simply called standard deviation and is denoted by σ.

17.3 MEAN DEVIATION FROM MEAN OF RAW AND GROUPED DATA

Mean Deviation from mean of raw data $= \dfrac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$

Mean deviation from mean of grouped data $= \dfrac{\sum_{i=1}^{n} [f_i |x_i - \overline{x}|]}{N}$

where $N = \sum_{i=1}^{n} f_i, \overline{x} = \dfrac{1}{N}\sum_{i=1}^{n} (f_i x_i)$

The following steps are employed to calculate the mean deviation from mean.

Step 1 : Make a column of deviation from the mean, namely $x_i - \overline{x}$ (In case of grouped data take x_i as the mid value of the class.)

Step 2 : Take absolute value of each deviation and write in the column headed $|x_i - \overline{x}|$.

For calculating the mean deviation from the mean of raw data use

Mean deviation of Mean $= \dfrac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$

For grouped data proceed to step 3.

Step 3 : Multiply each entry in step 2 by the corresponding frequency. We obtain $f_i(x_i - \overline{x})$ and write in the column headed $f_i|x_i - \overline{x}|$.

Step 4 : Find the sum of the column in step 3. We obtain $\sum_{i=1}^{n}[f_i|x_i - \overline{x}|]$

Step 5 : Divide the sum obtained in step 4 by N.

Now let us take few examples to explain the above steps.

Example 17.1 Find the mean deviation from the mean of the following data :

Size of items x_i	4	6	8	10	12	14	16
Frequency f_i	2	5	5	3	2	1	4

Mean is 10

Solution :

x_i	f_i	$x_i - \overline{x}$	$\lvert x_i - \overline{x}\rvert$	$f_i\lvert x_i - \overline{x}\rvert$
4	2	–5.7	5.7	11.4
6	4	–3.7	3.7	14.8
8	5	–1.7	1.7	8.5
10	3	0.3	0.3	0.9
12	2	2.3	2.3	4.6
14	1	4.3	4.3	4.3
16	4	6.3	6.3	25.2
	21			69.7

Mean deviation from mean $= \dfrac{\sum[f_i|x_i - \overline{x}|]}{21} = \dfrac{69.7}{21} = 3.319$

Example 17.2 Calculate the mean deviation from mean of the following distribution :

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of Students	5	8	15	16	6

Mean is 27 marks

Solution :

Marks	Class Marks x_i	f_i	$x_i - \overline{x}$	$\lvert x_i - \overline{x}\rvert$	$f_i\lvert x_i - \overline{x}\rvert$
0 – 10	5	5	–22	22	110
10 – 20	15	8	–12	12	96
20 – 30	25	15	–2	2	30
30 – 40	35	16	8	8	128
40 – 50	45	6	18	18	108
Total		50			472

Mean deviation from Mean $= \dfrac{\sum [f_i |x_i - \bar{x}|]}{N} = \dfrac{472}{50}$ Marks $= 9.44$ Marks

CHECK YOUR PROGRESS 17.1

1. The ages of 10 girls are given below :

   3 5 7 8 9 10 12 14 17 18

   What is the range ?

2. The weight of 10 students (in Kg) of class XII are given below :

   45 49 55 43 52 40 62 47 61 58

   What is the range ?

3. Find the mean deviation from mean of the data

   45 55 63 76 67 84 75 48 62 65

   Given mean = 64.

4. Calculate the mean deviation from mean of the following distribution.

Salary (in rupees)	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	4	6	8	12	7	6	4	3

Given mean = Rs. 57.2

5. Calculate the mean deviation for the following data of marks obtained by 40 students in a test

Marks obtained	20	30	40	50	60	70	80	90	100
No. of students	2	4	8	10	8	4	2	1	1

6. The data below presents the earnings of 50 workers of a factory

Earnings (in rupees) :	1200	1300	1400	1500	1600	1800	2000
No. of workers :	4	6	15	12	7	4	2

   Find mean deviation.

7. The distribution of weight of 100 students is given below :

Weight (in Kg)	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75	75 – 80
No. of students	5	13	35	25	17	5

   Calculate the mean deviation.

8. The marks of 50 students in a particular test are :

Marks	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of students	4	6	9	12	8	6	4	1

Find the mean deviation for the above data.

17.4 MEDIAN

17.4.1 MEDIAN OF GROUPED DATA

Median of Discrete Frequency Distribution :

Step 1 : Arrange the data in ascending order.

Step 2 : Find cumulative frequencies

Step 3 : Find $\frac{N}{2}$

Step 4 : The observation whose cumulative frequency is just greater than $\frac{N}{2}$ is the median of the data.

Example 17.3 Find the median of the data

x_i	8	9	10	12	14	16
f_i	6	2	2	2	6	8

Solution : The given data are already in ascending order. Let us now write the cumulative frequencies of observations

x_i	8	9	10	12	14	16
f_i	6	2	2	2	6	8
c.f.	6	8	10	12	18	26

$$N = 26, \qquad \therefore \frac{N}{2} = 13.$$

The observation whose c.f. is just greater than 13 is 14 (whose c.f. is 18)

$\therefore$ Median = 14.

17.4.2 MEDIAN OF CONTINUOUS FREQUENCY DISTRIBUTION

Step 1 : Arrange the data in ascending order

Step 2 : Write cumulative frequencies of the observations

Step 3 : Identify the class whose cumulative frequency is just greater than $\frac{N}{2}$. Call this class-interval as median class.

Step 4 : Find median by the formula

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i$$

Where

$l \rightarrow$ Lower limit of the median class
$N \rightarrow$ Number of observations N = Σf_i
$C \rightarrow$ Cumulative frequency of the class just preceding the median class
$f \rightarrow$ Frequency of the median class
$i \rightarrow$ Width of the median class

Example 17.4 Find the median marks obtained by 50 students from the following distribution :

Marks	0-10	10-20	20-30	30-40	40-50
Number of Students	8	8	14	16	4

Solution : The given intervals are already in ascending order. The following table has the row corresponding to the cumulative frequencies.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	8	8	14	16	4
Cummulative frequency	8	16	30	46	50

$$N = 50, \quad \frac{N}{2} = 25$$

The class corresponding to the c.f. just greater than 25 is 20-30.

$\therefore$ Median class is 20-30

where l = 20, N = 50, C = 16, f = 14, i = 10.

$$\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i = 20 + \frac{25-16}{14} \times 10$$

$$= 20 + \frac{9}{14} \times 10 = 20 + 6.43 = 26.43$$

Example 17.5 Find the median of the following:

Marks	Number of Students
0 – 9	3
10 – 19	5
20 – 29	8
30 – 39	9
40 – 49	13
50 – 59	6

Solution : The given class intervals are inclusive series Before finding the median we have to convert the inclusive series into exclusive series.

Method of converting an inclusive series into exclusive series.

(1) Find the half of the difference between the upper limit of a class and the lower limit of its succeeding (next) class.

(2) Subtract this half from the lower limit and add into the upper limit.

Mark	Exclusive Series	f.	c.f.
0-9	0.5-9.5	3	3
10-19	9.5-19.5	5	8
20-29	19.5-29.5	8	16
30-39	29.5-39.5	9	25
40-49	39.5-49.5	13	38
50-59	49.5-59.5	6	44

$$\frac{N}{2}=\frac{44}{2}=22$$

$\therefore$ Median class is 29.5 – 39.5 as its c.f. is 25, which is just greater than 22.

Now, l = 29.5, N = 44, C = 16, f = 9, i = 39.5 – 29.5 = 10

$$\therefore \quad \text{Median} = l+\frac{\frac{N}{2}-C}{f}\times i = 29.5+\frac{22-16}{9}\times 10$$

$$= 29.5+\frac{6}{9}\times 10 = 29.5+\frac{20}{3}$$

$$= 29.5 + 6.66 = 36.16$$

CHECK YOUR PROGRESS 17.2

Find the median of the following data :

1.

x_i	6	11	16	21	26
f_i	5	3	6	4	7

2.

x_i	5	10	15	20	25
f_i	5	25	29	17	9

3.

Marks	0-5	5-10	10-15	15-20	20-25
Number of Boys	5	9	10	14	12

4.

Age (in years)	17-21	21-26	26-31	31-36	36-41
Number of Boys	5	6	12	7	4

17.5 MEAN DEVIATION FROM MEDIAN

We know that for observations in data the central tendency give us the values about which the data concentrate or cluster. It is also essential to know that how far all observation are, from a measure of central tendency. In other words, in data it is required to know how dispersed the observations are from a given point (or a measure of central tendency). In most of the cases mean deviation from mean and median give us the desired disperson or deviation of the observations. Recall that mean deviation for data is defined as the mean of the absolute values of deviations from ‘*a*’.

Recall that the deviation of an observation x from a fixed point ‘a’ is the difference $x - a$.

So mean deviation about ‘a’ denoted by M.D (a) is given by

$$\text{M.D. (a)} = \frac{\text{Sum of the absolute values of deviations from 'a'}}{\text{Number of observations}}$$

Methematically we can write

$$\text{M.D.}(a) = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Like wise

$$\text{M.D. (Mean} = \overline{\text{X}}) = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{n}$$

and

$$\text{M.D.(Median M)} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \text{M}|$$

Example 17.6 Find mean deviation about median for the observation

7, 10, 15, 16, 8, 9, 5, 17, 14

Solution : In order to find median, arrange the given values in ascending order, so we have

5, 7, 8, 9, 10, 14, 15, 16, 17,

Algorithm to find mean deviation about mean/median :

Step 1 : Calculate the mean or median of the data

Step 2 : Find deviations of each observation x_i from mean/median

Step 3 : Find the absolute values of the deviations.
Assolute values can be obtained by dropping the minus sign if it is there

Step 4 : Calculate the mean of the obsolute values of the deviations. This mean will be the required Mean deviation.

$n = 9$,

Median $= \dfrac{n+1}{2}$ th observation

= 5th observation

$M = 10$.

Deviations of the observation from median i.e. 10 are

	5–10	7–10	8–10	9–10	10–10	14–10	15–10	16–10	17–10
i.e x_i-M are	–5	–3	–2	–1	0	4	5	6	7

Absolute values of the deviations i.e. $|x_i - M|$ are

5, 3, 2, 1, 0, 4, 5, 6, 7

Now $\quad$ $$\text{M.D. (M)} = \frac{\sum_{i=1}^{n} |x_i - M|}{n}$$

$$= \frac{5+3+2+1+0+4+5+6+7}{10} = \frac{33}{10} = 3.3.$$

17.5.1 MEAN DEVIATION OF GROUPED DATA FROM MEDIAN

Recall that data presented in the following form are called grouped data

(a) Discrete frequency distribution

Observation :	x_1	x_2	x_3	...	x_n
Frequencies :	f_1	f_2	f_3	...	f_n

(b) Continuous frequency distribution :

Observations	$l_1 - u_1$	$l_2–u_2$	$l_3 - u_3$	...	$l_n–u_n$
Frequencies	f_1	f_2	f_3	...	f_n

For example, marks obtained by 50 students

Marks	0-5	5-10	10-15	15-20	20-25	25-30
Number of Students	8	6	12	10	10	4

Let us now learn to find mean deviation about median by following examples.

Example 17.7 Find the mean deviation about the median for the following data :

x_i	25	20	15	10	5
f_i	7	4	6	3	5
c.f.	7	11	17	20	25

Here N = 25, and we know that median is the $\frac{25+1}{2}$ = 13th observation. This observation lies in the C.f 17, for which corresponding observation is 15.

$\therefore$ Median M = 15

Now deviations and their absolute values are given in following table.

x_i	f_i	x_i – M	$\|x_i$ – M$\|$	$f_i\|x_i$ – M$\|$
25	7	25 – 15 = 10	10	7 × 10 = 70
20	4	20 – 15 = 5	5	4 × 5 = 20
15	6	15 – 15 = 0	0	6 × 0 = 0
10	3	10 – 15 = – 5	5	3 × 5 = 15
5	5	5 – 15 = – 10	10	5 × 10 = 50
	N= Σf_i = 25			$\Sigma f_i\ \|x_i - M\|$ = 155

$\therefore$ Mean Deviation (M) = $\dfrac{\sum_{i=1}^{n} f_i\,|x_i - M|}{\sum_{i=1}^{n} f_i} = \dfrac{155}{25} = 6.2$

Example 17.8 Find the mean deviation about median for the following data :

Heights (in cm)	95-105	105-115	115-125	125-135	135-145	145-155
Number of Girls	9	15	23	30	13	10

Solution : Let us first find median :

Height (in cm)	Number of Girls (f)	Cumulative frequncy (c.f)
95-105	9	9
105-115	15	24
115-125	23	47
125-135	30	77
135-145	13	90
145-155	10	100

N = 100 $\Rightarrow \frac{N+1}{2} = \frac{101}{2} = 50.5$

$\frac{N}{2} = 50.5$ lies in c.f. 77.

$\therefore$ Median class is corresponding to the c.f. 77 i.e., 125 – 135

Now, Median = $l + \dfrac{\dfrac{N}{2} - C}{f} \times i$

where l = lower limit of the median class

N = sum of frequencies

C = c.f. of the class just preceding the median class

f = frequency of the median class

and i = width or class-size of the median class

Here, l = 125, N = 100, C = 47, f = 30, i = 10

$$\therefore \quad M = 125 + \frac{50 - 47}{30} \times 10 = 125 + \frac{3}{3} = 126$$

To find mean deviation let us form the following table :

Height (in cm)	Number of Girls (f)	Mid-value of the heights	Absolute Deviation $(x_i - M\|)$	$f_i\|x_i - M\|$
95-105	9	100	|100-126| = 26	9 × 26 = 234
105-115	15	110	|110-126| = 16	15 × 16 = 240
115-125	23	120	|120 – 126| = 6	23 × 6 = 138
125-135	30	130	|130-126| = 4	30 × 4 = 120
135-145	13	140	|140-126| = 14	13 × 14 = 182
145-155	10	150	|150-126| = 24	10 × 24 = 240
	Σf_i = 100			Σf_i |xi – M| = 1154

$$\therefore \quad \text{Mean Deviation (Median)} = \text{M.D.(M)} = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{\sum_{i=1}^{n} f_i} = \frac{1154}{100} = 11.54.$$

17.5.2 STEP TO FIND MEAN DEVIATION FROM MEDIAN OF A CONTINUOUS FREQUENCY DISTRIBUTION.

Step 1 : Arrange the intervals in ascending order

Step 2 : Write cumulative frequencies

Step 3 : Identify the median class, as the class having c.f. just greater than $\frac{N}{2}$, where N is the total number of obsservations (i.e. sum of all frequencies)

Step 4 : Find the corresponding values for the median class and put in the formula :

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i$$

where

$l \rightarrow$ lower limit of the madian class

$N \rightarrow$ Sum of frequencies

$C \rightarrow$ c.f. of the class just preceding the median class

$f \rightarrow$ frequency of the median class

$i \rightarrow$ width of the median class

Step 5 : Now form the table for following columns :

Given intervals	Frequencies	Mid-value x_i	Absolute Deviation from Median $\lvert x_i - M)$	$f_i \lvert x_i - M \rvert$

Step 6 : Now calculate

$$\text{M.D.(M)} = \frac{\sum_{i=1}^{n} f_i \lvert x_i - M \rvert}{\sum_{i=1}^{n} f_i}$$

CHECK YOUR PROGRESS 17.3

Find the mean deviation about median of the following data.

1.

x_i	11	12	13	14	16	17	18
f_i	2	3	2	3	1	2	1

2.

x_i	3	6	7	9	11	13
f_i	3	9	11	8	9	6

3.

Weight (in kg)	40-42	42-44	44-46	46-48	48-50
No. of Students	9	13	24	28	6

4.

Age (in years)	0-1	1-2	2-3	3-4	4-5
No. of Children given polio drops	100	155	210	315	65

17.6 VARIANCE AND STANDARD DEVIATION OF RAW DATA

If there are n observations, $x_1, x_2, \ldots, x_n$, then

$$\text{Variance}\left(\sigma^2\right) = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots\ldots + (x_n - \bar{x})^2}{n}$$

or $$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}; \text{ where } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The standard deviation, denoted by σ, is the positive square root of σ^2. Thus

$$\sigma = +\sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

The following steps are employed to calculate the variance and hence the standard deviation of raw data. The mean is assumed to have been calculated already.

Step 1 : Make a column of deviations from the mean, namely, $x_i - \bar{x}$.

Step 2 (check) : Sum of deviations from mean must be zero, i.e., $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$

Step 3: Square each deviation and write in the column headed $(x_i - \bar{x})^2$.

Step 4 : Find the sum of the column in step 3.

Step 5 : Divide the sum obtained in step 4 by the number of observations. We obtain σ^2.

Step 6 : Take the positive square root of σ^2. We obtain σ (Standard deviation).

Example 17.9 The daily sale of sugar in a certain grocery shop is given below :

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
75 kg	120 kg	12 kg	50 kg	70.5 kg	140.5 kg

The average daily sale is 78 Kg. Calculate the variance and the standard deviation of the above data.

Solution : $\bar{x} = 78$ kg (Given)

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
75	– 3	9
120	42	1764
12	– 66	4356
50	– 28	784
70.5	– 7.5	56.25
140.5	62.5	3906.25
	0	10875.50

Thus $$\sigma^2 = \frac{\sum_{i=1} (x_i - \bar{x})^2}{n} = \frac{10875.50}{6} = 1812.58 \text{ (approx.)}$$

and $$\sigma = 42.57 \text{ (approx.)}$$

Example 17.10 The marks of 10 students of section A in a test in English are given below :

7 10 12 13 15 20 21 28 29 35

Determine the variance and the standard deviation.

Solution : Here $\bar{x} = \frac{\sum x_i}{10} = \frac{190}{10} = 19$

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
7	– 12	144
10	– 9	81
12	– 7	49
13	– 6	36
15	– 4	16
20	+1	1
21	+2	4
28	+9	81
29	+10	100
35	+16	256
	0	768

Thus $\sigma^2 = \frac{768}{10} = 76.8$ and $\sigma = +\sqrt{76.8} = 8.76$ (approx)

CHECK YOUR PROGRESS 17. 4

1. The salary of 10 employees (in rupees) in a factory (per day) is

   50 60 65 70 80 45 75 90 95 100

   Calculate the variance and standard deviation.

2. The marks of 10 students of class X in a test in English are given below :

   9 10 15 16 18 20 25 30 32 35

   Determine the variance and the standard deviation.

3. The data on relative humidity (in %) for the first ten days of a month in a city are given below:

   90 97 92 95 93 95 85 83 85 75

   Calculate the variance and standard deviation for the above data.

4. Find the standard deviation for the data

   4 6 8 10 12 14 16

5. Find the variance and the standard deviation for the data

   4 7 9 10 11 13 16

6. Find the standard deviation for the data.

   40 40 40 60 65 65 70 70 75 75 75 80 85 90 90 100

17. 7 STANDARD DEVIATION AND VARIANCE OF RAW DATA AN ALTERNATE METHOD

If $\overline{x}$ is in decimals, taking deviations from $\overline{x}$ and squaring each deviation involves even more decimals and the computation becomes tedious. We give below an alternative formula for computing σ^2. In this formula, we by pass the calculation of $\overline{x}$.

We know
$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n} = \sum_{i=1}^{n} \frac{x_i^2 - 2x_i\overline{x} + \overline{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{2\overline{x}\sum_{i=1}^{n} x_i}{n} + \overline{x}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \overline{x}^2, \quad \left(\because \overline{x} = \frac{\sum x_i}{n}\right)$$

i.e.
$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

MODULE - V
Statistics and Probability

Notes

And $\sigma = +\sqrt{\sigma^2}$

The steps to be employed in calculation of σ^2 and, hence σ by this method are as follows :

Step 1 : Make a column of squares of observations i.e. x_i^2.

Step 2 : Find the sum of the column in step 1. We obtain $\sum_{i=1}^{n} x_i^2$

Step 3 : Substitute the values of $\sum_{i=1}^{n} x_i^2$, n and $\sum_{i=1}^{n} x_i$ in the above formula. We obtain σ^2.

Step 4 : Take the positive sauare root of σ^2. We obtain σ.

Example 17.11 We refer to Example 17.10 of this lesson and re-calculate the variance and standard deviation by this method.

Solution :

x_i	x_i^2
7	49
10	100
12	144
13	169
15	225
20	400
21	441
28	784
29	841
35	1225
190	4378

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

$$= \frac{4378 - \frac{(190)^2}{10}}{10}$$

$$= \frac{4378 - 3610}{10} = \frac{768}{10} = 76.8$$

and $\qquad \sigma = +\sqrt{76.8} = 8.76$ (approx)

We observe that we get the same value of σ^2 and σ by either methods.

17.8 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - I

We are given k classes and their corresponding frequencies. We will denote the variance and the standard deviation of grouped data by σ_g^2 and σ_g respectively. The formulae are given below :

$$\sigma_g^2 = \frac{\sum_{i=1}^{K}\left[f_i\,(x_i - \bar{x})^2\right]}{N}, \quad N = \sum_{i=1}^{K} f_i \quad \text{and} \quad \sigma_g = +\sqrt{\sigma_g^2}$$

The following steps are employed to calculate σ_g^2 and, hence σ_g : (The mean is assumed to have been calculated already).

Step 1 : Make a column of class marks of the given classes, namely x_i

Step 2 : Make a column of deviations of class marks from the mean, namely, $x_i - \bar{x}$. Of course the sum of these deviations need not be zero, since x_i's are no more the original observations.

Step 3 : Make a column of squares of deviations obtained in step 2, i.e., $(x_i - \bar{x})^2$ and write in the column headed by $(x_i - \bar{x})^2$.

Step 4 : Multiply each entry in step 3 by the corresponding frequency.

We obtain $f_i\,(x_i - \bar{x})^2$.

Step 5 : Find the sum of the column in step 4. We obtain $\sum_{i=1}^{k}\left[f_i\,(x_i - \bar{x})^2\right]$

Step 6 : Divide the sum obtained in step 5 by N (total no. of frequencies). We obtain σ_g^2.

Step 7 : $\sigma_g = +\sqrt{\sigma_g^2}$

Example 17.12 In a study to test the effectiveness of a new variety of wheat, an experiment was performed with 50 experimental fields and the following results were obtained :

Yield per Hectare (in quintals)	Number of Fields
31 – 35	2
36 – 40	3
41 – 45	8
46 – 50	12
51 – 55	16
56 – 60	5
61 – 65	2
66 – 70	2

The mean yield per hectare is 50 quintals. Determine the variance and the standard deviation of the above distribution.

Solution :

Yield per Hectare (in quintal)	No. of Fields	Class Marks	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
31 – 35	2	33	– 17	289	578
36 – 40	3	38	– 12	144	432
41 – 45	8	43	– 7	49	392
46 – 50	12	48	– 2	4	48
51 – 55	16	53	+3	9	144
56 – 60	5	58	+8	64	320
61 – 65	2	63	+13	169	338
66 – 70	2	68	+18	324	648
Total	50				2900

Thus $\sigma_g^2 = \dfrac{\sum_{i=1}^{n}\left[f_i(x_i - \bar{x})^2\right]}{N} = \dfrac{2900}{50} = 58$ and $\sigma_g = +\sqrt{58} = 7.61$ (approx)

17.9 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : – METHOD - II

If $\bar{x}$ is not given or if $\bar{x}$ is in decimals in which case the calculations become rather tedious, we employ the alternative formula for the calculation of σ_g^2 as given below:

$$\sigma_g^2 = \frac{\sum_{i=1}^{k}\left[f_i x_i^2\right] - \dfrac{\left(\sum_{i=1}^{k}\left[f_i x_i\right]\right)^2}{N}}{N}, \quad N = \sum_{i=1}^{k} f_i$$

and $$\sigma_g = +\sqrt{\sigma_g^2}$$

The following steps are employed in calculating σ_g^2, and, hence σ_g by this method:

Step 1 : Make a column of class marks of the given classes, namely, x_i.

Step 2 : Find the product of each class mark with the corresponding frequency. Write the product in the column $x_i f_i$.

Step 3 : Sum the entries obtained in step 2. We obtain $\sum_{i=1}^{k}(f_i x_i)$.

Step 4 : Make a column of squares of the class marks of the given classes, namely, x_i^2.

Step 5 : Find the product of each entry in step 4 with the corresponding frequency. We obtain $f_i {x_i}^2$.

Step 6 : Find the sum of the entries obtained in step 5. We obtain $\sum_{i=1}^{k}(f_i x_i^2)$.

Step 7 : Substitute the values of $\sum_{i=1}^{k}(f_i x_i^2)$, N and $\left(\sum_{i=1}^{k}(f_i x_i)\right)$ in the formula and obtain σ_g^2.

Step 8 : $\sigma_g = +\sqrt{\sigma_g^2}$.

Example 17.13 Determine the variance and standard deviation for the data given in Example 17.12 by this method.

Solution :

Yields per Hectare (in quintals)	f_i	x_i	$f_i x_i$	${x_i}^2$	$f_i {x_i}^2$
31 – 35	2	33	66	1089	2178
36 – 40	3	38	114	1444	4332
41 – 45	8	43	344	1849	14792
46 – 50	12	48	576	2304	27648
51 – 55	16	53	848	2809	44944
56 – 60	5	58	290	3364	16820
61 – 65	2	63	126	3969	7938
66 – 77	2	68	136	4624	9248
Total	50		2500		127900

Substituting the values of $\sum_{i=1}^{k}\left(f_i x_i^2\right)$, N and $\sum_{i=1}^{k}\left(f_i x_i\right)$ in the formula, we obtain

$$\sigma_g^2 = \frac{127900 - \frac{(2500)^2}{50}}{50} = \frac{2900}{50} = 58$$

and $\sigma_g = +\sqrt{58} = 7.61$ (approx.)

Again, we observe that we get the same value of σ_g^2, by either of the methods.

CHECK YOUR PROGRESS 17.5

1. In a study on effectiveness of a medicine over a group of patients, the following results were obtained :

Percentage of relief	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
No. of patients	10	10	25	15	40

Find the variance and standard deviation.

2. In a study on ages of mothers at the first child birth in a village, the following data were available :

Age (in years) at first child birth	18 – 20	20 – 22	22 – 24	24 – 26	26 – 28	28 – 30	30 – 32
No. of mothers	130	110	80	74	50	40	16

Find the variance and the standard deviation.

3. The daily salaries of 30 workers are given below:

Daily salary (In Rs.)	0 – 50	50 – 100	100 – 150	150 – 200	200 – 250	250 – 300
No. of workers	3	4	5	7	8	3

Find variance and standard deviation for the above data.

17.10 STANDARD DEVIATION AND VARIANCE: STEP DEVIATION METHOD

In Example 17.12, we have seen that the calculations were very complicated. In order to simplify the calculations, we use another method called the step deviation method. In most of the frequency distributions, we shall be concerned with the equal classes. Let us denote, the class size by h.

Now we not only take the deviation of each class mark from the arbitrary chosen 'a' but also divide each deviation by h. Let

$$u_i = \frac{x_i - a}{h} \qquad .....(1)$$

Then $$x_i = hu_i + a \qquad .....(2)$$

We know that $$\bar{x} = h\bar{u} + a \qquad .....(3)$$

Subtracting (3) from (2), we get

$$x_i - \bar{x} = h(u_i - \bar{u}) \qquad .....(4)$$

In (4), squaring both sides and multiplying by f_i and summing over k, we get

$$\sum_{i=1}^{k}\left[f_i(x_i - \bar{x})^2\right] = h^2\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right] \qquad .....(5)$$

Dividing both sides of (5) by N, we get

$$\frac{\sum_{i=1}^{k}\left[f_i(x_i - \bar{x})^2\right]}{N} = \frac{h^2}{N}\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right]$$

i.e. $$\sigma_x^2 = h^2\sigma_u^2 \qquad .....(6)$$

where $\sigma_x{}^2$ is the variance of the original data and $\sigma_u{}^2$ is the variance of the coded data or coded variance. $\sigma_u{}^2$ can be calculated by using the formula which involves the mean, namely,

$$\sigma_u^2 = \frac{1}{N}\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right], \qquad N = \sum_{i=1}^{k} f_i \qquad .....(7)$$

or by using the formula which does not involve the mean, namely,

$$\sigma_u^2 = \frac{\sum_{i=1}^{k}\left[f_i u_i^2\right] - \frac{\left(\sum_{i=1}^{k}\left[f_i u_i\right]\right)^2}{N}}{N}, \qquad N = \sum_{i=1}^{k} f_i \qquad .....(8)$$

Example 17.14 We refer to the Example 17.12 again and find the variance and standard deviation using the coded variance.

Solution : Here h = 5 and let a = 48.

Yield per Hectare (in quintal)	Number of fields f_i	Class marks x_i	$u_i = \frac{x_i - 48}{5}$	$f_i u_i$	$u_i{}^2$	$f_i u_i{}^2$
31 – 35	2	33	–3	–6	9	18
36 – 40	3	38	–2	–6	4	12

41 – 45	8	43	– 1	– 8	1	8
46 – 50	12	48	0	0	0	0
51 – 55	16	53	+1	16	1	16
56 – 60	5	58	+2	10	4	20
61 – 65	2	63	+3	6	9	18
66 – 70	2	68	+4	8	16	32
Total	50			20		124

Thus
$$\sigma_u^2 = \frac{\sum_{i=1}^{k} f_i u_i^2 - \frac{\left(\sum_{i=1}^{k} f_i u_i\right)^2}{N}}{N}$$

$$= \frac{124 - \frac{(20)^2}{50}}{50} = \frac{124 - 8}{50} \quad \text{or} \quad \sigma_u^2 = \frac{58}{25}$$

Variance of the original data will be

$$\sigma_x^2 = h^2 \sigma_u^2 = 25 \times \frac{58}{25} = 58$$

and
$$\sigma_x = +\sqrt{58}$$
$$= 7.61 \text{ (approx)}$$

We, of course, get the same variance, and hence, standard deviation as before.

Example 17.15 Find the standard deviation for the following distribution giving wages of 230 persons.

Wages (in Rs)	No. of persons	Wages (in Rs)	No. of persons
70 – 80	12	110 – 120	50
80 – 90	18	120 – 130	45
90 – 100	35	130 – 140	20
100 – 110	42	140 – 150	8

Solution :

Wages (in Rs.)	No. of persons f_i	class mark x_i	$u_i = \dfrac{x_i - 105}{10}$	u_i^2	$f_i u_i$	$f_i u_i^2$
70 – 80	12	75	–3	9	–36	108
80 – 90	18	85	–2	4	–36	72
90 – 100	35	95	–1	1	–35	35
100 – 110	42	105	0	0	0	0
110 – 120	50	115	+1	1	50	50
120 – 130	45	125	+2	4	90	180
130 – 140	20	135	+3	9	60	180
140 – 150	8	145	+4	16	32	128
Total	230				125	753

$$\sigma^2 = h^2\left[\frac{1}{N}\sum\left[f_i u_i^2\right] - \left(\frac{1}{N}\sum\left[f_i u_i\right]\right)^2\right]$$

$$= 100\left[\frac{753}{230} - \left(\frac{125}{230}\right)^2\right] = 100\,(3.27 - 0.29) = 298$$

$\therefore$ $\sigma = +\sqrt{298} = 17.3$ (approx)

CHECK YOUR PROGRESS 17.6

1. The data written below gives the daily earnings of 400 workers of a flour mill.

Weekly earning (in Rs.)	No. of Workers
80 – 100	16
100 – 120	20
120 – 140	25
140 – 160	40
160 – 180	80
180 – 200	65
200 – 220	60
220 – 240	35
240 – 260	30
260 – 280	20
280 – 300	9

Calculate the variance and standard deviation using step deviation method.

2. The data on ages of teachers working in a school of a city are given below:

Age (in years)	20 – 25	25 – 30	30 – 35	35 – 40
No. of teachers	25	110	75	120
Age (in years)	40 – 45	45 – 50	50 – 55	55 – 60
No. of teachers	100	90	50	30

Calculate the variance and standard deviation using step deviation method.

3. Calculate the variance and standard deviation using step deviation method of the following data :

Age (in years)	25 – 30	30 – 35	35 – 40
No. of persons	70	51	47
Age (in years)	40 – 50	45 – 50	50 – 55
No. of persons	31	29	22

17.11 PROPERTIES OF VARIANCE AND STANDARD DEVIA TION

Property I : The variance is independent of change of origin.

To verify this property let us consider the example given below.

Example : 17.16 The marks of 10 students in a particular examination are as follows:

10 12 15 12 16 20 13 17 15 10

Later, it was decided that 5 bonus marks will be awarded to each student. Compare the variance and standard deviation in the two cases.

Solution : ***Case – I***

x_i	f_i	$f_i x_i$	$x_i - \overline{x}$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
10	2	20	– 4	16	32
12	2	24	– 2	4	8
13	1	13	– 1	1	1
15	2	30	1	1	2
16	1	16	2	4	4
17	1	17	3	9	9
20	1	20	6	36	36
	10	140			92

Here $$\overline{x} = \frac{140}{10} = 14$$

Variance $= \dfrac{\sum\left[f_i\,(x_i - \bar{x})^2\right]}{10}$

$= \dfrac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Case – II (By adding 5 marks to each x_i)

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
15	2	30	–4	16	32
17	2	34	–2	4	8
18	1	18	–1	1	1
20	2	40	1	1	2
21	1	21	2	4	4
22	1	22	3	9	9
25	1	25	6	36	36
	10	190			92

$$\bar{x} = \frac{190}{10} = 19$$

$\therefore$ Variance $= \dfrac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Thus, we see that there is no change in variance and standard deviation of the given data if the origin is changed i.e., if a constant is added to each observation.

Property II : The variance is not independent of the change of scale.

Example 17.17 In the above example, if each observation is multiplied by 2, then discuss the change in variance and standard deviation.

Solution : In case-I of the above example , we have variance = 9.2, standard deviation = 3.03.

Now, let us calculate the variance and the Standard deviation when each observation is multiplied by 2.

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
20	2	40	–8	64	128
24	2	48	–4	16	32
26	1	26	–2	4	4
30	2	60	2	4	8
32	1	32	4	16	16
34	1	34	6	36	36
40	1	40	12	144	144
	10	280			368

$$\bar{x} = \frac{280}{10} = 28, \text{ Variance} = \frac{368}{10} = 36.8$$

$$\text{Standard deviation} = +\sqrt{36.8} = 6.06$$

Here we observe that, the variance is four times the original one and consequently the standard deviation is doubled.

In a similar way we can verify that if each observation is divided by a constant then the variance of the new observations gets divided by the square of the same constant and consequently the standard deviation of the new observations gets divided by the same constant.

Property III : Prove that the standard deviation is the least possible root mean square deviation.

Proof : Let $\bar{x} - a = d$

By definition, we have

$$s^2 = \frac{1}{N}\sum\left[f_i(x_i - a)^2\right] = \frac{1}{N}\sum\left[f_i(x_i - \bar{x} + \bar{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i\left[(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - a) + (\bar{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i(x_i - \bar{x})^2 + \frac{2}{N}(\bar{x} - a)\sum f_i(x_i - \bar{x}) + \frac{(\bar{x} - a)^2}{N}\sum f_i$$

$$= \sigma^2 + 0 + d^2$$

$\therefore$ The algebraic sum of deviations from the mean is zero

or $\qquad s^2 = \sigma^2 + d^2$

Clearly s^2 will be least when d = 0 i.e., when $a = \bar{x}$.

Hence the root mean square deviation is the least when deviations are measured from the mean i.e., the standard deviation is the least possible root mean square deviation.

Property IV : The standard deviations of two sets containing n_1, and n_2 numbers are σ_1 and σ_2 respectively being measured from their respective means m_1 and m_2. If the two sets are grouped together as one of $(n_1 + n_2)$ numbers, then the standard deviation σ of this set, measured from its mean m is given by

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

Example 17.18 The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3; the standard deviations are 8 and 7. Find the standard deviation of the sample of size 150 by combining the two samples.

Solution : Here we have

$$n_1 = 50, n_2 = 100, m_1 = 54.1, m_2 = 50.3$$

$$\sigma_1 = 8 \text{ and } \sigma_2 = 7$$

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{(n_1 + n_2)} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

$$= \frac{(50 \times 64) + (100 \times 49)}{150} + \frac{50 \times 100}{(150)^2}(54.1 - 50.3)^2$$

$$= \frac{3200 + 4900}{150} + \frac{2}{9}(3.8)^2 = 57.21$$

$\therefore$ $\quad \sigma = 7.56$ (approx)

Example 17.19 Find the mean deviation (M.D) from the mean and the standard deviation (S.D) of the A.P.

$$a, a + d, a + 2d, \ldots\ldots, a + 2n.d$$

and prove that the latter is greater than the former.

Solution : The number of items in the A.P. is (2n + 1)

$\therefore$ $\quad \bar{x} = a + nd$

Mean deviation about the mean

$$= \frac{1}{(2n+1)}\sum_{r=0}^{2n}\left|(a + rd) - (a + nd)\right|$$

$$= \frac{1}{(2n+1)}.2\left[nd + (n-1)d + \ldots\ldots + d\right]$$

$$= \frac{2}{(2n+1)}\left[1 + 2 + \ldots\ldots + (n-1) + n\right]d$$

$$= \frac{2n(n+1)}{(2n+1)2}.d \qquad = \frac{n(n+1)d}{(2n+1)} \qquad \ldots\ldots(1)$$

Now $\quad \sigma^2 = \frac{1}{(2n+1)}\sum_{r=0}^{2n}\left[(a + rd) - (a + nd)\right]^2$

$$= \frac{2d^2}{(2n+1)}\left[n^2 + (n-1)^2 + \ldots . + 2^2 + 1^2\right]$$

$$= \frac{2d^2}{(2n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)d^2}{3}$$

$$\therefore \qquad \sigma = d.\sqrt{\left(\frac{n(n+1)}{3}\right)} \qquad \text{.....(2)}$$

We have further, (2) > (1)

if $$d\sqrt{\left(\frac{n(n+1)}{3}\right)} > \frac{n(n+1)}{(2n+1)}d$$

or if $$(2n+1)^2 > 3n(n+1)$$

or if $n^2 + n + 1 > 0$, which is true for $n > 0$

Hence the result.

Example 17.20 Show that for any discrete distribution the standard deviation is not less than the mean deviation from the mean.

Solution : We are required to show that

$$\text{S.D.} \geq \text{M.D. from mean}$$

or $$(\text{S. D})^2 \geq (\text{M.D. from mean})^2$$

i.e. $$\frac{1}{N}\sum\left[f_i(x_i - \bar{x})^2\right] \geq \left[\frac{1}{N}\sum\left[f_i|(x_i - \bar{x})|\right]\right]^2$$

or $$\frac{1}{N}\sum\left[f_i d_i^2\right] \geq \left[\frac{1}{N}\sum\left[f_i|d_i|\right]\right]^2, \text{ where } d_i = x_i - \bar{x}$$

or $$N\sum\left(f_i d_i^2\right) \geq \left[\sum\{f_i|d_i|\}\right]^2$$

or $$(f_1 + f_2 + \ldots.)\left(f_1 d_1^2 + f_2 d_2^2 + \ldots\ldots\right) \geq [f_1|d_1| + f_2|d_2| + \ldots..]^2$$

or $$f_1 f_2\left(d_1^2 + d_2^2\right) + \ldots.. \geq 2f_1 f_2\, d_1 d_2 + \ldots..$$

or $$f_1 f_2 (d_1 - d_2)^2 + \ldots.. \geq 0$$

which is true being the sum of perfect squares.

17.12 ANALYSIS OF FREQUENCY DISTRIBUTIONS WITH EQUAL MEANS

The variability of two series with same mean can be compared when the measures of variation are absolute and are free of units. For this, coefficient of variation (C.V.) is obtained which is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100, \ \overline{x} \neq 0$$

where σ and $\overline{x}$ are standard deviation and mean of the data. The coefficients of variation are compared to compare the variability of two series. The series with greater C.V. is said to be more variable than the other. The series having less C.V. is said to be more consistent than the other.

For series with same means, we can have

$$\text{C.V. (1st distribution)} = \frac{\sigma_1}{\overline{x}} \times 100 \qquad \text{...(1)}$$

$$\text{C.V. (2nd distribution)} = \frac{\sigma_2}{\overline{x}} \times 100 \qquad \text{...(2)}$$

where σ_1, σ_2 are standard deviation of the Ist and 2nd distribution respectively, $\overline{x}$ is the equal mean of the distributions.

From (1) and (2), we can conclude that two C.V.'s can be compared on the basis of the values of σ_1 and σ_2 only.

Example 17.21 The standard deviation of two distributions are 21 and 14 and their equal mean is 35. Which of the distributions is more variable?

Solution : Let

$$\sigma_1 = \text{Standard dev. of 1st series} = 21$$

$$\sigma_2 = \text{Standard dev. of 2nd series} = 14$$

$$\overline{x} = 35$$

$$\text{C.V. (Series I)} = \frac{\sigma_1}{\overline{x}} \times 100 = \frac{21}{35} \times 100 = 60$$

$$\text{C.V. (Series II)} = \frac{\sigma_1}{\overline{x}} \times 100 = \frac{14}{35} \times 100 = 40$$

C.V. of series I > C.V. of series II

$\Rightarrow$ Series with S.D = 21 is more variable.

Example 17.22 Monthly wages paid to workers in two factories A and B and other data are given below :

	Factory A	**Factory B**
Mean of monthly wages	` 15550	` 15550
Variance of the distribution of wages	100	121

Which factory A or B shows greater variablility in individual wages?

Solution : Given

$$\sigma_A = \sqrt{\text{variance}} = \sqrt{100} = 10$$

$$\sigma_B = \sqrt{\text{variance}} = \sqrt{121} = 11$$

$$\bar{x} = \text{` } 15550$$

Now,
$$\text{C.V. (A)} = \frac{\sigma_A}{\bar{x}} \times 100 = \frac{10}{15550} \times 100$$
$$= 0.064$$

$$\text{C.V.(B)} = \frac{\sigma_B}{\bar{x}} \times 100 = \frac{11}{15550} \times 100 = 0.07$$

Clearly C.V. (B) > C.V.(A)

$\therefore$ Factory B has greater variability in the individual wages.

Example 17.23 Which of the following series X or Y is more consistent?

X	58	52	50	51	49	35	54	52	53	56
Y	101	104	103	104	107	106	105	105	107	108

Solution : From the given data we have following table

X	Y	$D_i = X - \bar{X}$	D_i^2	$d_i = Y - \bar{Y}$	d_i^2
58	101	7	49	–4	16
52	104	1	1	–1	1
50	103	–1	1	–2	4
51	104	0	0	–1	1
49	107	–2	4	2	4
35	106	–16	256	1	1
54	105	3	9	0	0
52	105	1	1	0	0
53	107	2	4	2	4
56	108	5	25	3	9
$\Sigma X = 510$	$\Sigma Y = 1050$		$\Sigma D_i^2 = 350$		$\Sigma d_i^2 = 40$

Now,
$$\bar{X} = \frac{\Sigma X_i}{10} = \frac{510}{10} = 51$$

$$\bar{Y} = \frac{\Sigma Y_i}{10} = \frac{1050}{10} = 105$$

$$\sigma_x = \sqrt{\frac{\Sigma (X - \bar{X})^2}{N}} = \sqrt{\frac{\Sigma D_i^2}{N}} = \sqrt{\frac{350}{10}}$$

$$= 5.9$$

$$\sigma_y = \sqrt{\frac{\Sigma (Y - \bar{Y})^2}{N}} = \sqrt{\frac{\Sigma d_i^2}{N}} = \sqrt{\frac{40}{10}} = 2$$

Now,
$$\text{C.V.(X)} = \frac{\sigma_X}{X} \times 100 = \frac{5.9}{51} \times 100 = 11.5$$

$$\text{C.V.(Y)} = \frac{\sigma_Y}{Y} \times 100 = \frac{2}{105} \times 100 = 1.9$$

Clearly C.V.(Y) < C.V.(X) $\therefore$ Series Y is more consistent.

CHECK YOUR PROGRESS 17.7

1. From the data given below which section is more variable?

Marks	0-10	10-20	20-30	30-40	40-50
Section A	9	10	40	33	8
Section B	8	15	43	25	9

2. Which of the factory give better consistent wages to workers?

Wages (in `) per day	100-150	150-200	200-250	250-300	300-350
Factory A	35	45	50	42	28
Factory B	16	50	55	13	46

3. Two schools show following results of board examination in a year

	School A	School B
Average Marks Obtained	250	225
No. of Students Appeared	62	62
Variance of distribution of marks	2.25	2.56

Which school has greater variability in individual marks?

MODULE - V
Statistics and Probability

Notes

LET US SUM UP

- Range : The difference between the largest and the smallest value of the given data.

- Mean deviation from mean $= \dfrac{\sum_{i=1}^{n}(f_i|x_i - \bar{x}|)}{N}$

  where $N = \sum_{i=1}^{n} f_i$, $\bar{x} = \dfrac{1}{N}\sum_{i=1}^{n}(f_i x_i)$

- Mean deviation from median $= \dfrac{\sum_{i=1}^{n} fi|xi - m|}{N}$

  Where $N = \dfrac{\sum_{i=1}^{n} fi}{N}$,

  $M = l + \dfrac{\frac{N}{2} - C}{f} \times i$

- Variance $(\sigma^2) = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$ [for raw data]

- Standard derivation $(\sigma) = +\sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$

- Variance for grouped data

  $\sigma_g^2 = \dfrac{\sum_{i=1}^{k}\left[f_i (x_i - \bar{x})^2\right]}{N}$, x_i is the mid value of the class.

  Also, $\sigma_x{}^2 = h^2\sigma_u{}^2$ and $\sigma_u{}^2 = \dfrac{1}{N}\sum_{i=1}^{k}\left[f_i\left(u_i - \bar{u}^2\right)\right]$

  $N = \sum_{i=1}^{k} f_i$

or $\sigma_u^{\,2} = \dfrac{\sum_{i=1}^{k}\left(f_i u_i^{\,2}\right) - \dfrac{\left[\sum_{i=1}^{k}\left(f_i u_i^{\,2}\right)\right]^2}{N}}{N}$ where $N = \sum_{i=1}^{k} f_i$

- Standard deviation for grouped data $\sigma_g = +\sqrt{\sigma_g^2}$
- If two frequency distributions have same mean, then the distribution with greater Coefficient of variation (C.V) is said to be more variable than the other.

SUPPORTIVE WEB SITES

http:// en.wikipedia.org/wiki/Statistical_**dispersion**

simon.cs.vt.edu/SoSci/converted/**Dispersion**_I/activity.html

TERMINAL EXERCISE

1. Find the mean deviation for the following data of marks obtained (out of 100) by 10 students in a test

   55 45 63 76 67 84 75 48 62 65

2. The data below presents the earnings of 50 labourers of a factory

Earnings (in Rs.)	1200	1300	1400	1500	1600	1800
No. of Labourers	4	7	15	12	7	5

   Calculate mean deviation.

3. The salary per day of 50 employees of a factory is given by the following data.

Salary (in Rs.)	20 – 30	30 – 40	40 – 50	50 – 60
No. of employees	4	6	8	12
Salary (in rupees)	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	7	6	4	3

   Calculate mean deviation.

4. Find the batting average and mean deviation for the following data of scores of 50 innings of a cricket player:

Run Scored	0 – 20	20 – 40	40 – 60	60 – 80
No. of Innings	6	10	12	18
Run scored	80 – 100	100 – 120		
No. of innings	3	1		

5. The marks of 10 students in a test of Mathematics are given below:

6	10	12	13	15	20	24	28	30	32

Find the variance and standard deviation of the above data.

6. The following table gives the masses in grams to the nearest gram, of a sample of 10 eggs.

46	51	48	62	54	56	58	60	71	75

Calculate the standard deviation of the masses of this sample.

7. The weekly income (in rupees) of 50 workers of a factory are given below:

Income	400	425	450	500	550	600	650
No of workers	5	7	9	12	7	6	4

Find the variance and standard deviation of the above data.

8. Find the variance and standard deviation for the following data:

Class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
Frequency	7	8	25	15	45

9. Find the standard deviation of the distribution in which the values of x are 1,2,......, N. The frequency of each being one.

10. The following values are calculated in respect of heights and weights of students :

	Weight	Height
Mean	52.5 Kg	160.5 cm
Standard Dev.	11.5	12.2

Which of the attribute weight or height show greater variation?

11. The following are the wickets taken by a bowler in 20 matches, for Player A

No. of Wickets	0	1	2	3	4
No. of Matches	2	6	7	4	1

For the bowler B, mean number of wickets taken in 20 matches is 1.6 with standard deviation 1.25. Which of the players is more consistent?

Find the median of the following distributions (12-14) :

12.

x_i	14	20	26	29	34	46
f_i	4	6	7	8	9	6

13.

Age (in years)	15-19	20-24	25-29	30-34	35-39
Number	8	7	9	11	5

14.

Height (in cm)	95-104	105-114	115-124	125-134	135-144
Number of Boys	10	8	18	8	16

Find mean deviation from median (15-18) :

15.

x_i	5	15	25	35	45	55
f_i	5	23	30	20	16	6

16.

x_i	105	107	109	111	113	115
f_i	8	6	2	2	2	6

17.

Income (per month) (` in '000)	0-5	6-10	11-15	16-20	21-25
Number of Persons	5	6	12	14	26

18.

Age (in years)	0-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
No. of Persons	5	6	12	14	26	32	16	29

ANSWERS

CHECK YOUR PROGRESS 17.1

1. 15   2. 22   3. 9.4   4. 15.44

5. 13.7   6. 136   7. 5.01   8. 14.4

CHECK YOUR PROGRESS 17.2

1. 16   2. 15   3. 15.35 marks   4. 28 years

CHECK YOUR PROGRESS 17.3

1. 1.85   2. 2.36   3. 3..73   4. 0.977

CHECK YOUR PROGRESS 17.4

1. Variance = 311, Standard deviation = 17.63
2. Variance = 72.9, Standard deviation = 8.5
3. Variance = 42.6, Standard deviation = 6.53
4. Standard deviation = 4
5. Variance = 13.14, Standard deviation = 3.62
6. Standard deviation = 17.6

CHECK YOUR PROGRESS 17.5

1. Variance = 734.96, Standard deviation = 27.1
2. Variance = 12.16, Standard deviation = 3.49
3. Variance = 5489 , Standard deviation = 74.09

CHECK YOUR PROGRESS 17.6

1. Variance = 2194, Standard deviation = 46.84
2. Variance = 86.5 , Standard deviation = 9.3
3. Variance = 67.08 , Standard deviation = 8.19

CHECK YOUR PROGRESS 17.7

1. Section A
2. Factory A
3. School B

TERMINAL EXERCISE

1. 9.4 2. 124.48 3. 15.44 4. 52, 19.8

5. Variance = 72.29, Standard Deviation = 8.5 6. 8.8

7. Variance = 5581.25,Standard Deviation = 74.7

8. Variance = 840, Standard Deviation = 28.9

9. Standard deviation $= \sqrt{\dfrac{N^2 - 1}{12}}$

10. Weight 11. Player B 12. 29 13. 27.27

14. 121.16 15. 10.3 16. 3.38 17. 5.2

18. 0.62

18

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RANDOM EXPERIMENTS AND EVENTS

In day-to-day life we see that before commencement of a cricket match two captains go for a toss. Tossing of a coin is an activity and getting either a 'Head' or a ''Tail' are two possible outcomes. (Assuming that the coin does not stand on the edge). If we throw a die (of course fair die) the possible outcomes of this activity could be any one of its faces having numerals, namely 1,2,3,4,5 and 6..... at the top face.

An activity that yields a result or an outcome is called an experiment. Normally there are variety of outcomes of an experiment and it is a matter of chance as to which one of these occurs when an experiment is performed. In this lesson, we propose to study various experiments and their outcomes.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of a random experiments and cite examples thereof;
- explain the role of chance in such random experiments;
- define a sample space corresponding to an experiment;
- write a sample space corresponding to a given experiment; and
- differentiate between various types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events.

EXPECTED BACKGROUND KNOWLEDGE

- Basic concepts of probability

18.1 RANDOM EXPERIMENT

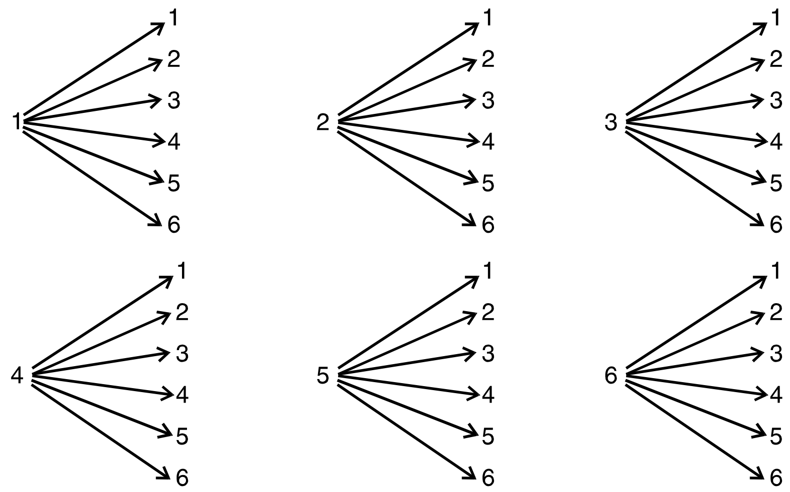
Let us consider the following activities :

(i) Toss a coin and note the outcomes. There are two possible outcomes, either a head (H) or a tail (T).

(ii) In throwing a fair die, there are six possible outcomes, that is, any one of the six faces 1,2,...... 6.... may come on top.

(iii) Toss two coins simultaneously and note down the possible outcomes. There are four possible outcomes, HH,HT,TH,TT.

(iv) Throw two dice and there are 36 possible outcomes which are represented as below :



i.e. outcomes are (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

$$\begin{array}{ccc} (2,1),(2,2),\ldots,(2,6) \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ (6,1),(6,2),\ldots,(6,6) \end{array}$$

Each of the above mentioned activities fulfil the following two conditions.

(a) The activity can be repeated number of times under identical conditions.

(b) Outcome of an activity is not predictable beforehand, since the chance play a role and each outcome has the same chance of being selection. Thus, due to the chance playing a role, an activity is

(i) repeated under identical conditions, and

(ii) whose outcome is not predictable beforehand is called a random experiment.

Example 18.1 Is drawing a card from well shuffled deck of cards, a random experiment ?

Solution :

(a) The experiment can be repeated, as the deck of cards can be shuffled every time before drawing a card.

(b) Any of the 52 cards can be drawn and hence the outcome is not predictable beforehand.

Hence, this is a random experiment.

Example 18.2 Selecting a chair from 100 chairs without preference is a random experiment. Justify.

Solution :

(a) The experiment can be repeated under identical conditions.

(b) As the selection of the chair is without preference, every chair has equal chances of selection. Hence, the outcome is not predictable beforehand. Thus, it is a random experiment.

Can you think of any other activities which are not random in nature.

Let us consider some activities which are not random experiments.

(i) Birth of Manish : Obviously this activity, that is, the birth of an individual is not repeatable and hence is not a random experiment.

(ii) Multiplying 4 and 8 on a calculator.

Although this activity can be repeated under identical conditions, the outcome is always 32. Hence, the activity is not a random experiment.

18.2 SAMPLE SPACE

We throw a die once, what are possible outcomes ? Clearly, a die can fall with any of its faces at the top. The number on each of the faces is, therefore, a possible outcome. We write the set S of all possible outcomes as , S = {1, 2, 3, 4, 5, 6}

Again, if we toss a coin, the possible outcomes for this experiment are either a head or a tail. We write the set S of all possible outcomes as , S = {H, T}.

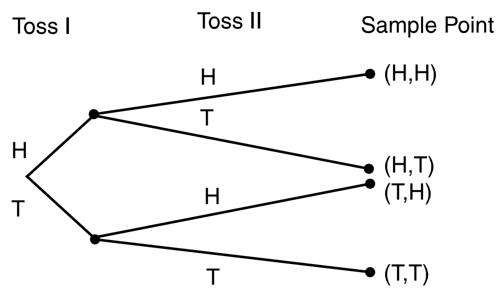
The set S associated with an experiment satisfying the following properties :

(i) each element of S denotes a possible outcome of the experiment.

(ii) any trial results in an outcome that corresponds to one and only one element of the set S is called the sample space of the experiment and the elements are called sample points. Sample space is generally denoted by S.

Example 18.3 Write the sample space in two tosses of a coin.

Solution : Let H denote a head and T denote a tail in the experiment of tossing of a coin.



S = { (H, H), (H, T), (T, H), (T, T) }.

Note : If two coins are tossed simultaneously then the sample space S can be written as

S = { H H, H T, T H, T T}.

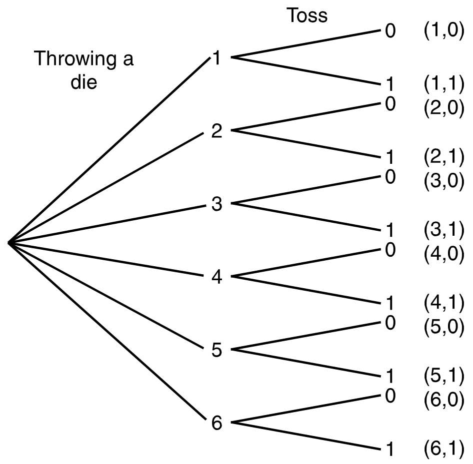
Example 18.4 Consider an experiment of rolling a fair die and then tossing a coin.

Write the sample space.

Solution : In rolling a die possible outcomes are 1, 2, 3, 4, 5 and 6. On tossing a coin the possible outcomes are either a head or a tail. Let H (head) = 0 and T (tail) = 1.

MODULE - V
Statistics and Probability

Notes

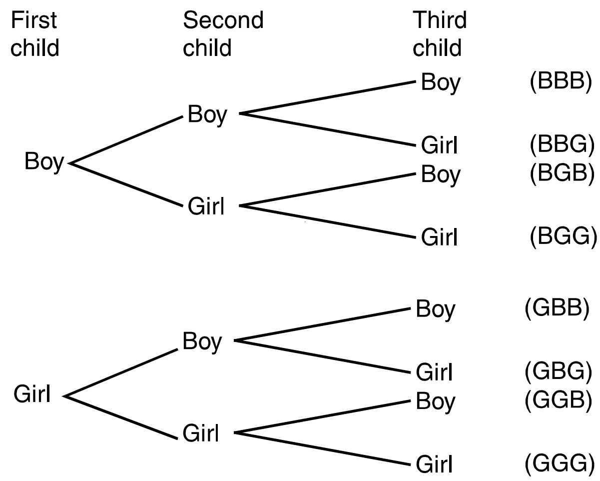


$S = \{(1, 0), (1, 1), (2, 0), (2, 1),(3, 0), (3, 1),(4, 0), (4, 1), (5, 0), (5, 1), (6, 0), (6, 1)\}$

$\therefore \quad n(S) = 6 \times 2 = 12$

Example 18.5 Suppose we take all the different families with exactly 3 children. The experiment consists in asking them the sex (or genders) of the first, second and third chid. Write down the sample space.

Solution : Let us write 'B' for boy and 'G' for girl and construct the following tree diagram.



The sample space is

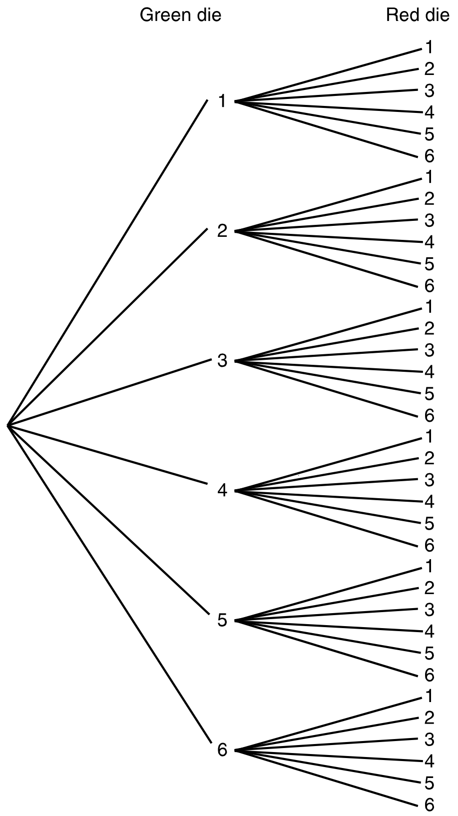
S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

The advantage of writing the sample space in the above form is that a question such as "Was the second child a girl" ? or " How many families have first child a boy ?" and so forth can be answered immediately.

$$n(S) = 2 \times 2 \times 2 = 8$$

Example 18.6 Consider an experiment in which one die is green and the other is red. When these two dice are rolled, what will be the sample space ?

Solution : This experiment can be displayed in the form of a tree diagram, as shown below :



Let g_i and r_j denote, the number that comes up on the green die and red die respectively. Then an out-come can be represented by an ordered pair (g_i, r_j), where i and j can assume any of the values 1, 2, 3, 4, 5, 6.

Thus, a sample space S for this experiment is the set, $S = \{(g_i, r_j) : 1 \le i \le 6, 1 \le j \le 6\}$.

Also, notice that the multiplication principle (principle of counting) shows that the number of elements in S is 36, since there are 6 choices for g and 6 choices for r, and $6 \times 6 = 36$

$$\therefore \; n(S) = 36$$

Example 18.7 Write the sample space for each of the following experiments :

(i) A coin is tossed three times and the result at each toss is noted.

(ii) From five players A, B, C, D and E , two players are selected for a match.

(iii) Six seeds are sown and the number of seeds germinating is noted.

(iv) A coin is tossed twice. If the second throw results in a head, a die is thrown, otherwise a coin is tossed.

Solution :

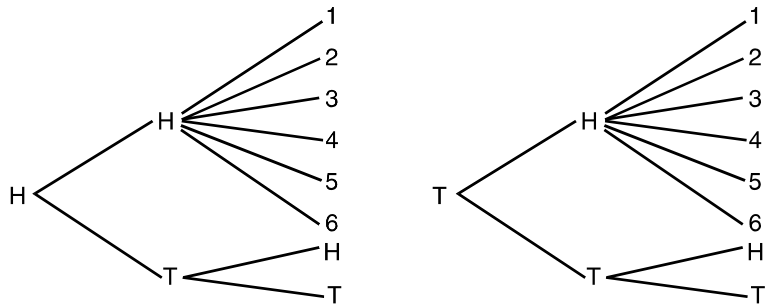
(i) S = { TTT, TTH, THT, HTT, HHT, HTH, THH, HHH}

number of elements in the sample space is $2 \times 2 \times 2 = 8$

(ii) S= { AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}. Here n (S) = 10

(iii) S = { 0, 1, 2, 3, 4, 5, 6}. Here n (S) = 7

(iv) This experiment can be displayed in the form of a tree-diagram as shown below :



Thus S = {HH1, HH2, HH3, HH4, HH5, HH6, HTH, HTT, TH1, TH2,TH3,TH4, TH5, TH6, TTH, TTT}

i.e. there are 16 outcomes of this experiment.

18.3. DEFINITION OF VARIOUS TERMS

Event : Let us consider the example of tossing a coin. In this experiment, we may be interested in 'getting a head'. Then the outcome 'head' is an event.

In an experiment of throwing a die, our interest may be in, 'getting an even number'. Then the outcomes 2, 4 or 6 constitute the event. We have seen that an experiment which, though repeated under identical conditions, does not give unique results but may result in any one of the several possible outcomes, which constitute the sample space.

Some outcomes of the sample space satisfy a specified description, which we call an 'event'.

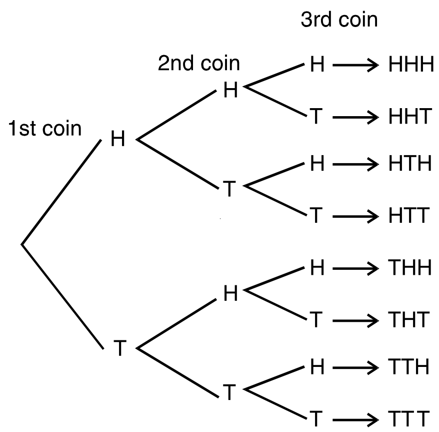
We often use the capital letters A, B, C etc. to represent the events.

Example 18.8 Let E denote the experiment of tossing three coins at a time. List all possible outcomes and the events that

(i) the number of heads exceeds the number of tails.

(ii) getting two heads.

Solution :



The sample space S is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\} \text{ (say)}$$

If E_1 is the event that the number of heads exceeds the number of tails, and E_2 the event getting two heads. Then

$$E_1 = \{ w_1, w_2, w_3, w_5 \}$$

and $$E_2 = \{ w_2, w_3, w_5 \}$$

18.3.1 Equally Likely Events

Outcomes of a trial are said to be equally likely if taking into consideration all the relevant evidences there is no reason to expect one in preference to the other.

Examples :

(i) In tossing an unbiased coin, getting head or tail are equally likely events.

(ii) In throwing a fair die, all the six faces are equally likely to come.

(iii) In drawing a card from a well shuffled deck of 52 cards, all the 52 cards are equally likely to come.

18.3.2 Mutually Exclusive Events

Events are said to be mutually exclusive if the happening of any one of the them preludes the happening of all others, i.e., if no two or more of them can happen simultaneously in the same trial.

Examples :

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive. If any one of these faces comes at the top, the possibility of others, in the same trial is ruled out.

(ii) When two coins are tossed, the event that both should come up tails and the event that there must be at least one head are mutually exclusive.

Mathematically events are said to be mutually exclusive if their intersection is a null set (i.e., empty)

18.3.3 Exhaustive Events

If we have a collection of events with the property that no matter what the outcome of the experiment, one of the events in the collection must occur, then we say that the events in the collection are exhaustive events.

For example, when a die is rolled, the event of getting an even number and the event of getting an odd number are exhaustive events. Or when two coins are tossed the event that at least one head will come up and the event that at least one tail will come up are exhaustive events.

Mathematically a collection of events is said to be exhaustive if the union of these events is the complete sample space.

18.3.4 Independent and Dependent Events

A set of events is said to be independent if the happening of any one of the events does not affect the happening of others. If, on the other hand, the happening of any one of the events influence the happening of the other, the events are said to be dependent.

Examples :

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second card is dependent on the first draw (in the sense that it cannot be the card drawn the first time).

CHECK YOUR PROGRESS 18.1

1. Selecting a student from a school without preference is a random experiment. Justify.
2. Adding two numbers on a calculator is not a random experiment. Justify.
3. Write the sample space of tossing three coins at a time.
4. Write the sample space of tossing a coin and a die.
5. Two dice are thrown simultaneously, and we are interested to get six on top of each of the die. Are the two events mutually exclusive or not ?
6. Two dice are thrown simultaneously. The events A, B, C, D are as below :

   A : Getting an even number on the first die.

   B : Getting an odd number on the first die.

   C : Getting the sum of the number on the dice < 7.

   D : Getting the sum of the number on the dice > 7.

   State whether the following statements are True or False.

   (i) A and B are mutually exclusive.

   (ii) A and B are mutually exclusive and exhaustive.

   (iii) A and C are mutually exclusive.

   (iv) C and D are mutually exclusive and exhaustive.
7. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. There will be how many sample points, in its sample space?
8. In a single rolling with two dice, write the sample space and its elements.
9. Suppose we take all the different families with exactly 2 children. The experiment consists in asking them the sex of the first and second child.

   Write down the sample space.

LET US SUM UP

- An activity that yields a result or an outcome is called an experiment.
- An activity repeated number of times under identical conditions and outcome of activity is not predictable is called Random Experiment.
- The set of possible outcomes of a random experiment is called sample space and elements of the set are called sample points.

- Some outcomes of the sample space satisfy a specified description, which is called an Event.
- Events are said to be Equally likely, when we have no preference for one rather than the other.
- If happening of an event prevents the happening of another event, then they are called Mutually Exclusive Events.
- The total number of possible outcomes in any trial is known as Exhaustive Events.
- A set of events is said to be Independent events, if the happening of any one of the events does not effect the happening of other events, otherwise they are called dependent events.

SUPPORTIVE WEB SITES

www.math.uah.edu/stat/prob/**Events**.html

http://en.wikipedia.org/wiki/**Experiment_**(probability_theory)

TERMINAL EXERCISE

1. A tea set has four cups and saucers. If the cups are placed at random on the saucers, write the sample space.
2. If four coins are tossed, write the sample space.
3. If n coins are tossed simultaneously, there will be how many sample points ?

   [Hint : try for n = 1,2, 3,4, ...]
4. In a single throw of two dice, how many sample points are there ?

ANSWERS

CHECK YOUR PROGRESS 18.1

1. Both properties are satisfied 2. Outcome is predictable

3. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

4. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T4, T6} 5. No.

6. (i) True (ii) True (ii) False (iv) True 7. 15

8. $$\{ (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

9. {MM, MF, FM, FF}

TERMINAL EXERCISE

1. $\{C_1S_1, C_1S_2, C_1S_3, C_1S_4, C_2S_1, C_2S_2, C_2S_3, C_2S_4,$
$C_3S_1, C_3S_2, C_3S_3, C_3S_4, C_4S_1, C_4S_2, C_4S_3, C_4S_4\}$

2. $2^4 = 16$, {HHHH, HHHT, HHTH, HTHH, HHTT, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

3. 2^n

4. $6^2 = 36$

19

PROBABILITY

311en19

In our daily life, we often used phrases such as 'It may rain today', or 'India may win the match' or ' I may be selected for this post.' These phrases involve an element of uncertainty. How can we measure this uncertainty? A measure of this uncertainty is provided by a branch of Mathematics, called the theory of probability. Probability Theory is designed to measure the degree of uncertainty regarding the happening of a given event. The dictionary meaning of probability is ' likely though not certain to occur. Thus, when a coin is tossed, a head is likely to occur but may not occur. Similarly, when a die is thrown, it may or may not show the number 6.

In this lesson we shall discuss some basic concepts of probability, addition theorem, dependent and independent events, multiplication theorem, Baye's theorem, ramdom variable, its probability distribution and binomial distribution.

OBJECTIVES

After studying this lesson, you will be able to :

- define probability of occurance of an event;
- cite through examples that probability of occurance of an event is a non-negative fraction, not greater than one;
- use permutation and combinations in solving problems in probability;
- state and establish the addition theorems on probability and the conditions under which each holds;
- generalize the addition theorem of probability for mutually exclusive events;
- understand multiplication law for independent and dependent events and solve problems releated to them.
- understand conditional probability and solve problems releated to it.
- understand Baye's theorem and solve questions related to it.
- define random variable and find its probability distribution.
- understand and find, mean and variance of random variable.
- understand binomial distribution and solve questions based on it.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of random experiments and events.
- The meaning of sample space.

- A standard deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each : spades, hearts, diamonds, clubs and cards in each suit are - ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, Queens and Jacks are called face cards and the other cards are called number cards.

19.1 EVENTS AND THEIR PROBABILITY

In the previous lesson, we have learnt whether an activity is a random experiment or not. The study of probability always refers to random experiments. Hence, from now onwards, the word experiment will be used for a random experiment only. In the preceeding lesson, we have defined different types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events and cited examples of the above mentioned events.

Here we are interested in the chance that a particular event will occur, when an experiment is performed. Let us consider some examples.

What are the chances of getting a ' Head' in tossing an unbiased coin ? There are only two equally likely outcomes, namely head and tail. In our day to day language, we say that the coin has chance 1 in 2 of showing up a head. In technical language, we say that the probability of getting a head is $\frac{1}{2}$.

Similarly, in the experiment of rolling a die, there are six equally likely outcomes 1, 2,3,4,5 or 6. The face with number '1' (say) has chance 1 in 6 of appearing on the top. Thus, we say that the probability of getting 1 is $\frac{1}{6}$.

In the above experiment, suppose we are interested in finding the probability of getting even number on the top, when a die is rolled. Clearly, the possible numbers are 2, 4 and 6 and the chance of getting an even number is 3 in 6. Thus, we say that the probability of getting an even number is $\frac{3}{6}$, i.e., $\frac{1}{2}$.

The above discussion suggests the following definition of probability.

If an experiment with '*n*' exhaustive, mutually exclusive and equally likely outcomes, *m* outcomes are favourable to the happening of an event A, the probability '*p*' of happening of A is given by

$$p = P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{m}{n} \qquad ....(i)$$

Since the number of cases favourable to the non-happening of the event A are $n - m$, the probability 'q' that 'A' will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - p \quad \text{[Using (i)]}$$

$$\therefore \qquad p + q = 1.$$

Obviously, p as well as q are non-negative and cannot exceed unity.

i.e., $0 \le p \le 1, \quad 0 \le q \le 1$

Thus, the probability of occurrence of an event lies between 0 and 1[including 0 and 1].

Remarks

1. Probability 'p' of the happening of an event is known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure.
2. Probability of an impossible event is 0 and that of a sure event is 1

   if P (A) = 1, the event A is certainly going to happen and

   if P (A) = 0, the event is certainly not going to happen.
3. The number (m) of favourable outcomes to an event cannot be greater than the total number of outcomes (n).

Let us consider some examples

Example 19.1 In a simultaneous toss of two coins, find the probability of

(i) getting 2 heads (ii) exactly 1 head

Solution : Here, the possible outcomes are

HH, HT, TH, TT.

i.e., Total number of possible outcomes = 4.

(i) Number of outcomes favourable to the event (2 heads) = 1 (i.e., HH).

$$\therefore \qquad P(\text{ 2 heads }) = \frac{1}{4}.$$

(ii) Now the event consisting of exactly one head has two favourable cases,

namely HT and TH . $\qquad \therefore \qquad P(\text{ exactly one head }) = \frac{2}{4} = \frac{1}{2}.$

Example 19.2 In a single throw of two dice, what is the probability that the sum is 9?

Solution : The number of possible outcomes is $6 \times 6 = 36$. We write them as given below :

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Now, how do we get a total of 9. We have :

$$3 + 6 = 9,\ 4 + 5 = 9,\ 5 + 4 = 9,\ 6 + 3 = 9$$

In other words, the outcomes (3, 6), (4, 5), (5, 4) and (6, 3) are favourable to the said event, i.e., the number of favourable outcomes is 4.

Hence, P (a total of 9) $= \frac{4}{36} = \frac{1}{9}$

Example 19.3 What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution : A leap year consists of 366 days consisting of 52 weeks and 2 extra days. These two extra days can occur in the following possible ways.

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday

Out of the above seven possibilities, two outcomes,

e.g., (i) and (vii), are favourable to the event

$\therefore \qquad P(\text{ 53 Sundays }) = \frac{2}{7}$

CHECK YOUR PROGRESS 19.1

1. A die is rolled once. Find the probability of getting 3.
2. A coin is tossed once. What is the probability of getting the tail ?
3. What is the probability of the die coming up with a number greater than 3 ?
4. In a simultaneous toss of two coins, find the probability of getting ' at least' one tail.
5. From a bag containing 15 red and 10 blue balls, a ball is drawn 'at random'. What is the probability of drawing (i) a red ball ? (ii) a blue ball ?
6. If two dice are thrown, what is the probability that the sum is (i) 6 ? (ii) 8? (iii) 10? (iv) 12?
7. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is divisible by 3 or by 4 ?
8. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is greater than 10 ?
9. What is the probability of getting a red card from a well shuffled deck of 52 cards ?
10. If a card is selected from a well shuffled deck of 52 cards, what is the probability of drawing

    (i) a spade ? (ii) a king ? (iii) a king of spade ?
11. A pair of dice is thrown. Find the probability of getting

(i) a sum as a prime number (ii) a doublet, i.e., the same number on both dice

(iii) a multiple of 2 on one die and a multiple of 3 on the other.

12. Three coins are tossed simultaneously. Find the probability of getting

(i) no head (ii) at least one head (iii) all heads

19.2. CALCULATION OF PROBABILITY USING COMBINATORICS (PERMUTATIONS AND COMBINATIONS)

In the preceding section, we calculated the probability of an event by listing down all the possible outcomes and the outcomes favourable to the event. This is possible when the number of outcomes is small, otherwise it becomes difficult and time consuming process. In general, we do not require the actual listing of the outcomes, but require only the total number of possible outcomes and the number of outcomes favourable to the event. In many cases, these can be found by applying the knowledge of permutations and combinations, which you have already studied.

Let us consider the following examples :

Example 19.4 A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

Solution : Total number of balls = 3 + 6 + 7 = 16

Now, out of 16 balls, 2 can be drawn in ${}^{16}C_2$ ways.

$$\therefore \text{ Exhaustive number of cases } = {}^{16}C_2 = \frac{16 \times 15}{2} = 120$$

Out of 6 white balls, 1 ball can be drawn in ${}^{6}C_1$ ways and out of 7 blue balls, one can be drawn is ${}^{7}C_1$ ways. Since each of the former case is associated with each of the later case, therefore total number of favourable cases are ${}^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$.

$$\therefore \text{ Required probability } = \frac{42}{120} = \frac{7}{20}$$

Remarks

When two or more balls are drawn from a bag containing several balls, there are two ways in which these balls can be drawn.

(i) **Without replacement :** The ball first drawn is not put back in the bag, when the second ball is drawn. The third ball is also drawn without putting back the balls drawn earlier and so on. Obviously, the case of drawing the balls without replacement is the same as drawing them together.

(ii) **With replacement :** In this case, the ball drawn is put back in the bag before drawing the next ball. Here the number of balls in the bag remains the same, every time a ball is drawn.

In these types of problems, unless stated otherwise, we consider the problem of without replacement.

Example 19.5 Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?

Solution : Six cards can be drawn from the pack of 52 cards in ${}^{52}C_6$ ways.

i.e., Total number of possible outcomes = ${}^{52}C_6$

3 red cards can be drawn in ${}^{26}C_3$ ways and

3 black cards can be drawn in ${}^{26}C_3$ ways.

$\therefore$ Total number of favourable cases = ${}^{26}C_3 \times {}^{26}C_3$

Hence, the required probability $= \dfrac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6} = \dfrac{13000}{39151}$

Example 19.6 Four persons are chosen at random from a group of 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is $\frac{10}{21}$.

Solution : Total number of persons in the group = 3 + 2 + 4 = 9. Four persons are chosen at random. If two of the chosen persons are children, then the remaining two can be chosen from 5 persons (3 men + 2 women).

Number of ways in which 2 children can be selected from 4, children = ${}^{4}C_2 = \dfrac{4 \times 3}{1 \times 2} = 6$

Number of ways in which remaining of the two persons can be selected

from 5 persons = ${}^{5}C_2 = \dfrac{5 \times 4}{1 \times 2} = 10$

Total number of ways in which 4 persons can be selected out of

9 persons = ${}^{9}C_4 = \dfrac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = 126$

Hence, the required probability $= \dfrac{{}^{4}C_2 \times {}^{5}C_2}{{}^{9}C_4} = \dfrac{6 \times 10}{126} = \dfrac{10}{21}$

CHECK YOUR PROGRESS 19.2

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn at random are both white?
2. A bag contains 5 red and 8 blue balls. What is the probability that two balls drawn are red and blue ?
3. A bag contains 20 white and 30 black balls. Find the probability of getting 2 white balls, when two balls are drawn at random

   (a) with replacement (b) without replacement

4. Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the three cards are jacks.
5. Two cards are drawn from a well-shuffled pack of 52 cards. Show that the chances of drawing both aces is $\frac{1}{221}$.
6. In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

   (i) one is boy and two are girls. (ii) all are boys. (iii) all are girls.
7. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the numbers on them are in A.P.
8. Two cards are drawn at random from 8 cards numbered 1 to 8. What is the probability that the sum of the numbers is odd, if the cards are drawn together ?
9. A team of 5 players is to be selected from a group of 6 boys and 8 girls. If the selection is made randomly, find the probability that there are 2 boys and 3 girls in the team.
10. An integer is chosen at random from the first 200 positive integers.Find the probability that the integer is divisible by 6 or 8.

19.3 EVENT RELATIONS

19.3.1 Complement of an event

Let us consider the example of throwing a fair die. The sample space of this experiment is

$$S = \{ 1 , 2, 3, 4, 5, 6 \}$$

If A be the event of getting an even number, then the sample points 2, 4 and 6 are favourable to the event A.

The remaining sample points 1, 3 and 5 are not favourable to the event A. Therefore, these will occur when the event A will not occur.

In an experiment, the outcomes which are not favourable to the event A are called complement of A and defined as follows :

'The outcomes favourable to the complement of an event A consists of all those outcomes which are not favourable to the event A, and are denoted by 'not' A or by $\bar{A}$.

19.3.2 Event 'A or B'

Let us consider the example of throwing a die. A is an event of getting a multiple of 2 and B be another event of getting a multiple of 3.

The outcomes 2, 4 and 6 are favourable to the event A and the outcomes 3 and 6 are favourable to the event B.

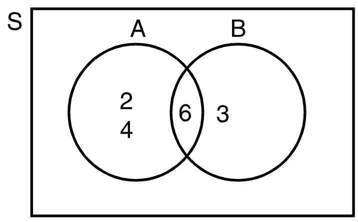


Fig. 19.1

The happening of event A or B is $A \cup B = \{ 2, 3, 4, 6 \}$

Again, if A be the event of getting an even number and B is another event of getting an odd number, then $A = \{ 2, 4, 6 \}$, $B = \{ 1, 3, 5 \}$

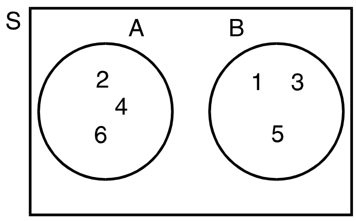


Fig. 19.2

$A \cup B = \{1, 2, 3, 4, 5, 6 \}$

Here, it may be observed that if A and B are two events, then the event 'A or B' ($A \cup B$) will consist of the outcomes which are either favourable to the event A or to the event B or to both the events.

Thus, the event 'A or B' occurs, if either A or B or both occur.

19.3.3 Event "A and B'

Recall the example of throwing a die in which A is the event of getting a multiple of 2 and B is the event of getting a multiple of 3. The outcomes favourable to A are 2, 4, 6 and the outcomes favourable to B are 3 , 6.

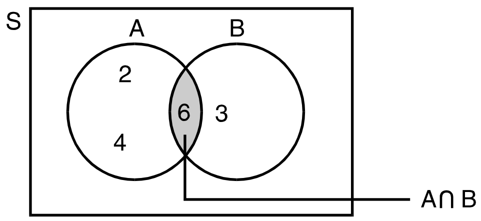


Fig. 19.3

Here, we observe that the outcome 6 is favourable to both the events A and B.

Draw a card from a well shuffled deck of 52 cards. A and B are two events defined as

A : a red card, B : a king

We know that there are 26 red cards and 4 kings in a deck of cards. Out of these 4 kings, two are red.

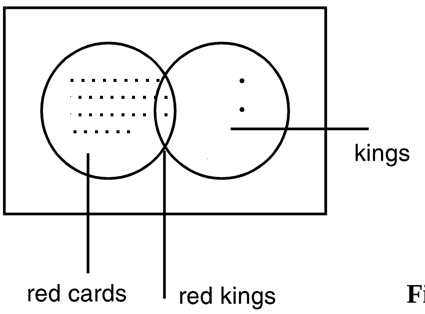


Fig. 19.4

Here, we see that the two red kings are favourable to both the events.

Hence, the event ' A and B' consists of all those outcomes which are favourable to both the events A and B. That is, the event 'A and B' occurs, when both the events A and B occur simultaneously. Symbolically, it is denoted as $A \cap B$.

19.4 ADDITIVE LAW OF PROBABILITY

Let A be the event of getting an odd number and B be the event of getting a prime number in a single throw of a die. What will be the probability that it is either an odd number or a prime number ?

In a single throw of a die, the sample space would be

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

The outcomes favourable to the events A and B are

$$A = \{ 1, 3, 5 \}, B = \{ 2, 3, 5 \}$$

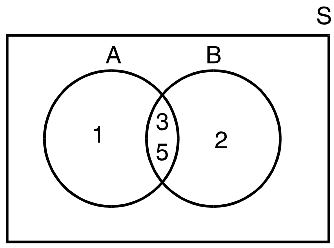


Fig. 19.5

The outcomes favourable to the event 'A or B' are

$$A \cup B = \{1, 2, 3, 5 \}.$$

Thus, the probability of getting either an odd number or a prime number will be

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$

To discover an alternate method, we can proceed as follows :

The outcomes favourable to the event A are 1, 3 and 5. $\therefore$ $P(A) = \frac{3}{6}$

Similarly, $P(B) = \frac{3}{6}$

The outcomes favourable to the event 'A and B' are 3 and 5. $\therefore$ $P(A \text{ and } B) = \frac{2}{6}$

Now, $P(A) + P(B) - P(A \text{ and } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6}$

$$\frac{4}{6} = \frac{2}{3} = P(A \text{ or } B)$$

Thus, we state the following law, called additive rule, which provides a technique for finding the probability of the union of two events, when they are not disjoint.

For any two events A and B of a sample space S,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \text{.....(ii)}$$

Example 19.7 A card is drawn from a well-shuffled deck of 52 cards. What is the probability that it is either a spade or a king ?

Solution : If a card is drawn at random from a well-shuffled deck of cards, the likelyhood of any of the 52 cards being drawn is the same. Obviously, the sample space consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52}, \qquad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

Hence, the probability that the card drawn is either a spade or a king is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 19.8 In an experiment with throwing 2 fair dice, consider the events

A : The sum of numbers on the faces is 8

B : Doubles are thrown.

What is the probability of getting A or B ?

Solution : In a throw of two dice, the sample space consists of $6 \times 6 = 36$ sample points.

The favourable outcomes to the event A (the sum of the numbers on the faces is 8) are

$$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

The favourable outcomes to the event B (Double means both dice have the same number) are

$$B = \{(1,1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$\therefore \qquad A \cap B = \{(4, 4)\}.$$

Now $P(A) = \frac{5}{36}$, $P(B) = \frac{6}{36}$, $P(A \cap B) = \frac{1}{36}$

Thus, the probability of A or B is

$$P(A \cup B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

19.5 ADDITIVE LAW OF PROBABILITY FOR MUTUALLY EXCLUSIVE EVENTS

We know that the events A and B are mutually exclusive, if and only if they have no outcomes in common. That is, for mutually exclusive events,

$$P\,(A \text{ and } B) = 0$$

Substituting this value in the additive law of probability, we get the following law :

$$P\,(A \text{ or } B) = P\,(A) + P\,(B) \qquad \text{.....(iii)}$$

Example 19.9 In a single throw of two dice, find the probability of a total of 9 or 11.

Solution : Clearly, the events - a total of 9 and a total of 11 are mutually exclusive.

Now $\quad P\,(\text{a total of } 9) = P\,[(3, 6), (4, 5), (5, 4), (6, 3)] = \dfrac{4}{36}$

$$P\,(\text{a total of } 11) = P\,[(5, 6), (6, 5)] = \frac{2}{36}$$

Thus, $\quad P\,(\text{a total of 9 or 11}) = \dfrac{4}{36} + \dfrac{2}{36} = \dfrac{1}{6}$

Example 19.10 Prove that the probability of the non-occurrence of an event A is $1 - P\,(A)$.

i.e., $\quad P\,(\text{not } A) = 1 - P\,(A) \quad$ or, $\quad P\,(\bar{A}) = 1 - P\,(A)$.

Solution : We know that the probability of the sample space S in any experiment is 1.

Now, it is clear that if in an experiment an event A occurs, then the event $(\bar{A})$ cannot occur simultaneously, i.e., the two events are mutually exclusive.

Also, the sample points of the two mutually exclusive events together constitute the sample space S. That is,

$$A \cup \bar{A} = S$$

Thus, $\quad P\left(A \cup \bar{A}\right) = P(S)$

$$\Rightarrow P\,(A) + P(\bar{A}) = 1 \quad (\because A \text{ and } \bar{A} \text{ are mutually exclusive and S is sample space})$$

$$\Rightarrow P\left(\bar{A}\right) = 1 - P\,(A),$$

which proves the result.

This is called the law of complementation.

Law of complimentation : $P\left(\bar{A}\right) = 1 - P\,(A)$

$$\frac{P(\bar{A})}{P(A)} \text{ or } P\,(A) \text{ to } P\,(\bar{A}).$$

Example 19.11 The probability of the event that it will rain is 0.3. Find the odds in favour of rain and odds against rain.

Solution : Let A be the event that it will rain. $\therefore$ $P(A) = .3$

By law of complementation, $P\left(\bar{A}\right) = 1 - .3 = .7$.

Now, the odds in favour of rain are $\frac{0.3}{0.7}$ or 3 to 7 (or 3 : 7).

The odds against rain are $\frac{0.7}{0.3}$ or 7 to 3.

When either the odds in favour of A or the odds against A are given, we can obtain the probability of that event by using the following formulae

If the odds in favour of A are *a* to *b*, then $P(A) = \frac{a}{a+b}$.

If the odds against A are a to b, then $P(A) = \frac{b}{a+b}$.

This can be proved very easily.

Suppose the odds in favour of A are *a* to *b*. Then, by the definition of odds,

$$\frac{P(A)}{P(\bar{A})} = \frac{a}{b}.$$

From the law of complimentation, $P\left(\bar{A}\right) = 1 - P(A)$

Therefore, $\frac{P(A)}{1 - P(A)} = \frac{a}{b}$ or $b\,P(A) = a - a\,P(A)$

or $(a+b)\,P(A) = a$ or $P(A) = \frac{a}{a+b}$

Similarly, we can prove that $P(A) = \frac{b}{a+b}$

when the odds against A are b to a.

Example 19.12 Are the following probability assignments consistent ? Justify your answer.

(a) $P(A) = P(B) = 0.6,$ $P(A \text{ and } B) = 0.05$

(b) $P(A) = 0.5,$ $P(B) = 0.4,$ $P(A \text{ and } B) = 0.1$

(c) $P(A) = 0.2,$ $P(B) = 0.7,$ $P(A \text{ and } B) = 0.4$

Solution : (a) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.6 + 0.6 - 0.05 = 1.15$$

Since P (A or B) > 1 is not possible, hence the given probabilities are not consistent.

(b) $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= 0.5 + 0.4 - 0.1 = 0.8$$

which is less than 1.

As the number of outcomes favourable to event 'A and B' should always be less than or equal to those favourable to the event A,

Therefore, $P(A \text{ and } B) \leq P(A)$

and similarly $P(A \text{ and } B) \leq P(B)$

In this case, P (A and B) = 0.1, which is less than both P (A) = 0.5 and P (B) = 0.4. Hence, the assigned probabilities are consistent.

(c) In this case, P (A and B) = 0.4, which is more than P (A) = 0.2.

$$[\because P(A \text{ and } B) \leq P(A)]$$

Hence, the assigned probabilities are not consistent.

Example 19.13 An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball ?

Solution : Urn contains 8 white balls and 2 green balls.

$\therefore$ Total number of balls in the urn = 10

Three balls can be drawn in ${}^{10}C_3$ ways = 120 ways.

Let A be the event " at least one green ball is selected".

Let us determine the number of different outcomes in A. These outcomes contain either one green ball or two green balls.

There are ${}^{2}C_1$ ways to select a green ball from 2 green balls and for this remaining two white balls can be selected in ${}^{8}C_2$ ways.

Hence, the number of outcomes favourable to one green ball

$$= {}^{2}C_1 \times {}^{8}C_2 = 2 \times 28 = 56$$

Similarly, the number of outcomes favourable to two green balls

$$= {}^{2}C_2 \times {}^{8}C_1 = 1 \times 8 = 8$$

Hence, the probability of at least one green ball is

P (at least one green ball)

$$= P(\text{one green ball}) + P(\text{two green balls})$$

$$= \frac{56}{120} + \frac{8}{120} = \frac{64}{120} = \frac{8}{15}$$

Example 19.14 Two balls are drawn at random with replacement from a bag containing 5 blue and 10 red balls. Find the probability that both the balls are either blue or red.

MODULE - V
Statistics and Probability

Notes

MODULE - V
Statistics and Probability

Notes

Solution : Let the event A consists of getting both blue balls and the event B is getting both red balls. Evidently A and B are mutually exclusive events.

By fundamental principle of counting, the number of outcomes favourable to $A = 5 \times 5 = 25.$

Similarly, the number of outcomes favourable to B = 10 × 10 = 100.

Total number of possible outcomes = 15 × 15 = 225.

$$\therefore \qquad P(A) = \frac{25}{225} = \frac{1}{9} \text{ and } P(B) = \frac{100}{225} = \frac{4}{9}.$$

Since the events A and B are mutually exclusive, therefore

$$P(A \text{ or } B) = P(A) + P(B)$$

$$= \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

Thus, P (both blue or both red balls) = $\frac{5}{9}$

CHECK YOUR PROGRESS 19.3

1. A card is drawn from a well-shuffled pack of cards. Find the probability that it is a queen or a card of heart.
2. In a single throw of two dice, find the probability of a total of 7 or 12.
3. The odds in favour of winning of Indian cricket team in 2010 world cup are 9 to 7. What is the probability that Indian team wins ?
4. The odds against the team A winning the league match are 5 to 7. What is the probability that the team A wins the league match.
5. Two dice are thrown. Getting two numbers whose sum is divisible by 4 or 5 is considered a success. Find the probability of success.
6. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both the cards are either black or red ?
7. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is an ace or a black card.
8. Two dice are thrown once. Find the probability of getting a multiple of 3 on the first die or a total of 8.
9. (a) In a single throw of two dice, find the probability of a total of 5 or 7.

   (b) A and B are two mutually exclusive events such that P (A) = 0.3 and P (B) = 0.4. Calculate P (A or B).
10. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective ?
11. Two dice are rolled once. Find the probability

(a) that the numbers on the two dice are different,

(b) that the total is at least 3.

12. A couple have three children. What is the probability that among the children, there will be at least one boy or at least one girl ?

13. Find the odds in favour and against each event for the given probability

(a) $P(A) = .7$ (b) $P(A) = \frac{4}{5}$

14. Determine the probability of A for the given odds

(a) 7 to 2 in favour of A (b) 10 to 7 against A.

15. If two dice are thrown, what is the probability that the sum is

(a) greater than 4 and less than 9 ?

(b) neither 5 nor 8 ?

16. Which of the following probability assignments are inconsistent ? Give reasons.

(a) $P(A) = 0.5$, $P(B) = 0.3$, $P(A \text{ and } B) = 0.4$

(b) $P(A) = P(B) = 0.4$, $P(A \text{ and } B) = 0.2$

(c) $P(A) = 0.85$, $P(B) = 0.8$, $P(A \text{ and } B) = 0.61$

17. Two balls are drawn at random from a bag containing 5 white and 10 green balls. Find the probability that the sample contains at least one white ball.

18. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both cards are of the same suit?

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

19.6 MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

Let us recall the definition of independent events.

Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Can you think of some examples of independent events ?

The event of getting 'H' on first coin and the event of getting 'T' on the second coin in a simultaneous toss of two coins are independent events.

What about the event of getting 'H' on the first toss and event of getting 'T' on the second toss in two successive tosses of a coin ? They are also independent events.

Let us consider the event of 'drawing an ace' and the event of 'drawing a king' in two successive draws of a card from a well-shuffled deck of cards without replacement.

Are these independent events ?

No, these are not independent events, because we draw an ace in the first draw with probability

$\frac{4}{52}$. Now, we do not replace the card and draw a king from the remaining 51 cards and this affect the probability of getting a king in the second draw, i.e., the probability of getting a king in the second draw without replacement will be $\frac{4}{51}$.

Note : If the cards are drawn with replacement, then the two events become independent.

Is there any rule by which we can say that the events are independent ?

How to find the probability of simultaneous occurrence of two independent events?

If A and B are independent events, then

$$P\ (A\ \text{and}\ B) = P(A)\ .\ P(B)$$

or

$$P(A \cap B) = P(A).\ P(B)$$

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

Note : The above law can be extended to more than two independent events, i.e.,

$$P(A \cap B \cap C...) = P(A) \cdot P(B) \cdot P(C)...$$

On the other hand, if the probability of the event 'A' and 'B' is equal to the product of the probabilities of the events A and B, then we say that the events A and B are independent.

Example 19.15 A die is tossed twice. Find the probability of a number greater than 4 on each throw.

Solution : Let us denote by A, the event 'a number greater than 4' on first throw. B be the event 'a number greater than 4' in the second throw. Clearly A and B are independent events.

In the first throw, there are two outcomes, namely, 5 and 6 favourable to the event A.

$\therefore$ $$P(A) = \frac{2}{6} = \frac{1}{3}$$

Similarly, $$P(B) = \frac{1}{3}$$

Hence, $P(A\ \text{and}\ B) = P(A)\ .\ P(B) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$.

Example 19.16 Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is $\frac{1}{3}$ and that of Tarun's selection is $\frac{1}{5}$. Find the probability that

(a) both of them will be selected. (b) none of them is selected.

(c) at least one of them is selected. (d) only one of them is selected.

Solution : Probability of Arun's selection $= P(A) = \frac{1}{3}$

Probability of Tarun's selection $= P(T) = \frac{1}{5}$

(a) P (both of them will be selected) = P (A) P (T)

$$= \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

(b) P (none of them is selected)

$$= P(\bar{A})P(\bar{T}) = \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = \frac{2}{3}\times\frac{4}{5} = \frac{8}{15}$$

(c) P (at least one of them is selected)

$$= 1 - P \text{ (None of them is selected)}$$

$$= 1 - P(\bar{A})P(\bar{T}) = 1 - \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 1 - \left(\frac{2}{3}\times\frac{4}{5}\right) = 1 - \frac{8}{15} = \frac{7}{15}$$

(d) P (only one of of them is selected)

$$= P(A)P(\bar{T}) + P(\bar{A})\,P(T)$$

$$= \frac{1}{3}\times\frac{4}{5} + \frac{2}{3}\times\frac{1}{5} = \frac{6}{15} = \frac{2}{5}$$

Example 19.17 A problem in statistics is given to three students, whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that problem will be solved ?

Solution : Let p_1, p_2 and p_3 be the probabilities of three persons of solving the problem.

Here, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{4}$.

The problem will be solved, if at least one of them solves the problem.

$\therefore$ P (at least one of them solves the problem)

$$= 1 - P \text{ (None of them solves the problem)} \qquad .....(1)$$

Now, the probability that none of them solves the problem will be

P (none of them solves the problem) $= (1-p_1)(1-p_2)(1-p_3)$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Putting this value in (1), we get

$$\text{P (at least one of them solves the problem)} = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, the probability that the problem will be solved is $\frac{3}{4}$.

Example 19.18 Two balls are drawn at random with replacement from a box containing 15 red and 10 white balls. Calculate the probability that

(a) both balls are red.

(b) first ball is red and the second is white.

(c) one of them is white and the other is red.

Solution :

(a) Let A be the event that first drawn ball is red and B be the event that the second ball drawn is red. Then as the balls drawn are with replacement,

therefore $\quad P(A) = \frac{15}{25} = \frac{3}{5},\ P(B) = \frac{3}{5}$

As A and B are independent events

therefore $\quad$ P (both red) = P (A and B)

$$= P(A) \times P(B) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

(b) Let A : First ball drawn is red.

B : Second ball drawn is white.

$$\therefore \quad P(\text{A and B}) = P(A) \times P(B) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

(c) If WR denotes the event of getting a white ball in the first draw and a red ball in the second draw and the event RW of getting a red ball in the first draw and a white ball in the second draw. Then as 'RW' and WR' are mutually exclusive events, therefore

$\therefore$ P (a white and a red ball)

$$= \text{P (WR or RW)}$$

$$= P(WR) + P(RW)$$

$$= P(W)\,P(R) + P(R)\,P(W)$$

$$= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5}$$

$$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Example 19.19 A dice is thrown 3 times. Getting a number '5 or 6' is a success. Find the probability of getting

(a) 3 successes (b) exactly 2 successes (c) at most 2 successes (d) at least 2 successes.

Solution : Let S denote the success in a trial and F denote the ' not success' i.e. failure. Therefore,

$$P(S) = \frac{2}{6} = \frac{1}{3}, \quad P(F) = 1 - \frac{1}{3} = \frac{2}{3}$$

(a) As the trials are independent, by multiplication theorem for independent events,

$$P(SSS) = P(S) \; P(S) \; P(S) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(SSF) = P(S)\,P(S)\,P(F) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$$

Since the two successes can occur in ${}^3C_2$ ways

$\therefore$ $\quad$ $P(\text{exactly two successes}) = {}^3C_2 \times \frac{2}{27} = \frac{2}{9}$

(c) $P(\text{at most two successes}) = 1 - P(3\,\text{successes}) = 1 - \frac{1}{27} = \frac{26}{27}$

(d) $P(\text{at least two successes}) = P(\text{exactly 2 successes}) + P(\text{3 successes})$

$$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$$

Example 19.20 A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(i) A : the card drawn is a spade

B : the card drawn is an ace

(ii) A : the card drawn is black

B : the card drawn is a king

(iii) A : the card drawn is a king or a queen

B : the card drawn is a queen or a jack

Solution : (i) There are 13 cards of spade in a pack. $P(A) = \frac{13}{52} = \frac{1}{4}$

There are four aces in the pack. $P(B) = \frac{4}{52} = \frac{1}{13}$

$$A \cap B = \{\text{ an ace of spade }\}$$

$\therefore$ $$P(A \cap B) = \frac{1}{52}$$

Now $$P(A)\;P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$$

Since $$P(A \cap B) = P(A).\ P(B)$$

Hence, the events A and B are independent.

(ii) There are 26 black cards in a pack.

$\therefore$ $$P(A) = \frac{26}{52} = \frac{1}{2}$$

There are four kings in the pack. $\therefore$ $$P(B) = \frac{4}{52} = \frac{1}{13}$$

$$A \cap B = \{\text{ 2 black kings }\} \;\therefore\; P(A \cap B) = \frac{2}{52} = \frac{1}{26}$$

Now, $$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$$

Since $$P(A \cap B) = P(A)\;P(B)$$

Hence, the events A and B are independent.

(iii) There are 4 kings and 4 queens in a pack of cards.

$\therefore$ Total number of outcomes favourable to the event A is 8.

$\therefore$ $$P(A) = \frac{8}{52} = \frac{2}{13}$$

Similarly, $$P(B) = \frac{2}{13},\quad A \cap B = \{\text{ 4 queens }\}$$

$\therefore$ $$P(A \cap B) = \frac{4}{52} = \frac{1}{13}$$

$\therefore$ $$P(A) \times P(B) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}$$

Here, $$P(A \cap B) \neq P(A).P(B)$$

Hence, the events A and B are not independent.

CHECK YOUR PROGRESS 19.4

1. A husband and wife appear in an interview for two vacancies in the same department. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) Only one of them will be selected ?

(b) Both of them will be selected ?

(c) None of them will be selected ?

(d) At least one of them will be selected ?

2. Probabilities of solving a specific problem independently by Raju and Soma are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved.

(b) exactly one of them solves the problem.

3. A die is rolled twice. Find the probability of a number greater than 3 on each throw.

4. Sita appears in the interview for two posts A and B, selection for which are independent. The probability of her selection for post A is $\frac{1}{5}$ and for post B is $\frac{1}{7}$. Find the probability that she is selected for

(a) both the posts

(b) at least one of the posts.

5. The probabilities of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

6. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet ?

7. Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3:2 against a person B now 45 living till he is 75. Find the chance that at least one of these persons will be alive 30 years hence.

8. A bag contains 13 balls numbered from 1 to 13. Suppose an even number is considered a 'success'. Two balls are drawn with replacement, from the bag. Find the probability of getting

(a) Two successes (b) exactly one success

(c) at least one success (d) no success

9. One card is drawn from a well-shuffled deck of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(a) A : The drawn card is red

B : The drawn card is a queen

(b) A : The drawn card is a heart B: The drawn card is a face card

19.7 CONDITIONAL PROBABILITY

Suppose that a fair die is thrown and the score noted. Let A be the event, the score is 'even'. Then

$$A = \{2, 4, 6\}, \therefore \; P(A) = \frac{3}{6} = \frac{1}{2}.$$

Now suppose we are told that the score is greater than 3. With this additional information what will be P (A) ?

Let B be the event, 'the score is greater than 3'. Then B is {4, 5, 6 }. When we say that B has occurred, the event 'the score is less than or equal to 3' is no longer possible. Hence the sample space has changed from 6 to 3 points only. Out of these three points 4, 5 and 6; 4 and 6 are even scores.

Thus, given that B has occurred, P (A) must be $\frac{2}{3}$.

Let us denote the probability of A given that B has already occurred by $P(A \mid B)$.

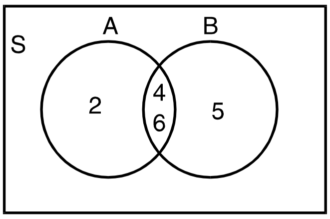


FIg. 19.7

Again, consider the experiment of drawing a single card from a deck of 52 cards. We are interested in the event A consisting of the outcome that a black ace is drawn.

Since we may assume that there are 52 equally likely possible outcomes and there are two black aces in the deck, so we have

$$P(A) = \frac{2}{52} \; .$$

However, suppose a card is drawn and we are informed that it is a spade. How should this information be used to reappraise the likelihood of the event A ?

Clearly, since the event B "A spade has been drawn " has occurred, the event "not spade" is no longer possible. Hence, the sample space has changed from 52 playing cards to 13 spade cards. The number of black aces that can be drawn has now been reduced to 1.

Therefore, we must compute the probability of event A relative to the new sample space B.

Let us analyze the situation more carefully.

The event A is " a black ace is drawn'. We have computed the probability of the event A knowing that B has occurred. This means that we are computing a probability relative to a new sample space B. That is, B is treated as the universal set. We should consider only that part of A which is included in B.

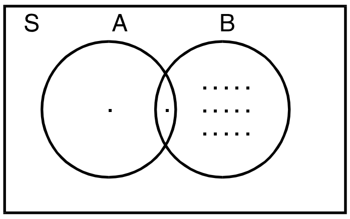


Fig. 19.8

Hence, we consider $A \cap B$ (see figure 31.8).

Thus, the probability of A. given B, is the ratio of the number of entries in $A \cap B$ to the number of entries in B. Since $n(A \cap B) = 1$ and $n(B) = 13$,

then $$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{13}$$

Notice that $$n(A \cap B) = 1 \Rightarrow P(A \cap B) = \frac{1}{52}$$

$$n(B) = 13 \Rightarrow P(B) = \frac{13}{52}$$

$$\therefore \quad P(A \mid B) = \frac{1}{13} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{P(A \cap B)}{P(B)}.$$

This leads to the definition of conditional probability as given below :

Let A an B be two events defined on a sample space S. Let P(B) > 0, then the conditional probability of A, provided B has already occurred, is denoted by P(A|B) and mathematically written as :

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Similarly, $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

The symbol $P(A \mid B)$ is usually read as "the probability of A given B".

Example 19.21 Consider all families "with two children (not twins). Assume that all the elements of the sample space {BB, BG, GB,GG} are equally likely. (Here, for instance, BG denotes the birth sequence "boy girls"). Let A be the event {BB} and B be the event that 'at least one boy'. Calculate P (A | B).

Solution : Here, $A = \{BB\}$, $B = \{BB, BG, GB\}$

$$A \cap B = \{BB\} \therefore \quad P(A \cap B) = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Hence, $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Example 19.22 Assume that a certain school contains equal number of female and male students. 5 % of the male population is football players. Find the probability that a randomly selected student is a football player male.

Solution : Let M = Male

F = Football player

We wish to calculate $P(M \cap F)$. From the given data,

$$P(M) = \frac{1}{2} \quad (\because \text{ School contains equal number of male and female students})$$

$$P(F|M) = 0.05$$

But from definition of conditional probability, we have

$$P(F \mid M) = \frac{P(M \cap F)}{P(M)}$$

$$\Rightarrow \quad P(M \cap F) = P(M) \times P(F \mid M)$$

$$= \frac{1}{2} \times 0.05 = 0.025$$

Example 19.23 If A and B are two events, such that $P(A) = 0.8$, $P(B) = 0.6$, $P(A \cap B) = 0.5$, find the value of

(i) $P(A \cup B)$ (ii) $P(B \mid A)$ (iii) $P(A \mid B)$.

Solution :(i) $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.8 + 0.6 - 0.5 = 0.9$$

(ii) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.5}{0.8} = \frac{5}{8}$$

(iii) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

Example 19.24 A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the probability that coin is tossed three times ?

Solution : Here, it is given that head does not occur on the first toss. That is, we may get the head on the second toss or on the third toss or even no head.

Let B be the event, " no heads on first toss".

Then $B = \{TH, TTH, TTT\}$

These events are mutually exclusive.

$$P(B) = P(TH) + P(TTH) + P(TTT) \qquad .....(1)$$

Now $P(TH) = \frac{1}{4}$ ($\because$ This event has the sample space of four outcomes)

and $P(TTH) = P(TTT) = \frac{1}{8}$ ($\because$ This event has the sample space of eight outcomes)

Putting these values in (1), we get

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Let A be the event "coin is tossed three times".

Then $A = \{TTH, TTT\}$

$\therefore$ We have to find $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Here, $A \cap B = A$, $\therefore$ $P(A \mid B) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

CHECK YOUR PROGRESS 19.5

1. A sequence of two cards is drawn at random (without replacement) from a well-shuffled deck of 52 cards. What is the probability that the first card is red and the second card is black ?

2. Consider a three child family for which the sample space is

   { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }

   Let A be the event " the family has exactly 2 boys " and B be the event " the first child is a boy". What is the probability that the family has 2 boys, given that first child is a boy ?

3. Two cards are drawn at random without replacement from a deck of 52 cards. What is the probability that the first card is a diamond and the second card is red ?

4. If A and B are events with $P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.1$, find the probability of A given B. Also find $P(B|A)$.

5. From a box containing 4 white balls, 3 yellow balls and 1 green ball, two balls are drawn one at a time without replacement. Find the probability that one white and one yellow ball is drawn.

19.8 THEOREMS ON MULTIPLICATION LAW OF PROBABILITY AND CONDITIONAL PROBABILITY.

Theorem 1 : For two events A and B,

$$P(A \cap B) = P(A).P(B \mid A),$$

and $$P(A \cap B) = P(B).P(A \mid B),$$

where P (B|A) represents the conditional probability of occurrence of B, when the event A has already occurred and P (A|B) is the conditional probability of happening of A, given that B has already happened.

Proof : Let n (S) denote the total number of equally likely cases, n (A) denote the cases favourable to the event A, n (B) denote the cases favourable to B and $n(A \cap B)$ denote the cases favourable to both A and B.

$$\therefore \qquad P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad \text{.....(1)}$$

For the conditional event A|B , the favourable outcomes must be one of the sample points of B, i.e., for the event A|B, the sample space is B and out of the n (B) sample points, $n(A \cap B)$ pertain to the occurrence of the event A, Hence,

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (1), we get $$P(A \cap B) = \frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)} = P(B).P(A \mid B)$$

Similarly, we can prove

$$P(A \cap B) = P(A).P(B \mid A)$$

Note : If A and B are independent events, then

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B)$$

$$P(A \cap B) = P(A).P(B)$$

Theorem 2 : Two events A and B of the sample space S are independent, if and only if

$$P(A \cap B) = P(A).P(B)$$

Proof : If A and B are independent events,

then $$P(A \mid B) = P(A)$$

We know that $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \qquad P(A \cap B) = P(A)P(B)$$

Hence, if A and B are independent events, then the probability of 'A and B' is equal to the product of the probability of A and probability of B.

Conversely, if $P(A \cap B) = P(A).P(B)$, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ gives}$$

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

That is, A and B are independent events.

19.9 INTRODUCTION TO BAYES' THEOREM

In conditional probability we have learnt to find probability of an event with the condition that some other event has already occurred. Consider an experiment of selecting one coin out of three coins : If I with P(H) = $\frac{1}{3}$ and P(T) = $\frac{2}{3}$, II with P(H) = $\frac{3}{4}$ and P(T) = $\frac{1}{4}$ and III with P(H) = $\frac{1}{2}$, P(T) = $\frac{1}{2}$ (a normal coin).

After randomly selecting one of the coins, it is tossed. We can find the probability of selecting one coin $\left(\text{i.e. } \frac{1}{3}\right)$ and can also find the probability of any outcome i.e. head or tail; given the coin selected. But can we find the probability that coin selected is coin I, II or III when it is known that the head occurred as outcome? For this we have to find the probability of an event which occurred prior to the given event. Such probability can be obtained by using Bayes' theorem, named after famous mathematician, Johan Bayes Let us first learn some basic definition before taking up Baye's theorem

Mutually exclusive and exhaustive events.

For a sample space S, the set of events $E_1, E_2, \ldots E_n$ is said to mutually exclusive and exhaustive if

(i) $E_i \cap E_j = \phi, \forall\, i \neq j = 1, 2, \ldots\ldots n$ i.e. none of two events can occur together.

(ii) $E_i \cup E_2 \cup \ldots\ldots \cup E_n = S$, all outcomes of S have been taken up in the events E_1, $E_2 \ldots E_n$

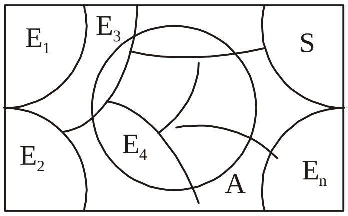
(iii) $P(E_i) > 0$ for all $i = 1, 2, \ldots\ldots n$

19.10 : THEOREM OF TOTAL PROBABILITY

Let $E_1, E_2, \ldots., E_n$ are mutually exclusive and exhaustive events for a sample space S with $P(E_i) > 0$, $\forall\, i = 1, 2, \ldots n$. Let A be any event associated with S, then

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A/E_i)$$



$U(A \cap E_n)$

Fig. 19.9

Proof : The events E_i and A are shown in the venn-diagram

Given $S = E_1 \cup E_2, \cup E_3, \ldots \cup E_n$ and $E_i \cap E_j \neq \phi$.

We can write

$$\begin{aligned} A &= A \cap S \\ &= A \cup (E_1 \cup E_2 \cup \ldots \cup E_n) \\ &= (A \cap E_1) \cap (A \cap E_2) \cup (A \cap E_3) \ldots (A \cap E_n) \end{aligned}$$

Since all E_i, are mutually exclusive, so $A \cap E_1$, $A \cap E_2$ ... will also be mutually exclusive

$$\begin{aligned} \Rightarrow \quad P(A) &= P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n) \\ &= P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/En) \end{aligned}$$

By using the multiplication rule of probability,

$$P(A) = \sum_{i=1}^{n} P(E_i)P(A/E_i)$$

19.11 : BAYE'S THEOREM

If $E_1, E_2, \ldots E_n$ are non-empty mutually exclusive and exhaustive events (i.e. $P(E_i) > 0 \ \forall\, i$) of a sample space S and A be any event of non-zero probability then

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \ \forall\ i = 1, 2, \ldots n$$

Proof : By law of total probabilities we know that

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \qquad \ldots(i)$$

Also by law of multiplication of probabilities we have

$$P(E_i|A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \text{ by using (i)}$$

This gives the proof of the Baye's theorem let us now apply the result of Baye's theorem to find probabilities.

Example 19.25 Given three identical coins (in shape and size) with following specifications

Coin I : with $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$

Coin II : with $P(H) = \frac{3}{4}$ $P(T) = \frac{1}{4}$

Coin III : with $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ (normal coin).

A Coin is selected at random and tossed. The out come found to be head. What is the probability that the selected coin was coin III?

Solution : Let E_1, E_2, E_3 be the events that coins I, II or III is selected, respectively.

Then $\quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, Let A be the event 'the coin drawn 'has head on tossing'.

Then $\quad P(A/E_1) =$ P(a head on coin I) $= \frac{1}{3}$

$P(A/E_2) =$ P(a head on coin II) $= \frac{3}{4}$

$P(A/E_3) =$ P(a head on coin III) $= \frac{1}{2}$

Now the probability that the coin tossed is Coin III = $P(E_3/A)$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{3}+\frac{3}{4}+\frac{1}{2}} = \frac{6}{4+9+6} = \frac{6}{19}$$

Example 19.26 Bag I contains 4 red and 3 black balls while another bag II contains 6 red and 5 black balls. One of the bags is selected at random and a ball is drawn from it. Find the probability that the ball is drawn from Bag II, if it is known that the ball drawn is red.

Solution : Let E_1 and E_2 be the events of selecting Bag I and Bag II, respectively and A be the event of selecting a red ball.

Then, $\quad P(E_1) = P(E_2) = \frac{1}{2}$

Also, $\quad P(A/E_1) =$ P(drawing a red ball from Bag I) $= \frac{4}{7}$

$P(A/E_2) =$ P(drawing a red ball from Bag II) $= \frac{6}{11}$

Now, By Baye's theorem

P(bag selected is Bag II when it is known that red ball is drawn) = $P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{4}{7}}{\frac{1}{2}\times\frac{6}{11}+\frac{1}{2}\times\frac{4}{7}}$$

$$= \frac{4/7}{\frac{6}{11}+\frac{4}{7}} = \frac{22}{43}$$

CHECK YOUR PROGRESS 19.6

1. Urn I contains 3 blue and 4 white balls and another Urn II contains 4 blue and 3 white balls. One Urn was selected at random and a ball was drawn from the selected Urn. The ball was found to be white. What is the probability that the ball was drawn from Urn-II?

2. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% by machine B were defective. All the items are put in one stock pile and then one item is randomly drawn from this and is found to be defective. Find the probability that the defective item was produced by machine A?

3. By examining the chest x-ray, the probability that T.B is detected when a person is actually suffering from it is 0.99.

   The probability that the doctor, diagnoses in correctly that a person has TB, on the basis of the x-ray is 0.001. In a certain city, 1 in 10000 persons suffer from TB. A person selected at random is diagnosed to have TB. What is the probability that person has actually TB?

19.12 : PROBABILITY DISTRIBUTION OF RANDOM VARIABLE

19.12.1 Variables : In earlier section you have learnt to find probabilities of various events with certain conditions. Let us now consider the case of tossing a coin four times. The outcomes can be shown in a sample space as :

S = {HHHH, HHHT, HHTH, HTHH, THHH, THHT, HHTT, HTTH, TTHH, HTHT, THTH, HTTT, THTT, TTHT, TTTH, TTTT}

On this sample space we can talk about various number associated with each outcome. For example, for each outcome, there is a number corresponding to number of heads we can call this number as X.

Clearly

X(HHHH) = 4, X(HHHT) = 3, X(HHTH) = 3

X(THHH) = 3, X(HHTT) = 2, X(HTTH) = 2

X(TTHH) = 2, X(HTHT) = 2, X(THTH) = 2

X(THHT) = 2, X(HTTT) = 1, X(THTT) = 1

X(TTHT) = 1, X(TTTH) = 1, X(TTTT) = 0

We find for each out come there corresponds values of X ranging from 0 to 4.

Such a variable X is called a random variable.

19.12.2 Definition

A random variable is a function whose domain is the sample space of a random experiment and range is real number values.

Example 19.27 Two dice are thrown simultaneously. Write the value of the random variable X : sum of number appearing on the upper faces of the dice.

Solution : The sample space of the experiment contains 36 elements.

S = {(1, 1), (1, 2), (1, 3) .... (1, 6)

(2, 1), (2, 2), (2, 3) .... (2, 6)

....

....

....

(6, 1), (6, 2), (6, 3) .... (6, 6)}

Clearly for each pair the sum of numbers appear ranging from 2 to 12. So the random variable *x* has the following values.

X(1, 1) = 2

X((1, 2), (2, 1)) = 3

X((1, 3), (2, 2), (3, 1)) = 4

X((1, 4), (2, 3), (3, 2), (4, 1) = 5

X((1, 5), (2, 4), (3, 3), (4, 2), (5, 1) = 6

X((1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), = 7

X((2, 6), (3, 5), (4, 4), (5, 3), (6, 2) = 8

X((3, 6), (4, 5), (5, 4), (6, 3) = 9

X((4, 6), (5, 5), (6, 4) = 10

X((5, 6), (6, 5)) = 11

X((6, 6)) = 12

19.12.3 Probability Distribution of a Random Variable

Let us now look at the experiment of drawing two cards successively with replacement from a well shuffled deck of 52 cards. Let us concentrate on the number of aces that can be there when two cards are successively drawn. Let it be denoted by X. Clearly X can take the values 0, 1 or 2.

The sample space for the experiment is given by S = {(Ace, Ace), (Ace, Non Ace), (Non Ace, Ace), (Non Ace, Non Ace)}

For X(Ace, Ace) = 2

X{(Ace, Non Ace) or (Non Ace, Ace)} = 1

and X{(Non Ace, Non Ace)} = 0

The probability that X can take the value 2 is P(Ace, Ace) = $\frac{4}{52} \times \frac{4}{52}$ as probability of an Ace is drawing one card is $\frac{4}{52}$.

Similarly

$$P(X = 1) = P[(\text{Ace, non Ace) or (Non Ace, ace})]$$

$$= P(\text{Ace, non Ace}) + P(\text{Non Ace, Ace})$$

$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{12}{169} + \frac{2}{169} = \frac{24}{169}$$

and $$P(X = 0) = P(\text{Non Ace, Non Ace}) = \frac{48}{52} \times \frac{48}{52} = \frac{144}{169}$$

The description given by the values of the random variable with the corresponding probabilities is called probability distribution.

19.12.4 Definition : The probability distribution of a random variable X is the distribution of probabilities to each value of X. A probability distribution of a random variable X is represented as

X_i	:	x_1	x_2	x_3	....	x_n
$P(X_i)$	:	P_1	P_2	P_3	....	P_n

where $P_i > 0, \sum_{i=1}^{n} P_i = 1, \forall\, i = 1, 2, 3, \ldots n$.

The real numbers x_1, x_2, ... x_n are the possible values of X and P_i is the probability of the random variable X_i taking the value X_i denoted as

$$P(X = x_i) = P_i$$

Thus the probability distribution of number of aces when two cards are successively drawn, with replacement from a deck of 52 cards is given by

X	:	0	1	2
$P(x_i)$	:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Note that in a probability distribution all probabilities must be between 0 and 1 and sum of all probabilities must be 1.

$$\Sigma P_i = \frac{144}{169} + \frac{24}{169} + \frac{1}{169} = \frac{144 + 24 + 1}{169} = 1$$

Example 19.28 Check whether the distribution given below is a probability distribution or not

X	2	1	0	–1	–2
P(X)	0.1	0.2	0.3	0.2	0.2

Solution : All probabilities P(X) are positive and less than 1.

Also,
$$\Sigma P(x_i) = 0.1 + 0.2 + 0.3 + 0.2 + 0.2$$
$$= 1.0$$

Hence, the given distribution is probability distribution of a the random variable X.

Example 19.29 A random variable X has the following probability distribution :

X	–1	–2	–3	–4	–5	–6
P(X)	$\frac{1}{3}$	k	$\frac{1}{4}$	$2k$	$\frac{1}{6}$	$\frac{k}{4}$

Find (1) k (2) $P(X > -4)$ (3) $P(X < -4)$

Solution : (1) The sum of probabilities in the given distribution, must be 1.

$$\Rightarrow \quad \frac{1}{3} + k + \frac{1}{4} + 2k + \frac{1}{6} + \frac{k}{4} = 1$$

$$\Rightarrow \quad \frac{4 + 12k + 3 + 24k + 2 + 3k}{12} = 1$$

$$39k + 9 = 12$$

$$\Rightarrow \quad 39k = 3$$

$$\therefore \quad k = \frac{1}{13}$$

(2)
$$P(X > -4) = P(x = -3) + P(x = -2) + P(x = -1)$$
$$= \frac{1}{4} + k + \frac{1}{3} = \frac{1}{4} + \frac{1}{13} + \frac{1}{3} = \frac{103}{156}$$

(3)
$$P(X < -4) = P(x = -5) + P(x = -6)$$
$$= \frac{1}{6} + \frac{k}{4} = \frac{1}{6} + \frac{1}{13 \times 4} = \frac{29}{156}$$

Example 19.30 Find the probability distribution of number of tails in the simultaneous tosses of three coins.

Solution : The sample space for simultaneous toss of three coins is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be the number of tails.

Clearly X can take values, 0, 1, 2 or 3.

Now,

$$P(X = 0) = P(HHH) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

$$P(X = 1) = P(HHT \text{ or } HTH \text{ or } THH)$$

$$= P(HHT) + P(HTH) + P(THH)$$

$$=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

$$P(X = 2) = P(HTT \text{ or } THT \text{ or } TTH)$$

$$= P(HTT) + P(THT) + P(TTH) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $P(X = 3) = P(TTT) = \frac{1}{8}$.

Hence, the required probability distribution is

X	:	0	1	2	3
P(X)	:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

CHECK YOUR PROGRESS 19.7

1. State which of the following are not probability distribution of a random variable. Justify your answer

(a)

x	100	200	300
P(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(b)

Y	0	1	2	3	4	5
P(y)	0.1	0.2	0.3	0.4	0.5	0.6

(c)

x_i	–1	–2	0	2	1
P_i	0.2	0.15	–0.5	0.45	0.7

(d)

x_i	2	3	4	5
P_i	0.4	0.1	0.2	0.2

2. Find the probability distribution of

(a) Number of red balls when two balls drawn are one after other with replacement from a bag containing 4 red and 3 white balls.

(b) Number of sixes when two dice are thrown simultaneously

(c) Number of doublets when two dice are thrown simultaneously

19.13 : MEAN AND VARIANCE OF A RANDOM VARIABLE

19.13.1 Mean

The mean of a random variable is denoted by μ and is defined as

$$\mu = \sum_{i=1}^{n} x_i P_i ,$$

where $\Sigma P_i = 1$, $P_i > 0$, $\forall\, i = 1, 2, \ldots n$.

In other words we can say that the mean of a random variable is the sum of the product of values of the variables with corresponding probabilities. Mean of a random variable X is also called Expectation of the random variable 'X', denoted by E(x)

So $$E(x) = \mu = \sum_{i=1}^{n} x_i P_i .$$

19.13.2 : VARIANCE

Recall in frequently distribution we have studied that variance is a measure of dispersion or variability in the values. The similar meaning is attached to variance of a random variable.

Definition : Let a probability distribution be given as

X_i	:	x_1	x_2	x_3	...	x_n
$P(X_i)$	:	P_1	P_2	P_3	...	P_n

Let $\mu = E(x)$ be the mean of x.

Then the varaiance of X, denoted by var(x) or σ_x^2 is defined as

$$\sigma_x^2 = \text{Var}(x) = \sum_{i=1}^{n} (x_i - \mu)^2 P_i$$

$$= \sum_{i=1}^{n} (x_1^2 p_i + \mu^2 p_i - 2\mu\, x_i p_i) = \sum_{i=1}^{n} (x_i^2 P_i + \mu^2 P_i - 2\mu x_i\ p_i)$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \sum_{i=1}^{n} \mu^2 P_i - \sum_{i=1}^{n} 2\mu x_i\ p_i = \sum_{i=1}^{n} x_i^2 p_i + \mu^2 \sum_{i=1}^{n} P_i - 2\mu \sum_{i=1}^{n} x_i\ p_i$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \mu^2 .1 - 2\, \mu.\mu = \sum_{i=1}^{n} x_i^2 p_i - \mu^2 \quad (\because\ \mu = \sum_{i=1}^{n} x_i^2 p_i \text{ and } \sum_{i=1}^{n} p_i = 1)$$

$$= \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i\right)^2$$

We can also write

$$\text{var}(x) = E(X^2) - [E(X)]^2$$

Example 19.31 Find the mean and variance of the following distribution

x	–2	–1	0	1	2
P(x)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Solution : Given distribution is

X_i	–2	–1	0	1	2
$P(X_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X_iP(X_i)$	$-\frac{2}{8}$	$-\frac{2}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$
$x^2P(X_i)$	$\frac{4}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{4}{8}$

Now, $$\mu = \Sigma P(X_i)X_i = -\frac{2}{8} - \frac{2}{8} + 0 + \frac{1}{8} + \frac{2}{8} = -\frac{1}{8}$$

$$\text{Var}(x) = \Sigma X_i^2 P(X_i) - [\Sigma\ P(X_i)X_i]^2$$

$$= \left[\frac{4}{8} + \frac{2}{8} + 0 + \frac{1}{8} + \frac{4}{8}\right] - \left(-\frac{1}{8}\right)^2$$

$$= \frac{11}{8} - \frac{1}{64} = \frac{87}{64}$$

CHECK YOUR PROGRESS 19.8

1. Find mean and variance in each of the following distributions

(a)

X	:	1	2	3	4
P(X)	:	0.3	0.2	0.4	0.1

(b)

y_i	–2	–1	0	1	2
$P(y_i)$	0.1	0.2	0.3	0.25	0.15

2. Find the mean number of heads in three tosses of a fair coin.
3. Let X denote the difference of two numbers obtained on throwing two fair dice. Find the mean and variance of X. (Take absolute value of the difference)
4. Find the mean of the numbers of tails obtained when a biased coin having 25% chances of head and 75% of tail, is tossed two times.
5. Find the mean and variance of the number of sixes when two dice are thrown.

19.14 BERNOULLI TRIALS

When an experiment is repeated under similar conditions, each repeat is called a trial of the experiment. For example, if a coin is tossed three times, we say that there are three trials of the tossing of the coin.

A particular event may be called success of a trial. Clearly non-happening of the event may be termed as a failure. For, example in throwing a die, if the occurrence of a number less then 4 is named as success then the non-occurrence of a number less than 4 is named as failure. Thus, each trial can have two outcomes namely, success or failure.

Two or more trials of a random experiment can be performed in two ways :

1. The probability of success or failure remain constant in each trial. For example tossing a coin n number of times, but in each trial probability of getting head is $\frac{1}{2}$. Such trials are called independent trials.
2. The probability of success/failure varies with each trial. For example in drawing card from a deck of cards one after the other without replacement, in such trials if success is taken to be drawing a card of spade, the probability of success in respective trials will change.

i.e.	Trial	1st	2nd	3rd, ...
	Probability	$\frac{13}{52}$	$\frac{12}{51}$	$\frac{11}{50}$, ...

The trials of first type i.e. independent trials with two out comes success or failure are called Bernoulli trials.

Definition : Trials of a random experiment are called Bernoulli trials, if each trial has exactly two outcomes and trials are finite and independent.

19.15 : BINOMIAL DISTRIBUTION

The probability distribution of number successes in Bernoulli trials of a random experiment may be obtained by the expansion of $(q + p)^n$ where

p = prob. of success in each trial

q = $1 - p$, = prob. of failure

n = number of trials

Such a probability distribution is called Binomial Distribution. In other words we can say that in n Bernoulli trials of a random experiment, the number of successes can have the value, 0, 1, 2, 3, ...n.

So the Binomial Distribution of number of success : X, is given by

$$P(X = 0) = \text{1st term of the expansion of } (q + p)^n$$

$$P(X = 1) = \text{2nd term of the expansion of } (q + p)^n$$

$$\vdots$$

$$P(X = r) = (r + 1)\text{th term of expansion of } (q + p)^n$$

$$\vdots$$

$$P(X = n) = (n + 1)\text{th term of expansion of } (q + p)n$$

We know that

$$(q + p)^n = {}^nC_0\, q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2} p^2 + \ldots + {}^nC_r\, q^{n-r} p^r + \ldots + {}^nC_n p^n$$

$$\Rightarrow \quad P(x = 0) = {}^nC_0\, q^n$$

$$P(x = 1) = {}^nC_1\, q^{n-1} p$$

$$P(x = 2) = {}^nC_2\, q^{n-2} p^2$$

$$\vdots$$

$$P(X = r) = {}^nC_r\, q^{n-r} p^r$$

$$\vdots$$

$$P(X = n) = {}^nC_n\, p^n.$$

A Binomial distribution with n Bernoulli trials and probability of success in each trial as P, is denoted by B(n, p)

Let us now understand Binomial Distribution with following examples.

Example 19.32 Write the Binomial Distribution of number of successes in 3 Bernoulli trials.

Solution : Let $\quad p$ = prob. of success (S) in each trial

q = prob. of failure (F) in each trial

Clearly $\quad q = 1 - p$

Number of successes in three trials can take the values 0, 1, 2 or 3

The sample space for three trials ....

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

where S and F denote success and failure.

Now $\quad P(S = 0) = P(FFF) = P(F)\,P(F)\,P(F) = q.q.q = q^3$

$$P(S = 1) = P(SFF, FSF \text{ or } FFS) = P(SFF) + P(FSF) + P(FFS)$$

$$= P(S).P(F)\,P(F) + P(F).P(S).P(F) + P(F)\,P(F)\,P(S)$$

$$= p.q.q. + q.p.q + q.q.p = 3\,q^2 p$$

$$\begin{aligned} P(S = 2) &= P(SSF \text{ or } SFS \text{ or } FSS) \\ &= P(SSF) + P(SFS) + P(FSS) \\ &= P(S).P(S).P(F) + P(S)\ P(F)P(S) + P(F)\ P(S)\ P(S) \\ &= p.p.q + p.q.p + q.p.p = 3qp^2 \end{aligned}$$

$$P(S = 3) = P(SSS) = P(S)\ .\ P(S)\ .\ P(S) = p.p.p = p^3$$

Hence the prob. distribution of number of successes is

X_i	:	0	1	2	3
$P(X_i)$	:	q^3	$3q^2p$	$3qp^2$	p^3

Also $(q + p)^3 = q^3 + 3q^2p + 3p^2q + p^3$

Note that probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Example 19.33 A die is thrown 5 times. If getting 'an even number' is a success, what is the probability of.

(a) 5 successes

(b) at least 4 successes

(c) at most 3 successes?

Solution : Given X : "an even number"

Then p = P(an even number) $= \dfrac{3}{6} = \dfrac{1}{2}$

$$q = \text{P(not an even number)} = \frac{3}{6} = \frac{1}{2}$$

Since the trials of throwing die are Bernoulli trials.

So, $\qquad P(r \text{ successes}) = {}^nC_r\, q^{n-r}\, p^r$

Here, $n = 5 = {}^5C_r\left(\dfrac{1}{2}\right)^{5-2}\left(\dfrac{1}{2}\right)^2 = {}^5C_r\left(\dfrac{1}{2}\right)^5$

(*a*) Now P(5 successes) $= {}^5C_5\left(\dfrac{1}{2}\right)^5 = \dfrac{1}{32}$

(b) P(at least 3 successes)

= P(3 success or 4 successes or 5 successes)

= P(3 successes) + P(4 successes) + P(5 successes).

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5 \left(\frac{5\times4\times3}{3\times2\times1}+5+1\right) = \left(\frac{1}{2}\right)^5 (10+5+1) = \frac{16}{32} = \frac{1}{2}$$

(c) P(at most 3 successes)

= P(0 successes or 1 success or 2 success or 3 successes)

= P(0 successes) + P(1 success) + P(2 successes) + P(3 successes)

$$= {}^5C_0\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32} + 5\times\frac{1}{32} + \frac{5\times4}{2\times1}\times\frac{1}{32} + \frac{5\times4\times3}{3\times2\times1}\times\frac{1}{32}$$

$$= \frac{1}{32}[1+5+10+10] = \frac{26}{32} = \frac{13}{16}.$$

CHECK YOUR PROGRESS 19.9

1. Find the following probabilities when a fair coin is tossed 10 times.
   (a) exactly 6 heads
   (b) at least 6 heads
   (c) at most 6 heads
2. A pair of dice is thrown 4 times. If getting a doublet (1, 1), (2, 2)... etc. is considered a success, find the probability of two successes.
3. From a bag containing 3 red and 4 black sells, five balls are drawn successively with replacement. If getting "a black ball" is considered "success", find the probability of getting 3 successes.
4. In a lot of bulbs manufactured in a factory, 5% are defective. What is the probability that a sample of 10 bulbs will include not more than one defective bulb?
5. Probability that a CFL produced by a factory will fuse after 1 year of use is 0.01. Find the probability that out of 5 such CFL's.
   (a) none
   (b) not more than one
   (c) more than one
   (d) at least one

   will fuse after 1 year of use.

LET US SUM UP

- **Complement of an event :** The complement of an event A consists of all those outcomes which are not favourable to the event A, and is denoted by 'not A' or by $\bar{A}$.

- **Event 'A or B' :** The event 'A or B' occurs if either A or B or both occur.
- **Event 'A and B' :** The event 'A and B' consists of all those outcomes which are favourable to both the events A and B.
- **Addition Law of Probability :** For any two events A and B of a sample space S

  $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- **Additive Law of Probability for Mutually Exclusive Events :** If A and B are two mutually exclusive events, then

  $$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

- **Odds in Favour of an Event :** If the odds for A are a to b, then $P(A) = \dfrac{a}{a+b}$

  If odds against A are a to b, then $P(A) = \dfrac{b}{a+b}$

- Two events are mutually exclusive, if occurrence of one precludes the possibility of simultaneous occurrence of the other.
- Two events are independent, if the occurence of one does not affect the occurence of other. If A and B are independent events, then P (A and B) = P (A). P (B) or $P(A \cap B) = P(A).P(B)$
- For two dependent events $P(A \cap B) = P(A).P\left(B/A\right) where \, P(A) > 0$

  or

  $P(A \cap B) = P(B) where \, P\left(A/B\right) / P(B) > 0$

- Conditional Probability $P\left(\dfrac{A}{B}\right) = \dfrac{A(A \cap B)}{P(B)}$ and $P\left(B/A\right) = \dfrac{P(A \cap B)}{P(A)}$
- Theorem of Total Probability

  $$P(A) = P(E_1).\, P\left(\frac{A}{E_1}\right) + P(E_2).\, P\left(\frac{A}{E_2}\right) + \cdots + P(E_n).P\left(\frac{A}{E_n}\right)$$

- Baye's Theorem: If $B_1, B_2 \cdots B_n$ are mutually exclusive events and A is any event that occurs with B_1 or B_2 or B_n then $P\left(\dfrac{Bi}{A}\right) = \dfrac{P(Bi), P\left(A/Bi\right)}{\sum_{i=1}^{n} P(Bi), P\left(A/Bi\right)}, i = 1, 2, \cdots n$
- Mean and Variance of a Random Varibale

  $$\mu = E(x) = \frac{n}{z} XiPi \, , \; \sigma^2 = \frac{n}{z}(xi - \mu)^2 = \frac{n}{z} xi^2 pi - \mu^2$$

- Binomial Distribution, $P(x = r) = nc_r \, p^r . q^{n-r}$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Probability

http://mathworld.wolfram.com/Probability

http://en.wikipedia.org/wiki/probability _distribution

http://en.wikipedia.org/wiki/Probability_theory

http://en.wikipedia.org/wiki/Bernoulli_distribution

TERMINAL EXERCISE

1. In a simultaneous toss of four coins, what is the probability of getting
   (a) exactly three heads ?
   (b) at least three heads ?
   (c) atmost three heads ?
2. Two dice are thrown once. Find the probability of getting an odd number on the first die or a sum of seven.
3. An integer is chosen at random from first two hundred integers. What is the probability that the integer chosen is divisible by 6 or 8 ?
4. A bag contains 13 balls numbered from 1 to 13. A ball is drawn at random. What is the probability that the number obtained it is divisible by either 2 or 3 ?
5. Find the probability of getting 2 or 3 heads, when a coin is tossed four times.
6. Are the following probability assignments consistent ? Justify your answer.
   (a) $P(A) = 0.6,\ P(B) = 0.5\ ,\ P(A \text{ and } B) = 0.4$
   (b) $P(A) = 0.2,\ P(B) = 0.3\ ,\ P(A \text{ and } B) = 0.4$
   (c) $P(A) = P(B) = 0.7,\ P(A \text{ and } B) = 0.2$
7. A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even ?
8. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?
9. A lady buys a dozen eggs, of which two turn out to be bad. She chose four eggs to scramble for breakfast. Find the chances that she chooses
   (a) all good eggs
   (b) three good and one bad eggs
   (c) two good and two bad eggs (d) at least one bad egg.

10. Two cards are drawn at random without replacement from a well-shuffled deck of 52 cards. Find the probability that the cards are both red or both kings.

11. Let A and B be two events such that $P(\vec{A}) = \frac{1}{2}, P(\vec{B}) = \frac{2}{3}, P(A \vec{\cap} B) = \frac{1}{4}$, Find $P(A/B)$ and $P(B/A)\cdot$

12. A bag contains 10 black and 5 white balls. Two balls are drawn from the bag successively whithout replacement. Find the probability that both the balls drawn are black.

13. Find the probability distribution of X; where X denotes the sum of numbers obtained when two dice are rolled.

14. An urn contains 4 black, 2 red and 2 white balls. Two balls (one after the other without replacement) are drawn randomly from the urn. Find the probability distribution of number of black balls.

15. Find the mean and variance of number of kings when two cards are simultaneously drawn from a deck of 52 cards.

16. Ten bolts are drawn successively with replacement from a bag containing 5% defective bolts. Find the probability that there is at least one defective bolt.

17. Find the mean of the Binomial $B\left(4, \frac{1}{3}\right)$.

18. A die is thrown again and again until three sixes are obtained. Find the probability of getting the third six in the sixth throw.

19. How many times must a man toss a fair coin so that the probability of having atleast one head is more than 90%?

20. Find the probability of getting 5 exactly twice in seven throws of a die.

21. Find the mean number of heads in three tosses of a fair coin.

22. A factory produces nuts, by using three machine A, B and C, manufacturing 20%, 40% and 40% of the nuts. 5%, 4% and 2% of their outputs are respectively found to be defective, A nut is drawn of randomly from the product and is found to be defective. What is the probability that it is manufactured by the machine C?

ANSWERS

CHECK YOUR PROGRESS 19.1

1. $\frac{1}{6}$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{3}{4}$

5. (i) $\frac{3}{5}$ (ii) $\frac{2}{5}$

6. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{36}$

7. $\frac{5}{9}$ 8. $\frac{1}{12}$ 9. $\frac{1}{2}$

10. (i) $\frac{1}{4}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{52}$

11. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}$ (iii) $\frac{11}{36}$

12. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{1}{8}$

CHECK YOUR PROGRESS 19.2

1. $\frac{1}{8}$ 2. $\frac{20}{39}$ 3.(a) $\frac{4}{25}$ (b) $\frac{38}{245}$

4. $\frac{1}{5525}$ 6. (i) $\frac{3}{10}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{30}$

7. $\frac{10}{133}$ 8. $\frac{4}{7}$ 9. $\frac{60}{143}$ 10. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.3

1. $\frac{4}{13}$ 2. $\frac{7}{36}$ 3. $\frac{9}{16}$ 4. $\frac{7}{12}$

5. $\frac{4}{9}$ 6. $\frac{1}{2}$ 7. $\frac{7}{13}$ 8. $\frac{5}{12}$

9. (a) $\frac{5}{18}$ (b) 0.7 10. $\frac{4}{11}$

11. (a) $\frac{5}{6}$ (b) $\frac{35}{36}$ 12. $\frac{3}{4}$

13. (a) The odds for A are 7 to 3 . The odds against A are 3 to 7

13. (b) The odds for A are 4 to 1 and The odds against A are 1 to 4

14. (a) $\frac{7}{9}$ (b) $\frac{7}{17}$ 15. (a) $\frac{5}{9}$ (b) $\frac{3}{4}$ 16. (a), (c) 17. $\frac{4}{7}$ 18. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.4

1. (a) $\frac{2}{7}$ (b) $\frac{1}{35}$ (c) $\frac{24}{35}$ (d) $\frac{11}{35}$ 2. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ 3. $\frac{1}{4}$

4. (a) $\frac{1}{35}$ (b) $\frac{11}{35}$ 5. $\frac{1}{2}$ 6. (a) $\frac{5}{144}$ (b) $\frac{1}{1014}$ 7. $\frac{53}{80}$

8. (a) $\frac{36}{169}$ (b) $\frac{84}{169}$ (c) $\frac{120}{169}$ (d) $\frac{49}{169}$

9. (a) Independent (b) Independent

CHECK YOUR PROGRESS 19.5

1. $\frac{13}{51}$ 2. $\frac{1}{2}$ 3. $\frac{25}{204}$ 4. $\frac{1}{2}, \frac{1}{4}$ 5. $\frac{3}{7}$

CHECK YOUR PROGRESS 19.6

1. $\frac{3}{7}$ 2. $\frac{3}{4}$ 3. $\frac{10}{111}$

CHECK YOUR PROGRESS 19.7

1. (a) Yes (b) No, as ΣP_i is not 1

   (c) No, as one of the P_i is –ve (d) No as ΣP_i is not 1

2. (a)

x	:	0	1	2
$P(x)$	:	$\frac{9}{49}$	$\frac{24}{49}$	$\frac{16}{49}$

(b)

X_i	:	0	1	2
$P(X_i)$	:	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

CHECK YOUR PROGRESS 19.8

1. (a) $\mu = 2.3$, Var $= 1.01$ (b) $\mu = 0.15$, Var $= 0.4275$

2. $\mu = \frac{3}{2}$

3.

X	:	0	1	2	3	4	5
$P(X_i)$	:	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

4. Mean $\mu = \frac{3}{2}$, Var. $(X_i) = \frac{3}{8}$

5. Mean $= \frac{1}{3}$, Var. $= \frac{5}{18}$

CHECK YOUR PROGRESS 19.9

1. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

2. $\frac{25}{216}$

3. $\frac{90 \times 64}{7^5}$

4. $\left(\frac{29}{20}\right)\left(\frac{19}{20}\right)^9$

5. (a) $\left(\frac{99}{100}\right)^5$ (b) $\left(\frac{99}{100}\right)^5 + 5.\frac{99^4}{100^5}$

(c) $1 - \left\{\left(\frac{99}{100}\right)^5 + \frac{5 \times 99^4}{100^5}\right\}$ (d) $1 - \left(\frac{99}{100}\right)^5$

TERMINAL EXERCISE

1. (a) $\frac{1}{4}$ (b) $\frac{5}{16}$ (c) $\frac{15}{16}$

2. $\frac{7}{12}$

3. $\frac{1}{4}$

4. $\frac{8}{13}$

5. $\frac{5}{8}$

6. Only (a) is consistent

7. $\frac{456}{625}$

8. $\frac{5}{8}$

9. (a) $\frac{14}{33}$ (b) $\frac{16}{33}$ (c) $\frac{1}{11}$ (d) $\frac{19}{33}$

10. $\frac{55}{221}$

11. $\frac{3}{4}, \frac{1}{2}$

12. $\frac{3}{7}$

13.

X_i :	2	3	4	5	6	7	8	9	10	11	12
$P(X_i)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

14.

x :	0	1	2
$P(x)$:	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{3}{14}$

15. Mean $= \frac{34}{221}$, variance $= \frac{6800}{(221)^2}$.

16. $1 - \left(\frac{19}{20}\right)^{10}$

17. $\frac{4}{3}$

18. $\frac{625}{23328}$

19. $n = 4$

20. $\frac{7}{12} \times \left(\frac{5}{6}\right)^5$

21. 1.5

22. $\frac{4}{17}$

20

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MATRICES

In the middle of the 19th Century, Arthur Cayley (1821-1895), an English mathematician created a new discipline of mathematics, called **matrices**. He used matrices to represent simultaneous system of equations. As of now, theory of matrices has come to stay as an important area of mathematics. The matrices are used in game theory, allocation of expenses, budgeting for by-products etc. Economists use them in social accounting, input-output tables and in the study of inter-industry economics. Matrices are extensively used in solving the simultaneous system of equations. Linear programming has its base in matrix algebra. Matrices have found applications not only in mathematics, but in other subjects like Physics, Chemistry, Engineering, Linear Programming etc.

In this lesson we will discuss different types of matrices and algebraic operations on matrices in details.

OBJECTIVES

After studying this lesson, you will be able to:

- define a matrix, order of a matrix and cite examples thereof;
- define and cite examples of various types of matrices-square, rectangular, unit, zero, diagonal, row, column matrix;
- state the conditions for equality of two matrices;
- define transpose of a matrix;
- define symmetric and skew symmetric matrices and cite examples;
- find the sum and the difference of two matrices of the same order;
- multiply a matrix by a scalar;
- state the condition for multiplication of two matrices; and
- multiply two matrices whenever possible.
- use elementary transformations
- find inverse using elementary trnsformations

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number system
- Solution of system of linear equations

20.1 MATRICES AND THEIR REPRESENTATIONS

Suppose we wish to express that Anil has 6 pencils. We may express it as [6] or (6) with the understanding that the number inside [ ] denotes the number of pencils that Anil has. Next suppose that we want to express that Anil has 2 books and 5 pencils. We may express it as [2 5] with the understanding that the first entry inside [ ] denotes the number of books; while the second entry, the number of pencils, possessed by Anil.

Let us now consider, the case of two friends Shyam and Irfan. Shyam has 2 books, 4 notebooks and 2 pens; and Irfan has 3 books, 5 notebooks and 3 pens.

A convenient way of representing this information is in the tabular form as follows:

	Books	Notebooks	Pens
Shyam	2	4	2
Irfan	3	5	3

We can also briefly write this as follows:

$$\begin{array}{lc} & \begin{array}{ccc}\text{First Column} & \text{Second Column} & \text{Third Column}\\ \downarrow & \downarrow & \downarrow\end{array}\\ \begin{array}{l}\text{First Row}\\ \text{Second Row}\end{array} & \begin{bmatrix}2 & 4 & 2\\ 3 & 5 & 3\end{bmatrix}\end{array}$$

This representation gives the following information:

(1) The entries in the first and second rows represent the number of objects (Books, Notebooks, Pens) possessed by Shyam and Irfan, respectively

(2) The entries in the first, second and third columns represent the number of books, the number of notebooks and the number of pens, respectively.

Thus, the entry in the first row and third column represents the number of pens possessed by Shyam. Each entry in the above display can be interpreted similarly.

The above information can also be represented as

	Shyam	Irfan
Books	2	3
Notebooks	4	5
Pens	2	3

which can be expressed in three rows and two columns as given below:

$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$ The arrangement is called a **matrix.** Usually, we denote a matrix by a capital letter of English alphabets, i.e. *A*, *B*, *X*, etc. Thus, to represent the above information in the form of a matrix, we write

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}$$

Note: Plural of matrix is matrices.

20.1.1 Order of a Matrix Observe the following matrices (arrangement of numbers):

(a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & i \\ i & 1+i \\ 1+i & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 2 & 3 & 4 & 5 \\ 4 & -1 & -2 & 0 \end{pmatrix}$

In matrix (a), there are two rows and two columns, this is called a 2 by 2 matrix or a matrix of order 2×2. This is written as 2×2 matrix. In matrix (b), there are three rows and two columns. It is a 3 by 2 matrix or a matrix of order 3×2. It is written as 3×2 matrix. The matrix (c) is a matrix of order 3×4.

Note that there may be any number of rows and any number of columns in a matrix. If there are *m* rows and *n* columns in matrix *A*, its order is $m \times n$ and it is read as an $m \times n$ matrix.

Use of two suffixes *i* and *j* helps in locating any particular element of a matrix. In the above $m \times n$ matrix, the element a_{ij} belongs to the *i*th row and *j*th column.

$$A = \begin{bmatrix} a_{11}\ a_{12}\ a_{13} \cdots a_{1j} & \cdots a_{1n} \\ a_{21}\ a_{22}\ a_{23} \cdots a_{2j} & \cdots a_{2n} \\ a_{31}\ a_{32}\ a_{33} \cdots a_{3j} & \cdots a_{3n} \\ a_{i1}\ a_{i2}\ a_{i3} \cdots\ a_{ij} & \cdots a_{in} \\ a_{m1}\ a_{m2}\ a_{m3} \cdots a_{mj} & \cdots a_{mn} \end{bmatrix}$$

A matrix of order $m \times n$ can also be written as

$A = [a_{ij}]$, $i = 1, 2, ..., m$; and
$j = 1, 2, ..., n$

Example 20.1 Write the order of each of the following matrices:

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$ (iv) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{pmatrix}$

Solution: The order of the matrix

(i) is 2×2 (ii) is 3×1

(iii) is 1×3 (iv) is 2×3

Example 20.2 For the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & 6 \end{bmatrix}$$

(i) find the order of A

(ii) write the total number of elements in A

(iii) write the elements a_{23}, a_{32}, a_{14} and a_{34} of A

(iv) express each element 3 in A in the form a_{ij}.

Solution: The order of the matrix

(i) Since A has 3 rows and 4 columns, A is of order 3×4.

(ii) number of elements in $A = 3 \times 4 = 12$

(iii) $a_{23} = 2$; $a_{32} = 2$; $a_{14} = 4$ and $a_{34} = 6$

(iv) a_{22}, a_{31} and a_{33}

Example 20.3 If the element in the ith row and jth column of a 2×3 matrix A is given by $\dfrac{i+2j}{2}$, write the matrix A.

Solution: Here, $a_{ij} = \dfrac{i+2j}{2}$ (Given)

$$a_{11} = \frac{1+2\times1}{2} = \frac{3}{2};\quad a_{12} = \frac{1+2\times2}{2} = \frac{5}{2};\quad a_{13} = \frac{1+2\times3}{2} = \frac{7}{2}$$

$$a_{21} = \frac{2+2\times1}{2} = 2;\quad a_{22} = \frac{2+2\times2}{2} = 3;\quad a_{23} = \frac{2+2\times3}{2} = 4$$

Thus, $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 \end{bmatrix}$$

Example 20.4 There are two stores A and B. In store A, there are 120 shirts, 100 trousers and 50 cardigans; and in store B, there are 200 shirts, 150 trousers and 100 cardigans. Express this information in tabular form in two different ways and also in the matrix form.

Solution:

Tabular Form 1

	Shirts	Trousers	Cardigans
Store A	120	100	50
Store B	200	150	100

Matrix Form

$$\Rightarrow \begin{pmatrix} 120 & 100 & 50 \\ 200 & 150 & 100 \end{pmatrix}$$

Tabular Form 2

	Store A	Store B
Shirts	120	200
Trousers	100	150
Cardigans	50	100

Matrix Form

$$\Rightarrow \begin{pmatrix} 120 & 200 \\ 100 & 150 \\ 50 & 100 \end{pmatrix}$$

CHECK YOUR PROGRESS 20.1

1. Marks scored by two students A and B in three tests are given in the adjacent table. Represent this information in the matrix form, in two ways

	Test 1	Test 2	Test 3
A	56	65	71
B	29	37	57

2. Three firms X, Y and Z supply 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively, to a contractor. Express this information in the matrix form in two ways.

3. In family P, there are 4 men, 6 women and 3 children; and in family Q, there are 4 men, 3 women and 5 children. Express this information in the form of a matrix of order 2×3.

4. How many elements in all are there in a

   (a) 2×3 matrix (b) 3×4 matrix (c) 4×2 matrix

   (d) 6×2 matrix (e) $a \times b$ matrix (f) $m \times n$ matrix

5. What are the possible orders of a matrix if it has
(a) 8 elements (b) 5 elements (c) 12 elements (d) 16 elements

6. In the matrix A,

$$A = \begin{bmatrix} 5 & 1 & 8 & 0 & 5 \\ 7 & 6 & 7 & 4 & 6 \\ 3 & 9 & 3 & -3 & 9 \\ 4 & 4 & 8 & 5 & 1 \end{bmatrix}$$

find: (a) number of rows;

(b) number of columns;

(c) the order of the matrix A;

(d) the total number of elements in the matrix A;

(e) $a_{14}, a_{23}, a_{34}, a_{45}$ and a_{33}

7. Construct a 3 $\times$ 3 matrix whose elements in the ith row and jth column is given by

(a) $i - j$ (b) $\frac{i^2}{j}$ (c) $\frac{(i+2j)^2}{2}$ (d) $3j - 2i$

8. Construct a 3×2 matrix whose elements in the ith row and jth column is given by

(a) $i + 3j$ (b) $5.i.j.$ (c) i^j (d) $i + j - 2$

20.2 TYPES OF MATRICES

Row Matrix : A matrix is said to be a *row matrix* if it has only one row, but may have any number of columns, e.g. the matrix $\begin{bmatrix} 1 & 6 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix.

T[he order of a row matrix is 1 $\times$ n.

Column Matrix : A matrix is said to be a *column matrix* if it has only one column, but may have any number of rows, e.g. the matrix $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}$ is a column matrix. The order of a column matrix is $m \times 1$

Square Matrix : A matrix is said to be a *square matrix* if number of rows is equal to the number of columns, e.g. the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ having 3 rows and 3 columns is a square matrix. The order of a square matrix is $n \times n$ or simply n.

The diagonal of a square matrix from the top extreme left element to the bottom extreme right element is said to be the principal diagonal. The principal diagonal of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ contains elements 2, 1 and 9.

Note: In any given matrix $A = [a_{ij}]$ of order $m \times n$, the elements of the principal diagonal are $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$

Rectangular Matrix : A matrix is said to be a *rectangular matrix* if the number of rows is not equal to the number of columns, e.g. the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$ having 3 rows and 4 columns is a rectangular matrix.It may be noted that a row matrix of order $1 \times n \ (n \neq 1)$ and a column matrix of order $m \times 1 \ (m \neq 1)$ are rectangular matrix.

Zero or Null Matrix : A matrix each of whose element is zero is called a *zero or null matrix*, e.g. each of the matrix

$$[0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix. Zero matrix is denoted by O.

Note: A zero matrix may be of any order $m \times n$.

Diagonal Matrix : A square matrix is said to be a *diagonal matrix*, if all elements other than those occuring in the principal diagonal are zero, i.e., if $A = [a_{ij}]$ is a square matrix of order $m \times n$, then it is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices.

Note: A diagonal matrix $A = [a_{ij}]_{n \times n}$ is also written as $A =$ diag $[a_{11}, a_{12}, a_{13}, \ldots, a_{nn}]$

Scalar Matrix : A diagonal matrix is said to be a *scalar matrix* if all the elements in its principal diagonal are equal to some non-zero constant, say k e.g., the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix.

Note: A square zero matrix is not a scalar matrix.

Unit or Identity Matrix : A scalar matrix is said to be a *unit or identity matrix*, if all of its elements in the principal diagonal are unity. It is denoted by I_n, if it is of order n e.g., the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: A square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 0, \text{ when } i \neq j \\ 1, \text{ when } i = j \end{cases}$

Equal Matrices : Two matrices are said to be equal if they are of the same order and if their corresponding elements are equal.

If A is a matrix of order $m \times n$ and B is a matrix of order $p \times r$, then $A = B$ if

(1) $m = p; n = r;$ and

(2) $a_{ij} = b_{ij}$ for all 3×2 and $j = 1, 2, 3, \ldots, n$

Two matrices X and Y given below are not equal, since they are of different orders, namely 2×3 and 3×2 respectively.

$$X = \begin{bmatrix} 7 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}, \quad Y = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

Also, the two matrices P and Q are not equal, since some elements of P are not equal to the corresponding elements of Q.

$$P = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

Example 20.5 Find whether the following matrices are equal or not:

(i) $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}$

Solution:

(i) Matrices A and B are of the same order 2×2. But some of their corresponding elements are different. Hence, $A \neq B$.

(ii) Matrices P and Q are of different orders, So, $P \neq Q$.

(iii) Matrices X and Y are of the same order 3×3, and their corresponding elements are also equal.

So, $X = Y$.

Example 20.6 Determine the values of x and y, if

(i) $\begin{bmatrix} x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \end{bmatrix}$ (iii) $\begin{bmatrix} x & 2 \\ 3 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Since the two matrices are equal, their corresponding elements should be equal.

(i) $x = 2$ (ii) $x = 4, \ y = 3$ (iii) $x = 1, \ y = -5$

Example 20.7 For what values of a, b, c and d, are the following matrices equal?

(i) $A = \begin{bmatrix} a & -2 & 2b \\ 6 & 3 & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 4 \\ 6 & 5c & 2 \end{bmatrix}$

(ii) $P = \begin{bmatrix} a & b-2d \\ -3 & 2b \\ a+c & 7 \end{bmatrix}$, $Q = \begin{bmatrix} 5 & 1 \\ -3 & 6 \\ 4 & 7 \end{bmatrix}$

Solution:

(i) The given matrices A and B will be equal only if their corresponding elements are equal, i.e. if

$$a=1,\ 2b=4,\ 3=5c, \text{ and } d=2$$

$$\Rightarrow\ a=1,\ b=2,\ c=\frac{3}{5} \text{ and } d=2$$

Thus, for $a=1,\ b=2,\ c=\frac{3}{5}$ and $d=2$ matrices A and B are equal.

(ii) The given matrices P and Q will be equal if their corresponding elements are equal, i.e. if

$$2b=6,\ b-2d=1,\ a=5 \text{ and } a+c=4$$

$$\Rightarrow\ a=5,\ b=3,\ c=-1 \text{ and } d=1$$

Thus, for $a=5,\ b=3,\ c=-1$ and $d=1$, matrices P and Q are equal.

CHECK YOUR PROGRESS 20.2

1. Which of the following matrices are

(a) row matrices (b) column matrices (c) square matrices (d) diagonal matrices (e) scalar matrices (f) identity matrices and (g) zero matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix},\ B = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$E = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 9 & 8 \\ 1 & 0 & 2 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G = \begin{bmatrix} 3 & 4 & 10 & 8 \end{bmatrix},\ H = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 4 & 9 \end{pmatrix},\ I = \begin{pmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 0 \end{pmatrix}$$

2. Find the values of a, b, c and d if

(a) $\begin{bmatrix} b & 2c \\ b+d & c-2a \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 8 & 2 \end{bmatrix}$ (b) $\begin{pmatrix} a+2 & 4 \\ b+3 & 25 \end{pmatrix} = \begin{pmatrix} 4 & 2c \\ 6 & 5d \end{pmatrix}$

(c) $\begin{pmatrix} 2a & b \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ d & 3c \end{pmatrix}$

3. Can a matrix of order 1×2 be equal to a matrix of order 2×1?

4. Can a matrix of order 2×3 be equal to a matrix of order 3×3?

20.3 TRANSPOSE OF A MATRIX

Associated with each given matrix there exists another matrix called its *transpose*. The transpose of a given matrix A is formed by interchanging its rows and columns and is denoted by A' or A^t, e.g. if

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 6 & 1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \\ -3 & 3 & 1 \end{bmatrix}$$

In general, If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose A' of A is the $n \times m$ matrix; and, (a_{ij})th element of $A = (a_{ji})$th element of A'

20.3.1 Symmetric Matrix

A square matrix A is said to be a *symmetric matrix* if $A' = A$.

For example,

$$If\ A = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}, \text{ then } A' = \begin{pmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{pmatrix}$$

Since $A' = A$, A is a symmetric matrix.

Note: (1) In a symmetric matrix $A = [a_{ij}]_{n \times n}$,
$a_{ij} = a_{ji}$ for all i and j

(2) A rectangular matrix can never be symmetric.

20.3.2 Skew-Symmetric Matrix

A square matrix A is said to be a *skew symmetric* if $A' = -A$, i.e. $a_{ij} = -a_{ji}$ for all i and j.

For example,

If $A = \begin{bmatrix} 0 & c & d \\ -c & 0 & f \\ -d & -f & 0 \end{bmatrix}$, *then* $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$

But $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$, which is the same as A'

$A' = -A$

Hence, A is a skew symmetric matrix

Note: In a skew symmetric matrix $A = \left[a_{ij}\right]_{n \times n}$, $a_{ij} = 0$, for $i = j$

i.e. all elements in the principal diagonal of a skew symmetric matrix are zeroes.

20.4 SCALAR MULTIPLICATION OF A MATRIX

Let us consider the following situation:

The marks obtained by three students in English, Hindi, and Mathematics are as follows:

	English	Hindi	Mathematics
Elizabeth	20	10	15
Usha	22	25	27
Shabnam	17	25	21

It is also given that these marks are out of 30 in each case. In matrix form, the above information can be written as

$$\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix}$$ (It is understood that rows correspond to the names and columns correspond to the subjects)

If the maximum marks are doubled in each case, then the marks obtained by these girls will also be doubled. In matrix form, the new marks can be given as:

$$\begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} \text{ which is equal to } \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

So, we write that

$$2\times\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix} = \begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} = \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

Now consider another matrix

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix}$$

Let us see what happens, when we multiply the matrix A by 5

$$\text{i.e.}\quad 5\times A = 5A = 5\times\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times(-2) & 5\times 0 \\ 5\times 1 & 5\times 6 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ -10 & 0 \\ 5 & 30 \end{bmatrix}$$

When *a* matrix is multiplied by a scalar, then each of its element is multiplied by the same scalar.

For example,

$$\text{if}\quad A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix} \text{ then } kA = \begin{bmatrix} k\times 2 & k\times(-1) \\ k\times 6 & k\times 3 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ 6k & 3k \end{bmatrix}$$

$$\text{When } k = -1,\ kA = (-1)A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$$

So, $(-1)A = -A$

Thus, if $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

Example 20.8 If $A = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, find

(i) $2A$ (ii) $\frac{1}{2}A$ (iii) $-A$ (iv) $\frac{2}{3}A$

Solution:

(i) Here, $2A = 2 \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times(-2) & 2\times 3 & 2\times 4 \\ 2\times(-1) & 2\times 0 & 2\times 1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 8 \\ -2 & 0 & 2 \end{bmatrix}$

(ii) $\frac{1}{2}A = \frac{1}{2} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times(-2) & \frac{1}{2}\times 3 & \frac{1}{2}\times 4 \\ \frac{1}{2}\times(-1) & \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(iii) $-A = (-1) \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -1 \end{bmatrix}$

iv) $\frac{2}{3}A = \frac{2}{3} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{bmatrix}$

CHECK YOUR PROGRESS 20.3

1. If $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, find:

   (a) $4A$ (b) $-A$ (c) $\frac{1}{2}A$ (d) $-\frac{3}{2}A$

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find:

   (a) $5A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $-\frac{1}{2}A$

3. If $A = \begin{bmatrix} -1 & 0 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$, find $(-7)A$

4. If $X = \begin{pmatrix} 3 & 0 & 1 \\ 4 & -2 & 0 \\ -1 & 0 & 5 \end{pmatrix}$, find:

(a) $5X$ (b) $-4X$ (c) $\frac{1}{3}X$ (d) $-\frac{1}{2}X$

5. Find A' (transpose of A):

(a) $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 4 & 10 & 9 \\ 6 & 8 & 7 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & -2 \\ 4 & -1 \\ -6 & 9 \end{pmatrix}$ (d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. For any matrix A, prove that $(A')' = A$

7. Show that each of the following matrices is a symmetric matrix:

(a) $\begin{bmatrix} 2 & -4 \\ -4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Show that each of the following matrices is a skew symmetric matrix:

(a) $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ (b) $\begin{bmatrix} 0 & i & 4 \\ -i & 0 & 2-i \\ -4 & -2+i & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix}$

20.5 ADDITION OF MATRICES

Two students A and B compare their performances in two tests in Mathematics, Physics and English. The maximum marks in each test in each subject are 50. The marks scored by them are as follows:

First Test Second Test

$$\begin{array}{cc} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 50 & 38 & 33 \\ 47 & 40 & 36 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 45 & 32 & 30 \\ 42 & 30 & 39 \end{bmatrix} \end{array}$$

How can we find their total marks in each subject in the two tests taken together ?

Observe that the new matrix giving the combined information of two matrices

$$\begin{array}{cc} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 50+45 & 38+32 & 33+30 \\ 47+42 & 40+30 & 36+39 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 95 & 70 & 63 \\ 89 & 70 & 75 \end{bmatrix} \end{array}$$

This new matrix is called the *sum* of the given matrices.

If *A* and *B* are any two given matrices of the same order, then their sum is defined to be a matrix *C* whose respective elements are the sum of the corresponding elements of the matrices *A* and *B* and we write this as *C* = *A* + *B*.

1. The order of the matrix *C* will also be the same as that of *A* and *B*.
2. It is not possible to add two matrices of different orders.

Example 20.9 If 2×2 and $B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices *A* and *B* are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 1+5 & 3+2 \\ 4+1 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 2 \end{bmatrix}$$

Example 20.10 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 0+3 & 1+0 & -1+4 \\ 2+1 & 3+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

20.5.1 Properties of Addition

Recall that in case of numbers, we have

(i) $x + y = y + x,$ i.e., addition is commutative

(ii) $x + (y + z) = (x + y) + z,$ i.e., addition is associative

(iii) $x + 0 = x,$ i.e., additive identity exists

(iv) $x + (-x) = 0$, i.e., additive inverse exists

Let us now find if these properties hold true in case of matrices too:

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$, Then,

$$A + B = \begin{bmatrix} 1+0 & 2-2 \\ -1+1 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$B + A = \begin{bmatrix} 0+1 & -2+2 \\ 1+(-1) & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

We see that $A + B$ and $B + A$ denote the same matrix. Thus, in general,

For any two matrices A and B of the same order, $A + B = B + A$

i.e. matrix addition is commutative

Let $A = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. Then,

$$A+(B+C) = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1+1 & -4+0 \\ 0+2 & 2+3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 3+(-4) \\ -2+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

and $$(A+B)+C = \begin{bmatrix} 0+1 & 3+(-4) \\ -2+0 & 1+2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1+0 \\ -2+2 & 3+3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

We see that $A + (B + C)$ and $(A + B) + C$ denote the same matrix. Thus, in general

For any three matrices A, B and C of the same order,

$A + (B + C) = (A + B) + C$ i.e., matrix addition is associative.

Recall that we have talked about zero matrix. A zero matrix is that matrix, all of whose elements are zeroes. It can be of any order.

Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$A+O = \begin{bmatrix} 2+0 & -2+0 \\ 4+0 & 5+0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix} = A$$

and $$O+A = \begin{bmatrix} 0+2 & 0-2 \\ 0+4 & 0+5 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix} = A$$

We see that $A + O$ and $O + A$ denote the same matrix A.

Thus, we find that $A + O = A = O + A$, where O is a zero matrix.

The matrix O, which is a zero matrix, is called the additive identity.

Additive identity is a zero matrix, which when added to a given matrix, gives the same given matrix, i.e., $A + O = A = O + A$.

Example 20.11 If $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, then

find: (a) $A + B$ (b) $B + C$ (c) $(A + B) + C$ (d) $A + (B + C)$

Solution:

(a) $A+B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}2+(-3) & 0+1\\ 1+1 & 3+2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}$

(b) $B+C=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}(-3)+(-1) & 1+0\\ 1+0 & 2+3\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$

(c) $(A+B)+C=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$ ... [From (a)]

$= \begin{bmatrix}(-1)+(-1) & 1+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$

(d) $A+(B+C)=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$ ... [From (b)]

$=\begin{bmatrix}2+(-4) & 0+1\\ 1+1 & 3+5\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$

Example 20.12 If $A=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

then find (a) $A+O$ (b) $O+A$

What do you observe?

Solution: (a) $A+O=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$=\begin{bmatrix}-2+0 & 3+0 & 5+0\\ 1+0 & -1+0 & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

(b) $O+A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

$=\begin{bmatrix}0+(-2) & 0+3 & 0+5\\ 0+1 & 0+(-1) & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

From (a) and (b), we see that

$A+O=O+A=A$

MODULE - VI
Algebra -II

Notes

20.6 SUBTRACTION OF MATRICES

Let A and B two matrices of the same order. Then the matrix $A-B$ is defined as the subtraction of B from A. $A-B$ is obtained by subtracting corresponding elements of B from the corresponding elements of A.

We can write $A-B = A+(-B)$

Note : $A-B$ and $B-A$ do not denote the same matrix, except when $A = B$.

Example 20.13 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find

(a) $A-B$ (b) $B-A$

Solution : (a) We know that

$$A-B = A+(-B)$$

Since $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, we have $-B = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$

Substituting it in (i), we get

$$A-B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+(-2) \\ 2+(-1) & (-1)+(-4) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

(b) Similarly,

$$B-A = B+(-A)$$

$$B-A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+(-1) & 2+0 \\ 1+(-2) & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Remarks : To obtain $A-B$, we can subtract directly the elements of B from the corresponding elements of A. Thus,

$$A-B = \begin{bmatrix} 1-3 & 0-2 \\ 2-1 & -1-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

and $$B-A = \begin{bmatrix} 3-1 & 2-0 \\ 1-2 & 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Example 20.14 If $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$; $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A+B = O$, find B.

Solution : Here, it is given that A+B = O

$$\therefore \quad \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+a & 3+b \\ -1+c & 4+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad 2+a = 0 \quad ; \quad 3+b=0$

$-1+c=0 \quad ; \quad 4+d=0$

$\Rightarrow \quad a = -2; \; b = -3; \quad c = 1$ and $d = -4$

$$\therefore \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & -2 \end{pmatrix}$$

In general, given a matrix A, there exists another matrix $B = (-1)\,A$ such that $A + B = O$, then such a matrix B is called the additive inverse of the matrix of A.

CHECK YOUR PROGRESS 20.4

1. If $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$ then find :

   (a) $A+B$ (b) $2A+B$ (c) $A+3B$ (d) $2A+3B$

2. If P= $\begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{bmatrix}$ and Q = $\begin{pmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{pmatrix}$, then find :

   (a) P−Q (b) Q−P (c) P− 2Q (d) 2Q−3P

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 & 0 \\ 1 & 6 & 1 \\ 2 & 0 & 7 \end{bmatrix}$, then find:

   (a) $A+B$ (b) $A-B$ (c) $-A+B$ (d) $3A+2B$

4. If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, find the zero matrix O satisfying A+O = A.

5. If A = $\begin{bmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$ then find :

   (a) −A (b) A + (−A) (c) (−A)+A

6. If A = $\begin{bmatrix} 1 & 9 \\ 3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 1 \\ 7 & 9 \end{bmatrix}$, then find :

   (a) 2A (b) 3B (c) 2A+3B (d) If 2A+3B+5X=O, what is X ?

7. If A = $\begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \end{bmatrix}$ and B= $\begin{bmatrix} -1 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, then find :

   (a) A′ (b) B′ (c) A+B (d) (A+B)′ (e) A′+B′

   What do you observe ?

8. If A = $\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, then find :

   (a) A−B (b) B−C (c) A−C (d) 3B−2C (e) A−B−C (f) 2A−B−3C

20.7 MULTIPLICATION OF MATRICES

Salina and Rakhi are two friends. Salina wants to buy 17 kg wheat, 3 kg pulses and 250gm ghee; while Rakhi wants to buy 15 kg wheat, 2kg pulses and 500 gm ghee. The prices of wheat, pulses and ghee per kg respectively are Rs. 8.00, Rs. 27.00 and Rs. 90.00.How much money will each spend? Clearly, the money needed by Salina and Rakhi will be :

Salina

Cost of 17 kg wheat	$\Rightarrow 17 \times$ Rs. 8	= Rs. 136.00
Cost of 3 kg pulses	$\Rightarrow 3 \times$ Rs. 27	= Rs. 81.00
Cost of 250 gm ghee	$\Rightarrow \frac{1}{4} \times$ Rs. 90	= Rs. 22.50
	Total	= Rs. 239.50

Rakhi

Cost of 15 kg wheat $\Rightarrow 15 \times$ Rs. 8 $\quad$ = Rs. 120.00

Cost of 2 kg pulses $\Rightarrow 2 \times$ Rs. 27 $\quad$ = Rs. 54.00

Cost of 500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 90 $\quad$ = Rs. 45.00

Total = Rs. 219.00

In matrix form, the above information can be represented as follows:

Requirements $\qquad$ Price $\qquad$ Money Needed

$$\begin{bmatrix} \text{Wheat} & \text{pulses} & \text{ghee} \\ 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8 \\ 27 \\ 90 \end{bmatrix} = \begin{bmatrix} 17 \times 8 + 3 \times 27 + 0.250 \times 90 \\ 15 \times 8 + 2 \times 27 + 0.500 \times 90 \end{bmatrix} = \begin{bmatrix} 239.50 \\ 219.00 \end{bmatrix}$$

Another shop in the same locality quotes the following prices.

Wheat : Rs. 9 per kg.; pulses : Rs.26 per kg; ghee : Rs. 100 per kg.

The money needed by Salina and Rakhi to buy the required quantity of articles from this shop will be

Salina

17 kg wheat $\Rightarrow 17 \times$ Rs. 9 = Rs. 153.00

3 kg pulses $\Rightarrow 3 \times$ Rs. 26 = Rs. 78.00

250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 100 = Rs. 25.00

Total = Rs. 256.00

Rakhi

15 kg wheat $\Rightarrow 15 \times$ Rs. 9 = Rs. 135.00

2 kg pulses $\Rightarrow 2 \times$ Rs. 26 = Rs. 52.00

500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 100= Rs. 50.00

Total = Rs. 237.00

In matrix form, the above information can be written as follows :

Requirements $\qquad$ Price $\qquad$ Money needed

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 9.00 \\ 26.00 \\ 100.00 \end{bmatrix} = \begin{bmatrix} 17 \times 9.00 + 3 \times 26.00 + 0.250 \times 100 \\ 15 \times 900 + 2 \times 26.00 + 0.500 \times 100 \end{bmatrix} = \begin{bmatrix} 256.00 \\ 237.00 \end{bmatrix}$$

To have a comparative study, the two information can be combined in the following way:

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 27 & 26 \\ 90 & 100 \end{bmatrix} = \begin{bmatrix} 239.50 & 256.00 \\ 219.00 & 237.00 \end{bmatrix}$$

Let us see how and when we write this product :

i) The three elements of first row of the first matrix are multiplied respectively by the corresponding elements of the first column of the second matrix and added to give element of the first row and the first column of the product matrix. In the same way, the product of the elements of the second row of the first matrix to the corresponding elements of the first column of the second matrix on being added gives the element of the second row and the first column of the product matrix; and so on.

ii) The number of column of the first matrix is equal to the number of rows of the second matrix so that the first matrix is compatible for multiplication with the second matrix.

Thus, If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$, then

$$A \times B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \times \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{bmatrix}$$

Definition : If A and B are two matrices of order $m \times p$ and $p \times n$ respectively, then their product will be a matrix C of order $m \times n$; and if a_{ij}, b_{ij} and c_{ij} are the elements of the ith row and jth column of the matrices A, B and C respectively, then

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 20.15 If $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, then find:

(a) AB (b) BA

Is $AB = BA$?

Solution : Order of A is 1×3

Order of B is 3×1

$\therefore$ Number of columns of A = Number of rows of B

$\therefore$ AB exists

$$\text{Now, } AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \left[1\times(-2)+(-1)\times 0+2\times 2\right] = \left[-2+0+4\right] = \left[2\right]$$

Thus, $AB = \left[2\right]$, a matix of order 1×1

Again, number of columns of B = number of rows of A.

$\therefore$ BA exists

Now,

$$BA = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2\times 1 & (-2)\times(-1) & (-2)\times 2 \\ 0\times 1 & 0\times(-1) & 0\times 2 \\ 2\times 1 & 2\times(-1) & 2\times 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

Thus, $BA = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, a matrix of order 3×3

From the above, we find that $AB \neq BA$

Example 20.16 Find AB and BA, if possible for the matrices A and B:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : Here, Number of columns of A $\neq$ Number of rows of B

$\therefore$ AB does not exist.

Further, Number of columns of B $\neq$ Number of rows of A

$\therefore$ BA does not exist.

Example 20.17 If A= $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, then find AB and BA. Also find if AB=BA.

Solution : Here, Number of columns of A = Number of rows of B

$\therefore$ AB exists.

Further,Number of columns of B = Number of rows of A

$\therefore$ BA also exists.

Now, $AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 2 + 2\times 2 & 1\times 1 + 2\times 2 \\ -1\times 2 + 0\times 2 & -1\times 1 + 0\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 1+4 \\ -2+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}_{2\times 2}$$

and $BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2\times 1 + 1\times(-1) & 2\times 2 + 1\times 0 \\ 2\times 1 + 2\times(-1) & 2\times 2 + 2\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+0 \\ 2-2 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}_{2\times 2}$$

Thus, $AB \neq BA$

Remarks : We observe that AB and BA are of the same order 2×2, but still $AB \neq BA$.

Example 20.18 If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, find AB and BA. Is $AB = BA$?

Solution : Here, both A and B are of order 2×2. So, both AB and BA exist. Now

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2} \text{ and}$$

$$BA = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2}$$

Here, both AB and BA are of the same order and $AB = BA$.
Hence, if two matrcies A and B are multiplied, then the following five cases arise:

(i) Both AB and BA exist, but are of different orders

(ii) Only one of the products AB or BA exists.

(iii) Neither AB nor BA exist.

(iv) Both AB and BA exist and are of the same order, but $AB \neq BA$.

(v) Both AB and BA exist and are of the same order. Also, $AB = BA$.

Example 20.19 If $A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, verify that $A^2-2A-3I=O$

Solution: Here,

$$A^2=AA=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}9+0 & 0+0\\ 0+0 & 0+9\end{bmatrix}=\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}$$

$$2A=2\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$$

and $\quad 3I = 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$$A^2-2A-3I = -\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}-\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}-\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}-\left(\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\right)$$

$$=\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}-\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}9-9 & 0-0\\ 0-0 & 9-9\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

Hence, verified.

Example 20.20 Solve the matrix equation :

$$\begin{bmatrix}2 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}$$

Solution : Here,

$$\text{L.H.S.}=\begin{bmatrix}2 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2x-3y\\ x+y\end{bmatrix}\Rightarrow\begin{bmatrix}2x-3y\\ x+y\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}$$

$$\Rightarrow 2x-3y=1;\ x+y=3$$

Solving these equations, we get

x= 2 and y = 1

Example 20.21 If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then find AB

Soution : Here,

$$A-B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1\times(-1)+1\times 1 & 1\times 1+(-1)\\ 1\times(-1)+1\times 1 & 1\times 1+(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-1+1 & 1-1\\ -1+1 & 1-1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

Hence, we conclude that the product of two non-zero matrices can be a zero matrix, whereas in numbers, the product of two non-zero numbers is always non-zero.

Example 20.22 For $A=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}$, $B=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, find

(a) $(AB)C$ (b) $A(BC)$

Is $(AB)C=A(BC)$?

Solution : (a) $(AB)C=\left\{\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\right\}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$

$$=\begin{bmatrix}4+2 & 0-4\\ 12-5 & 0+10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\ 7 & 10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0-12\\ -7+0 & 0+30\end{bmatrix}=\begin{bmatrix}-6 & -12\\ -7 & 30\end{bmatrix}$$

(b) $A(BC)=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\left\{\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\right\}$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4+0 & 0+0 \\ 1+0 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-2 & 0-12 \\ -12+5 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

From (a) and (b), we find that $(AB)C = A(BC)$, i.e., matrix multiplication is associative.

CHECK YOUR PROGRESS 20.5

1. If $A = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, find AB and BA. Is $AB=BA$?

2. If A= $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$, find AB and BA. Is AB = BA?

3. If A = $\begin{bmatrix} a \\ b \end{bmatrix}$ and B= $\begin{bmatrix} x & y & z \end{bmatrix}$, find AB and BA, whichever exists.

4. If A = $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$, find BA. Does AB exists?

5. If A= $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

   (a) Does AB exist? Why? (b) Does BA exist? Why?

6. If A = $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$, find AB and BA. Is AB=BA?

7. If A = $\begin{pmatrix} 1 & 2 & 0 \\ -3 & 5 & 4 \\ 5 & 3 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find AB and BA . Is AB=BA?

8. If A= $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, find AB and BA. Is AB=BA?

9. Find the values of *x* and *y* if

(a) $\begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

10. For $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$, verify that AB=O

11. For A= $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, verify that $A^2 - 5A + I = O$, where I is a unit matrix of order 2.

12. If A = $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, B= $B = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, and C = $= \begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$, find :

(a) A(BC) (b) (AB)C (c) (A+B)C

(d) AC+BC (e) $A^2 - B^2$ (f) (A−B) (A+B)

13. If A = $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, B= $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$ and C = $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, find : (a) AC (b) BC

Is AC = BC ? What do you conclude?

14. If A = $\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$, B = $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ and C= $\begin{pmatrix} 3 & 8 \\ 7 & -1 \end{pmatrix}$, find :

(a) B+C (b) A (B+C) (c) AB (d) AC (e) AB+AC

What do you observe?

15. For matices A = $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$, verify that $(AB)' = B'A'$

16. If $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, find X such that $AX = B$.

17. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, show that , $A^2-(a+d)A = (bc-ad)I$

18 If $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, is it true that

(a) $(A+B)^2 = A^2+B^2+2AB$? (b) $(A-B)^2 = A^2+B^2-2AB$?

(c) $(A+B)(A-B) = A^2-B^2$?

20.8 INVERTIBLE MATRICES

Definition : A square matrix of order n is invertible if there exists a square matrix B of the same order such that

$AB = I_n = BA$, Where I_n is identify matrix of order n.

In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$.

Theorem 1 : Every invertible matrix possesses a unique inverse.

Proof : Let A be an invertible matrix of order

Let B and C be two inverses of A.

Then,

$$AB = BA = I_n \qquad ...(1)$$

and

$$AC = CA = I_n \qquad ...(2)$$

Now, $AB = I_n$

$\Rightarrow$ $C(AB) = C\,I_n$ [Pre-multiplying by C]

$\Rightarrow$ $(CA)B = C\,I_n$ [by associativity]

$\Rightarrow$ $I_n B = C\,I_n$ $(\because CA = I_n$ from $(ii)]$

$\Rightarrow$ $B = C$ $[\because I_n B = B,\ C\,I_n = C]$

Hence, an invertible matrix possesses a unique inverse.

CORROLLARY **If A is an invertible matrix then $(A^{-1})^{-1} = A$**

Proof : We have, $A\,A^{-1} = I = A^{-1}A$

$\Rightarrow$ A is the inverse of A^{-1} i.e., $A = (A^{-1})^{-1}$

Theorem 2 : A square matrix is invertible iff it is non-singular.

Proof : Let A be an invertible matrix. Then, there exists a matrix B such that

$$AB = I_n = BA$$

$\Rightarrow \quad |AB| = |I_n|$

$\Rightarrow \quad |A|\,|B| = 1$

$[\because\ |AB| = |A|\,|B|]$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow$ A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n, then,

$$\Rightarrow \quad A\left(\frac{1}{|A|}adj\ A\right) = I_n = \left(\frac{1}{|A|}adj\ A\right)A \quad \left[\because |A| \neq 0 \therefore \frac{1}{|A|}\text{exists}\right]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}adj\ A \qquad \text{[By def. of inverse]}$$

Hence, A is an invertible matrix.

Remark : This theorem provides us a formula for finding the inverse of a non-singular square matrix.

The inverse of A is given by

$$A^{-1} = \frac{1}{|A|}adj\ A$$

20.9 ELEMENTARY TRANSFORMATIONS OR ELEMENTARY OPERATIONS OF A MATRIX

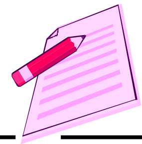
The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)

If i^{th} row (column) of a matrix is interchanged with the jth row (column), it is dennoted by $R_i \leftrightarrow R_j$ or $(C_i \leftrightarrow C_j)$.

for example, $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$

we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$



Notes

(ii) Multiplying all elements of any row (column) of a matrix by a non-zero scalar

If the elements of ith row (column) are multiplied by a non-zero scalar k, it is denoted by $R_i \to k\ R_i\ [C_i \to k\ C_i]$

For example

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_1 \to 2R_1$ we get $B = \begin{bmatrix} 6 & 4 & -2 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$

(iii) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar k

If k times the elements of jth row (column) are added to the corresponding elements of the ith row (column), it is denoted by $R_i \to R_i + kR_j$ $(C_i \to C_i + k\ C_j)$.

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then the application of elementary operation $R_3 \to R_3 + 2R_1$, gives the matrix

$$B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$$

20.9.1 INVERSE OF A MATRIX BY ELEMENTARY OPERATIONS

We can find the inverse of a matrix, if it exists, by using either elementary row operations or column operations but not both simultaneously.

Let A be an invertible square matrix of order n, if we want to find A^{-1} by using elementary raw operations then we write

$$A = I_n\ A \qquad ...(i)$$

As an elementary row operation on the product of two matrices can be affected by subjecting the pre factor to the same elementary row operation, we shall use elementary row operations on (i) so that its L.H.S reduces to In and R.H.S (after applying corresponding elementary row operations on the prefactor I_n), we get

$$I_n = BA \qquad ...(ii)$$

Which means matrix B and matrix A are inverse of each other i.e. $A^{-1} = B$

Similarly if we want to find A^{-1} by using elementary column operations, we write

$$A = A\, I_n \qquad \text{...(iii)}$$

Now use elementary column operations on (iii) so that its L.H.S reduces to I_n and R.H.S (after applying corresponding elementary column operations on the post factor I_n) takes the shape

$$I_n = AB$$

$$\text{Then} \quad A^{-1} = B$$

The method is explained below with the help of some examples.

Example 20.23 Find the inverse of matrix A, using elementary column operations where,

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Solution : Writing

$$A = A\, I_2 \quad \Rightarrow \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & 1 \end{bmatrix} \quad \text{Operating } C_2 \to C_2 + 3C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to \frac{1}{2} C_1$$

$$\Rightarrow I_2 = AB, \text{ where } B = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to C_1 - \frac{1}{2} C_2$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Example 20.24 Find the inverse of the matrix A using elementary row operations, where

$$A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

Solution : Writing

$$A = I_2A$$

$$\Rightarrow \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to \frac{1}{10} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 + 5\,R_1,$$

As the matrix in L.H.S contain, a row in which all elements are 0. So inverse of this matrix does not exist. Because in such case the matrix in L.H.S can not be conversed into a unit matrix.

Example 20.25 Find the inverse of the matrix A, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Solution : We have

$$A = I\,A$$

$$\text{or} \quad \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 - R_2,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to \frac{1}{2} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + R_2,\ R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_3 \to \frac{1}{4} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + \frac{1}{2} R_3,\ R_2 \to R_2 - \frac{1}{2} R_3$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$$

CHECK YOUR PROGESS 20.6

1. Find inverse of the following matrices using elementary operations :

(a) $\begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 10 \\ 3 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

LET US SUM UP

- A rectangular array of numbers, arranged in the form of rows and columns is called a *matrix*. Each number is called an *element* of the matrix.
- The *order* of a matrix having 'm' rows and 'n' columns is $m \times n$.
- If the number of rows is equal to the number of columns in a matrix, it is called a square matrix.
- A diagonal matrix is a square matrix in which all the elements, except those on the diagonal, are zeroes.
- A unit matrix of any order is a diagonal matrix of that order whose all the diagonal elements are 1.
- Zero matrix is a matrix whose all the elements are zeroes.
- Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.
- A transpose of a matrix is obtained by interchanging its rows and columns.
- Matrix A is said to be symmetric if $A' = A$ and skew symmetric if $A' = -A$.
- Scalar multiple of a matrix is obtained by multiplying each elements of the matrix by the scalar.
- The sum of two matrices (of the same order) is a matrix obtained by adding corresponding elements of the given matrices.
- Difference of two matrices A and B is nothing but the sum of matrix A and the negative of matrix B.
- Product of two matrices A of order $m \times n$ and B of order $n \times p$ is a matrix of order $m \times p$, whose elements can be obtained by multiplying the rows of A with the columns of B element wise and then taking their sum.
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=xZBbfLLfVV4

http://www.youtube.com/watch?v=ArcrdMkEmKo

http://www.youtube.com/watch?v=S4n-tQZnU6o

http://www.youtube.com/watch?v=obts_JDS6_Q

http://www.youtube.com/watch?v=01c12NaUQDw

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=sys

TERMINAL EXERCISE

1. How many elements are there in a matrix of order

   (a) 2×1 (b) 3×2 (c) 3×3 (d) 3×4

2. Construct a matrix of order 3×2 whose elements a_{ij} are given by

   (a) $a_{ij} = i - 2j$ (b) $a_{ij} = 3i - j$ (c) $a_{ij} = i + \frac{3}{2}j$

3. What is the order of the matrix?

   (a) $A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 6 & 1 \end{bmatrix}$

4. Find the value of x, y and z if

   (a) $\begin{bmatrix} x & y \\ z & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} x-2 & 3 \\ 0 & y+5 \end{bmatrix} = \begin{bmatrix} 1 & z \\ y+z & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}$, find :

   (a) A+B (b) 2A (c) 2A−B

6. Find X, if

   (a) $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} + X = \begin{bmatrix} 10 & -2 \\ 1 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. Find the values of a and b so that

$$\begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} a-b & 2 & -2 \\ 4 & a & b \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 2a+b & 5 \end{bmatrix}$$

8. For matrices A, B and C

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 5 & 7 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 6 \\ 7 & 1 \\ 4 & 1 \end{bmatrix}$$

   verify that A+(B+C) = (A+B)+C

9. If $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$, find AB and BA. Is AB= BA?

10. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$, find AB and BA. Is AB = BA?

11. If $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$, find A^2.

12. Find A (B+C), if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & 0 & 3 \\ 4 & 0 & -3 \end{bmatrix}$$

13. If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} x & 1 \\ y & -1 \end{pmatrix}$ and $(A+B)^2 = A^2 + B^2$, find the values of x and y.

14. Show that $A = \begin{pmatrix} -8 & 5 \\ 2 & 4 \end{pmatrix}$ satisfies the matrix equation $A^2 + 4A - 2I = O$.

Find inverse of the following matrices using elementary transformations.

15. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

20. $\begin{bmatrix} \cos x & \sin x \\ \sin x & \cos x \end{bmatrix}$

21. $\begin{bmatrix} 1 & \tan\dfrac{x}{2} \\ -\tan\dfrac{x}{2} & 1 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

ANSWERS

CHECK YOUR PROGRESS 20.1

1. $\begin{bmatrix} 56 & 65 & 71 \\ 29 & 37 & 57 \end{bmatrix}$; $\begin{bmatrix} 56 & 29 \\ 65 & 37 \\ 71 & 57 \end{bmatrix}$ 2. $\begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$; $\begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 6 & 3 \\ 4 & 3 & 5 \end{bmatrix}$

4. (a) 6   (b) 12   (c) 8   (d) 12   (e) ab   (f) *mn*

5. (a) 1×8; 2×4; 4×2; 8×1 (b) 1×5; 5×1
   (c) 1×12; 2×6; 3×4 ; 4×3; 6×2; 12×1
   (d) 1×16; 2×8; 4×4; 8×2; 16×1

6. (a) 4   (b) 5   (c) 4×5 (d) 20
   (e) $a_{14} = 0$; $a_{23} = 7$; $a_{34} = -3$; $a_{45} = 1$ and $a_{33} = 3$

7. (a) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 4 & 2 & \frac{4}{3} \\ 9 & \frac{9}{2} & 3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \\ \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{bmatrix}$

8. (a) $\begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 10 \\ 10 & 20 \\ 15 & 30 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

CHECK YOUR PROGRESS 20.2

1. (a) G   (b) B   (c) A, D, E and F   (d) A, D and F
   (e) D and F   (f) F   (g) C

2. (a) $a = 2$, $b = 10$, $c = 6$, $d = -2$
   (b) $a = 2$, $b = 3$, $c = 2$, $d = 5$
   (c) $a = \frac{3}{2}$, $b = -2$, $c = 2$, $d = -4$

MODULE - VI
Algebra -II

Notes

3. No 4. No

CHECK YOUR PROGRESS 20.3

1. (a) $\begin{bmatrix} 28 & 8 \\ 8 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & -2 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{7}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-21}{2} & -3 \\ -3 & \frac{-9}{2} \end{bmatrix}$

2. (a) $\begin{bmatrix} 0 & -5 & 10 \\ 15 & 5 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -6 \\ -9 & -3 & -12 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$

(d) $\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{-3}{2} & \frac{-1}{2} & -2 \end{bmatrix}$ 3. $\begin{bmatrix} 7 & 0 \\ -28 & -14 \\ 0 & 7 \end{bmatrix}$

4. (a) $\begin{bmatrix} 15 & 0 & 5 \\ 20 & -10 & 0 \\ -5 & 0 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 0 & -4 \\ -16 & 8 & 0 \\ 4 & 0 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{5}{3} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-3}{2} & 0 & \frac{-1}{2} \\ -2 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-5}{2} \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 10 & 8 \\ 9 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & -6 \\ -2 & -1 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CHECK YOUR PROGRESS 20.4

1. (a) $\begin{bmatrix} 3 & -2 \\ 8 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -3 \\ 13 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 14 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & -5 \\ 19 & 10 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 3 & 6 \\ -5 & 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 & -6 \\ 5 & -3 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 9 \\ -9 & 2 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & -11 & -15 \\ 11 & -10 & -10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & -6 & 3 \\ 5 & 5 & 3 \\ 6 & 5 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 3 \\ 3 & -7 & 1 \\ 2 & 5 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -2 & -3 \\ -3 & 7 & -1 \\ -2 & -5 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -14 & 9 \\ 14 & 9 & 8 \\ 16 & 15 & 14 \end{bmatrix}$

4. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \\ 4 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 18 \\ 6 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 15 & 3 \\ 21 & 27 \end{bmatrix}$ (c) $\begin{bmatrix} 17 & 21 \\ 27 & 31 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-17}{5} & \frac{-21}{5} \\ \frac{-27}{5} & \frac{-31}{5} \end{bmatrix}$

7. (a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ 0 & -4 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$

We observe that $(A+B)' = B' + A'$

8. (a) $\begin{bmatrix} 0 & 2 \\ -5 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 8 \\ 3 & -7 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & 3 \\ -8 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -5 & 9 \\ -16 & -3 \end{bmatrix}$

CHECK YOUR PROGRESS 20.5

1. $AB = \begin{bmatrix}-6\end{bmatrix}$; $BA = \begin{bmatrix}0 & 0 & 0\\ -4 & -6 & 0\\ 2 & 3 & 0\end{bmatrix}$ $AB \neq BA$

2. $AB = \begin{bmatrix}2 & 2\\ 1 & -6\end{bmatrix}$; $BA = \begin{bmatrix}-3 & 13 & -4\\ 1 & -1 & -2\\ 2 & -6 & 0\end{bmatrix}$ $AB \neq BA$

3. $AB = \begin{bmatrix}ax & ay & az\\ bx & by & bz\end{bmatrix}$; BA does not exist.

4. $BA = \begin{bmatrix}-2\\ -1\end{bmatrix}$; AB does not exist.

5. Both AB and BA do not exist. AB does not exist since the number of columns of A is not equal to the number of rows of B. BA also does not exist since number of coluumns of B is not equal to the number of rows of A.

6. $AB = \begin{bmatrix}0 & 5\\ 6 & 15\end{bmatrix}$; $BA = \begin{bmatrix}-2 & -1\\ 4 & 17\end{bmatrix}$; $AB \neq BA$

7. $AB = \begin{bmatrix}4 & -3 & 7\\ 3 & 17 & 24\\ 14 & -13 & 17\end{bmatrix}$; $BA = \begin{bmatrix}16 & -8 & -11\\ 16 & 11 & 3\\ 10 & 21 & 11\end{bmatrix}$; $AB \neq BA$.

8. $AB = \begin{bmatrix}10 & 0\\ 0 & -1\end{bmatrix}$; $BA = \begin{bmatrix}10 & 0\\ 0 & -1\end{bmatrix}$; $AB \neq BA$.

9. (a) $x = 3,\ y = -1$ (b) $x = -1,\ y = 2$

12. (a) $\begin{bmatrix}-14 & 18\\ 2 & 6\end{bmatrix}$ (b) $\begin{bmatrix}-14 & 18\\ 2 & 6\end{bmatrix}$ (c) $\begin{bmatrix}2 & 6\\ 0 & 3\end{bmatrix}$

(d) $\begin{bmatrix}2 & 6\\ 0 & 3\end{bmatrix}$ (e) $\begin{bmatrix}5 & 0\\ 7 & 8\end{bmatrix}$ (f) $\begin{bmatrix}-2 & -3\\ 9 & 15\end{bmatrix}$

13. (a) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$; AC= BC

Here, A≠B and C≠O, yet AC= BC
i.e. cancellation law does not hold good for matrices.

14. (a) $\begin{pmatrix} 4 & 7 \\ 9 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -8 \\ -11 & 10 \end{pmatrix}$ (e) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

We observe that A(B+C) = AB+AC

16. x= $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ 18. (a) No (b) No (c) No

CHECK YOUR PROGRESS 20.6

1. (*a*) $\frac{1}{25}\begin{bmatrix} 3 & 1 \\ 4 & -7 \end{bmatrix}$ (*b*) $\frac{1}{23}\begin{bmatrix} 5 & -6 \\ 3 & 1 \end{bmatrix}$ (*c*) does not exist

(*d*) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$ (*e*) $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix}$

TERMIAL EXERCISE

1. (a) 2 (b) 6 (c) 9 (d) 12

2. (a) $\begin{pmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{5}{2} & 4 \\ \frac{7}{2} & 5 \\ \frac{9}{2} & 6 \end{pmatrix}$

3. (a) 3×1 (b) 1×3

(c) 3×2 (d) 2×3

4. (a) $x = 1, y = 2, z = 3$ (b) $x = 5, y = 1, z = 5$

(c) $x = 3, y = -3, z = 3$ (d) $x = 2, y = 1, z = 5$

5. (a) $\begin{pmatrix} 3 & 2 \\ 3 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -4 \\ 8 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -8 \\ 9 & 0 \end{pmatrix}$

6. (a) $\begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 4 & -3 \\ -3 & 0 & -1 \end{pmatrix}$

7. $a = \frac{3}{2}$ $\quad b = -\frac{3}{2}$

9. $AB = \begin{pmatrix} 13 & 11 \\ 38 & 43 \end{pmatrix}$; $\quad BA = \begin{pmatrix} 5 & 10 & 17 \\ 6 & 14 & 24 \\ 4 & 21 & 37 \end{pmatrix}$; $AB \neq BA$

10. $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $AB = BA$

11. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ 12. $\begin{pmatrix} 9 & 1 & 1 \\ -1 & -4 & 10 \end{pmatrix}$

13. $x = 1, y = -4.$

15. $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ $\quad$ 16. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ $\quad$ 18. $\frac{1}{22}\begin{bmatrix} -4 & +5 \\ +2 & +3 \end{bmatrix}$

19. $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad$ 20. $\begin{bmatrix} \cos x & -\sin x \\ -\sin x & \cos x \end{bmatrix}$

21. $\cos^2 \frac{x}{2}\begin{bmatrix} 1 & -\tan {x}/{2} \\ \tan {x}/{2} & 1 \end{bmatrix}$ $\quad$ 22. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

23. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $\quad$ 24. $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$

21

DETERMINANTS

311en21

Every square matrix is associated with a unique number called the determinant of the matrix.

In this lesson, we will learn various properties of determinants and also evaluate determinants by different methods.

OBJECTIVES

After studying this lesson, you will be able to :

- define determinant of a square matrix;
- define the minor and the cofactor of an element of a matrix;
- find the minor and the cofactor of an element of a matrix;
- find the value of a given determinant of order not exceeding 3;
- state the properties of determinants;
- evaluate a given determinant of order not exceeding 3 by using expansion method;

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of solution of equations
- Knowledge of number system including complex number
- Four fundamental operations on numbers and expressions

21.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

On solving this system of equations for x and y, we get

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \text{ and } y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \text{ provided } a_1 b_2 - a_2 b_1 \neq 0$$

The number $a_1 b_2 - a_2 b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

i.e. a_{11} belongs to the 1st row and 1st column
a_{12} belongs to the 1st row and 2nd column
a_{21} belongs to the 2nd row and 1st column
a_{22} belongs to the 2nd row and 2nd column

21.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

write the elements in the following manner :

$$\begin{matrix} a_1 & & a_{12} \\ & \times & \\ a_{21} & & a_{22} \end{matrix}$$

Multiply the elements by the arrow. The sign of the arrow going **downwards** is positive, i.e., a_{11} a_{22} and the sign of the arrow going **upwards** is negative, i.e., $-a_{21}$ a_{12}

Add these two products, i.e., a_{11} a_{22} $+(-a_{21}.a_{12})$ or a_{11} a_{22} $-a_{21}$ a_{12} which is the required value of the determinant.

Example 21.1 Evaluate :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)\,(a+b) - (2a)\,(2b)$

$= a^2+2ab + b^2 - 4ab = a^2+b^2 - 2ab = (a-b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2+x+1)\,(x-1) - (x^2-x+1)\,(x+1)$

$= (x^3-1) - (x^3+1) = -2$

Example 21.2 Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution :

(i) Now, $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x = -x - 9$

According to the question,

$-x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now, $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the equation

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x-1) + 1(x-1) = 0$

or, $(2x+1)(x-1) = 0$

or, $x = 1, -\frac{1}{2}$

21.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has $(3)^2 = 9$ elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements

$a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 ,we write a determinant of order3 as follows: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or |A|, |B| etc.

$$\Delta = |a_{ij}| \text{ , where } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

21.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix".

With the 1 $\times$ 1 matrix [a], we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3 $\times$ 3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we associate the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\, a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 21.3 If A = $\begin{pmatrix} 3 & 6 \\ 1 & 5 \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9$

Example 21.4 If A= $\begin{pmatrix} a+b & a \\ b & a-b \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix}$ $= (a+b)(a-b) - b \times a \quad = a^2 - b^2 - ab$

Note : 1. The determinant of a unit matrix I is 1.

2. A square matrix whose determinant is zero, is called the singular matrix.

21.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements a_{11}, a_{12} and a_{13} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternatively, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign. Therefore, a_{11} has been assigned positive sign.

Note : We can expand the determinant using any row or column. The value of the determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 21.5 Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1 \text{ x } (20-2) - 2 \times (10-3) + 3 \times (4-12)$

$= 18-14-24$

$= -20$

Example 21.6 Expand the determinant, by using the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\, 3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -2 \times (3-4) + 1 \times (1-6) - 3 \times (2-9)$

$= 2 - 5 + 21$

$= 18$

CHECK YOUR PROGRESS 21.1

1. Find |A|, if

(a) $A = \begin{pmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{pmatrix}$ (b) $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

(c) $A = \begin{pmatrix} \sin\alpha + \cos\beta & \cos\beta + \cos\alpha \\ \cos\beta - \cos\alpha & \sin\alpha - \sin\beta \end{pmatrix}$ (d) $\begin{pmatrix} a+bi & c+di \\ c-di & a-bi \end{pmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{pmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{pmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ (d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

21.6 MINORS AND COFACTORS

21.6.1 *Minor of a_{ij} in $|A|$*

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in $|A|$ is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of $|A|$ and is denoted by M_{ij}. For example, minor of 3 in the determinant

$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 21.7 Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1^{st} row and 1^{st} column. Thus to find the minor of a_{11}, we delete the 1^{st} row and 1^{st} column of $|A|$.

The minor M_{11} of a_{11} is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \; ; \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \; ; \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

21.6.2 *Cofactors of a_{ij} in $|A|$*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is usually denoted by C_{ij}, Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Example 21.8 Find the cofactors of the elements a_{11}, a_{12}, and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

The cofactor of any element a_{ij} is $(-1)^{i+j} M_{ij}$, then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$= (a_{22}\, a_{33} - a_{32}\, a_{23})$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{31}\, a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$$

and $$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32}\, a_{13} - a_{12}\, a_{33})$$

Example 21.9 Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution : The elements of the second row are $a_{21}=5$; $a_{22}=2$; $a_{23}=4$.

$$\text{Minor of } a_{21} \text{ (i.e., 5)} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$\text{Minor of } a_{22} \text{ (i.e., 2)} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$$

$$\text{and Minor of } a_{23} \text{ (i.e., 4)} = \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1} M_{21} = -(48) = -48$$

$$C_{22} = (-1)^{2+2} M_{22} = +(-13) = -13$$

and $$C_{23} = (-1)^{2+3} M_{23} = -(-42) = 42$$

CHECK YOUR PROGRESS 21.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinat

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Solve for x, the following equations:

(a) $\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

21.7 PROPERTIES OF DETERMINANTS

We shall now discuss some of the properties of determinants. These properties will help us in expanding the determinants.

Property 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first column, we have

$$\Delta \quad = 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix}$$

$$= 2\,(3-0) - 0\,(1-6) + 4\,(0+9)$$

$$= 6 + 36 = 42$$

Let Δ' be the determinant obtained by interchanging rows and columns of Δ. Then

$$\Delta' = \begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

Expanding the determinant Δ' by second column, we have (Recall that a determinant can be expanded by any of its rows or columns)

$$(-)\,0\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-)\,0\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + (-3)\,(-2-12) + 0$$

$$= 42$$

Thus, we see that $\Delta = \Delta'$

Property 2: If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$= 2\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} - 3\begin{vmatrix}1 & 3\\3 & 2\end{vmatrix} + 1\begin{vmatrix}1 & 2\\3 & 1\end{vmatrix}$$

$= 2\,(4-3) - 3\,(2-9) + 1\,(1-6)$

$= 2 + 21 - 5 = 18$

Let Δ' be the determinant obtained by interchanging C_1 and C_2

$$\text{Then } \Delta' = \begin{vmatrix}3 & 2 & 1\\2 & 1 & 3\\1 & 3 & 2\end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$3\begin{vmatrix}1 & 3\\3 & 2\end{vmatrix} - 2\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} + 1\begin{vmatrix}2 & 1\\1 & 3\end{vmatrix}$$

$= 3\,(2-9) - 2(4-3) + 1(6-1)$

$= -21-2+5 = -18$

Thus we see that $\Delta' = -\Delta$

Corollary

If any row (or a column) of a determinant is passed over 'n' rows (or columns), then the resulting determinant Δ' is $\Delta = (-1)^n\ \Delta$

For example , $$\begin{vmatrix}2 & 3 & 5\\1 & 5 & 6\\0 & 4 & 2\end{vmatrix} = (-1)^2\begin{vmatrix}1 & 5 & 6\\0 & 4 & 2\\2 & 3 & 5\end{vmatrix}$$

$= 2\,(10-24) - 3\,(2-0) + 5(4)$

$= -28 - 6 + 20 = -14$

Property 3: If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

Proof : Let $\Delta = \begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}$

be a determinant with identical columns C_1 and C_2 and let Δ' determinant obtained from Δ by interchanging C_1 and C_2

Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is the same as Δ, but by property 2, the value of the determinant changes in sign, if its any two adjacent rows (or columns) are interchnaged

Therefore $\Delta' = -\Delta$

Thus, we find that

or $2\Delta = 0 \Rightarrow \Delta = 0$

Hence the value of a determinant is zero, if it has two identical rows (or columns).

Property 4: If each element of a row (or column) of a determinant is multiplied by the same constant say, $k \neq 0$, then the value of the determinat is multiplied by that constant k.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2(3-0) - 1\,(0-0) + (-5)\,(0+12\,)$$
$$= 6 - 60 = -54$$

Let us multiply column 3 of Δ by 4. Then, the new determinant Δ' is :

$$\Delta' = \begin{vmatrix} 2 & 1 & -20 \\ 0 & -3 & 0 \\ 4 & 2 & -4 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$\Delta' = 2\,(12-0) - 1\,(0-0) + (-20)\,(0+12)$$
$$= 24 - 240 = -216$$
$$= 4\,\Delta$$

Corollary :
If any two rows (or columns) of a determinant are proportional, then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & ka_1 \\ a_2 & b_2 & ka_2 \\ a_3 & b_3 & ka_3 \end{vmatrix}$

Note that elements of column 3 are k times the corresponding elements of column 1

By Property 4, $\Delta = k \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$

$= k \times 0$ (by Property 2)

$= 0$

Property 5: If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.

Proof : Let $\Delta = \begin{vmatrix} a_1 + \alpha & b_1 + \beta & c_1 + \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, on expanding the determinant by the first row, we have

$$\Delta = (a_1 + \alpha)(b_2c_3 - b_3c_2) - (b_1 + \beta)(a_2c_3 - a_3c_2) + (c_1 + \gamma)(a_2b_3 - a_3b_2)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) + \alpha(b_2c_3 - b_3c_2)$$

$$-\beta(a_2c_3 - a_3c_2) + \gamma(a_2b_3 - a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, the determinant Δ can be expressed as the sum of the determinants of the same order.

Property 6: The value of a determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding elements of one or more other rows (or columns)

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' be the determinant obtained from Δ by corresponding elements of R_3

i.e. $R_1 \to R_1 + kR_3$

Then,
$$\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or, $\Delta' = \Delta + k \times 0$ (Row 1 and Row 3 are identical)

$\Delta' = \Delta$

21.8 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a determinant easily by applying the aforesaid properties. The purpose of simplification of a determinant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the determinant by that row (or column). We denote 1st, 2nd and 3rd row by R_1, R_2, and R_3 respectively and 1st,2nd and 3rd column by C_1, C_2 and C_3 respectively.

Example 21.10 Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

Solution : $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \to C_1 + (C_2 + C_3)$

$$\therefore \qquad \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \quad \text{(on expanding by } C_1\text{)}$$

(since w is a non-real cube root of unity, therefore, $1+w+w^2 = 0$)

Example 21.11 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution : $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} [R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 0 & 1 & -b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1, we have

$$\Delta = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix} = (a-c)(b-c)(b-a)$$

$$= (a-b)(b-c)(c-a)$$

Example 21.12

Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$

Solution : $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad R_1 \to R_1 - (R_2 + R_3)$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\,[b(a+b) - bc] - 2b[bc - c(c+a)]$$

$$= 2bc\,[a+b-c] - 2bc[b-c-a]$$

$$= 2bc\,[(a+b-c) - (b-c-a)]$$

$$= 2bc\,[a+b-c-b+c+a]$$

$$= 4abc$$

Example 21.13 Evaluate:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3 = 0,$$

Example 21.14 Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution : $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \quad C_3 \to C_2 + C_3$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab+bc+ca) \times 0$ (by Property 3)

$= 0$

Example 21.15 Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$= abc(-a)\begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)(-4bc)$$

$$= 4a^2b^2c^2$$

Example 21.16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2\,(a+3)$$

Solution : Δ $$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (a+3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} \qquad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a+3) \times (1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3)$$

CHECK YOUR PROGRESS 21.3

1. Show that $\begin{vmatrix} x+3 & x & x \\ x & x+3 & x \\ x & x & x+3 \end{vmatrix} = 27\,(x+1)$

2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc+ca+ab+abc$

4. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for x :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

21.11 Application of Determinants

Determinant is used to find area of a triangle.

21.11.1 Area of a Triangle

We know that area of a triangle ABC, (say) whose vertices are $(x_1 y_1)$, $(x_2 y_2)$ and $(x_3 y_3)$ is given by

$$\text{Area of } (\Delta ABC) = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] \quad \text{...}(i)$$

Also, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ [expanding along C_1]

$$= x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)$$

$$= x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \quad \text{...}(ii)$$

from (*i*) and (*ii*)

$$\text{Area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

Thus the area of a triangle having vertices as $(x_1\ y_1)$, $(x_2\ y_2)$ and $(x_3\ y_3)$ is given by

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

21.11.2 Condition of collinearity of three points :

Let A(x_1 y_1), B(x_2 y_2) and C(x_3 y_3) be three points then

A, B, C are collinear if area of ΔABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

21.11.2 Equation of a line passing through the given two points

Let the two points be P(x_1, y_1) and Q(x_2 y_2) and R(x y) be any point on the line joining P and Q since the points P, Q and R are collinear.

$$\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Thus the equation of the line joining points (x_1 y_1) and (x_1 y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Example 21.17 Find the area of the triangle with vertices P(5, 4), Q(–2, 4) and R(2, –6)

Solution : Let A be the area of the triangle PQR, then

$$A = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[5(4-(-6)) - 4(-2-2) + 1(12-8)\right]$$

$$= \frac{1}{2}[50+16+4] = \frac{1}{2}(70) = 35 \text{ sq units.}$$

Example 21.18 Show that points (a, b + c), (b, c + a) and (c, a + b) are collinear.

Solution : We have

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$c_2 \to c_2 + c_1$$

$$= \begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = (a + b + c) \times 0 = 0$$

Hence, the given points are collinear.

Example 21.19 Find equation of the line joining A(1, 3) and B(2, 1) using determinants.

Solution : Let P(x, y) be any point on the line joining A(1, 3) and B(2, 1). Then

$$\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x\,(3 - 1) - y\,(1 - 2) + 1\,(1 - 6) = 0$$

$$\Rightarrow \quad 2x + y - 5 = 0$$

This is the required equation of line AB.

CHECK YOUR PROGRESS 21.4

1. Find area of the ΔABC when A, B and C are (3, 8), (–4, 2) and (5, –1) respectively.
2. Show that points A(5, 5), B(–5, 1) and C(10, 7) are collinear.
3. Using determinants find the equation of the line joining (1, 2) and (3, 6).

LET US SUM UP

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.

- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$
- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a *singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.
- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=det

http://mathworld.wolfram.com/Determinant.html

http://en.wikipedia.org/wiki/Determinant

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Matrices_and_determinants.html

TERMINAL EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the first column.

3. Evaluate $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$

4. Solve for x, if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

5. Using properties of determinants, show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate: (a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

,w being an imaginary cube-root of unity

7. Find the area of the triangle with vertices at the points :

(i) (2, 7), (1, 1) and (10, 8) (ii) (–1,–8), (–2,–3) and (3, 2)

(ii) (0, 0) (6, 0) and (4, 3) (iv) (1, 4), (2, 3) and (–5, –3)

8. Using determinants find the value of k so that the following points become collinear

(i) $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$

(ii) $(k, -2), (5, 2)$ and $(6, 8)$

(iii) $(3, -2), (k, 2)$ and $(8, 8)$

(iv) $(1, -5)(-4, 5)(k, 7)$

9. Using determinants, find the equation of the line joining the points

(i) (1, 2) and (3, 6) (ii) (3, 1) and (9, 3)

10. If the points (a, 0), (0, b) and (1, 1) are collinear then using determinants show that $ab = a + b$

ANSWERS

CHECK YOUR PROGRESS 21.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$

2. (a) and (d)

3. (a) 18 (b) −54

(c) $adf + 2bce - ae^2 - fb^2 - de^2$ (d) $x-1$

CHECK YOUR PROGRESS 21.2

1. $M_{21} = 39;\ C_{21} = -39$

$M_{22} = 3;\ C_{22} = 3$

$M_{23} = -11;\ C_{23} = 11$

2. $M_{13} = -5;\ C_{13} = -5$

$M_{23} = -7;\ C_{23} = 7$

$M_{33} = 1;\ C_{33} = 1$

3. (a) 19 (b) 0 (c) −131

(d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) $x = 2$ (b) $x = 2, 3$ (c) $x = 2, -\dfrac{17}{7}$

CHECK YOUR PROGRESS 21.3

7. (a) a^3 (b) $2abc(a+b+c)^3$

8. $x = \dfrac{2}{3}, \dfrac{11}{3}, \dfrac{11}{3}$

CHECK YOUR PROGRESS 21.4

1. $\dfrac{75}{2}$ sq units (3) $y = 2x$

TERMINAL EXERCISE

1. $M_{11} = -2,\ M_{12} = -1,\ M_{13} = 1,\ M_{21} = -7,\ M_{22} = -5,\ M_{23} = -1,$
$M_{31} = -8,\ M_{32} = -7,\ M_{33} = -2$

$C_{11} = -2,\ C_{12} = 1,\ C_{13} = 1,\ C_{21} = 7,\ C_{22} = -5,\ C_{23} = 1,$
$C_{31} = -8,\ C_{32} = 7,\ C_{33} = -2$

2. 0

3. –31

4. $x = 0,\ x = 1$

6. ( ) -8 (b) 0

7. (*i*) $\frac{45}{2}$ sq units (*ii*) 5 sq units

(*iii*) 9 sq units (*iv*) $\frac{15}{2}$ sq units

8. (*i*) $k = -1, \frac{1}{2}$ (*ii*) $k = \frac{13}{3}$

(*iii*) k = 5 (*iv*) $k = -5$

9. (*i*) $y = 2x$ (ii) $x = 3y$

22

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INVERSE OF A MATRIX AND ITS APPLICATIONS

Let us Consider an Example:

Abhinav spends Rs. 120 in buying 2 pens and 5 note books whereas Shantanu spends Rs. 100 in buying 4 pens and 3 note books.We will try to find the cost of one pen and the cost of one note book using matrices.

Let the cost of 1 pen be Rs. x and the cost of 1 note book be Rs. y. Then the above information can be written in matrix form as:

$$\begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 120 \\ 100 \end{pmatrix}$$

This can be written as $AX = B$

where $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 120 \\ 100 \end{pmatrix}$

Our aim is to find $X = \begin{pmatrix} x \\ y \end{pmatrix}$

In order to find X, we need to find a matrix A^{-1} so that $X = A^{-1}B$

This matrix A^{-1} is called the inverse of the matrix A.

In this lesson, we will try to find the existence of such matrices.We will also learn to solve a system of linear equations using matrix method.

OBJECTIVES

After studying this lesson, you will be able to :

- define a minor and a cofactor of an element of a matrix;
- find minor and cofactor of an element of a matrix;

MODULE - VI
Algebra-II

Notes

- find the adjoint of a matrix;
- define and identify singular and non-singular matrices;
- find the inverse of a matrix, if it exists;
- represent system of linear equations in the matrix form $AX = B$; and
- solve a system of linear equations by matrix method.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a determinant.
- Determinant of a matrix.
- Matrix with its determinant of value 0.
- Transpose of a matrix.
- Minors and Cofactors of an element of a matrix.

22.1 DETERMINANT OF A SQUARE MATRIX

We have already learnt that with each square matrix, a determinant is associated. For any given matrix, say $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$

its determinant will be $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$. It is denoted by $|A|$.

Similarly, for the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{bmatrix}$, the corresponding determinant is

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{vmatrix}$$

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is non-zero, i.e. $|A| \neq 0$

Example 22.1 Determine whether matrix A is singular or non-singular where

(a) $A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

Solution: (a) Here, $|A| = \begin{vmatrix} -6 & -3 \\ 4 & 2 \end{vmatrix}$

$$= (-6)(2) - (4)(-3)$$
$$= -12 + 12 = 0$$

Therefore, the given matrix A is a singular matrix.

(b)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -7 + 4 - 3 = -6 \neq 0$$

Therefore, the given matrix is non-singular.

Example 22.2 Find the value of x for which the following matrix is singular:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$$

Solution: Here,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix}$$

$$= 1(-6-2) + 2(-3-x) + 3(2-2x)$$
$$= -8 - 6 - 2x + 6 - 6x$$
$$= -8 - 8x$$

Since the matrix A is singular, we have $|A| = 0$

$$|A| = -8 - 8x = 0$$

$$or \quad x = -1$$

Thus, the required value of x is –1.

Example 22.3 Given $A = \begin{pmatrix} 1 & 6 \\ 3 & 2 \end{pmatrix}$. Show that $|A| = |A'|$, where A' denotes the transpose of the matrix.

Solution: Here, $A = \begin{pmatrix} 1 & 6 \\ 3 & 2 \end{pmatrix}$

This gives $A' = \begin{pmatrix} 1 & 3 \\ 6 & 2 \end{pmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad \ldots(1)$

and $\quad |A'| = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad \ldots(2)$

From (1) and (2), we find that $|A| = |A'|$

22.2 MINORS AND COFACTORS OF THE ELEMENTS OF SQUARE MATRIX

Consider a matrix $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The determinant of the matrix obtained by deleting the ith row and jth column of A, is called the minor of a_{ij} and is denoted by M_{ij}.

Cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, M_{23} = Minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

and C_{23} = Cofactor of a_{23}

$$= (-1)^{2+3} M_{23} = (-1)^5 M_{23} = -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 22.4 Find the minors and the cofactors of the elements of matrix $A = \begin{pmatrix} 2 & 5 \\ 6 & 3 \end{pmatrix}$

Solution: For matrix A, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = 6 - 30 = -24$

M_{11} (minor of 2) = 3; $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = 3$

M_{12} (minor of 5) = 6; $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -6$

M_{21} (minor of 6) = 5; $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -5$

M_{22} (minor of 3) = 2; $C_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22} = 2$

Example 22.5 Find the minors and the cofactors of the elements of matrix

$$A = \begin{pmatrix} -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 1 & 3 \end{pmatrix}$$

Solution: Here, $M_{11} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17;\ C_{11} = (-1)^{1+1} M_{11} = 17$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14;\ C_{12} = (-1)^{1+2} M_{12} = -14$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2 - 20 = -18;\ C_{13} = (-1)^{1+3} M_{13} = -18$$

$$M_{21} = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9 - 6 = 3;\ C_{21} = (-1)^{2+1} M_{21} = -3$$

$$M_{22} = \begin{vmatrix} -1 & 6 \\ 4 & 3 \end{vmatrix} = (-3 - 24) = -27;\ C_{22} = (-1)^{2+2} M_{22} = -27$$

$$M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = (-1-12) = -13;\ C_{23} = (-1)^{2+3} M_{23} = 13$$

$$M_{31} = \begin{vmatrix} 3 & 6 \\ 5 & -2 \end{vmatrix} = (-6-30) = -36;\ C_{31} = (-1)^{3+1} M_{31} = -36$$

$$M_{32} = \begin{vmatrix} -1 & 6 \\ 2 & -2 \end{vmatrix} = (2-12) = -10;\ C_{32} = (-1)^{3+2} M_{32} = 10$$

and $M_{33} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-5-6) = -11;\ C_{33} = (-1)^{3+3} M_{33} = -11$

CHECK YOUR PROGRESS 22.1

1. Find the value of the determinant of following matrices:

(a) $A = \begin{pmatrix} 0 & 6 \\ 2 & 5 \end{pmatrix}$ (b) $B = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

2. Determine whether the following matrix are singular or non-singular.

(a) $A = \begin{pmatrix} 3 & 2 \\ -9 & -6 \end{pmatrix}$ (b) $Q = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \end{pmatrix}$

3. Find the minors of the following matrices:

(a) $A = \begin{pmatrix} 3 & -1 \\ 7 & 4 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & 6 \\ 2 & 5 \end{pmatrix}$

4. (a) Find the minors of the elements of the 2nd row of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 1 \end{pmatrix}$$

(b) Find the minors of the elements of the 3rd row of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & 0 & -3 \end{bmatrix}$$

5. Find the cofactors of the elements of each the following matrices:

(a) $A = \begin{bmatrix} 3 & -2 \\ 9 & 7 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 4 \\ -5 & 6 \end{bmatrix}$

6. (a) Find the cofactors of elements of the 2nd row of matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 4 & 1 & -2 \end{bmatrix}$$

(b) Find the cofactors of the elements of the 1st row of matrix

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -2 \\ -5 & -3 & 0 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 7 & 4 \end{bmatrix}$, verify that

(a) $|A| = |A'|$ and $|B| = |B'|$ (b) $|AB| = |A|\,|B| = |BA|$

22.3 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ be a matrix. Then $|A| = \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix}$

Let M_{ij} and C_{ij} be the minor and cofactor of a_{ij} respectively. Then

$$M_{11} = |7| = 7;\; C_{11} = (-1)^{1+1}\,|7| = 7$$

$$M_{12} = |5| = 5;\; C_{12} = (-1)^{1+2}\,|5| = -5$$

$$M_{21} = |1| = 1;\; C_{21} = (-1)^{2+1}\,|1| = -1$$

$$M_{22} = |2| = 2; \; C_{22} = (-1)^{2+2}\,|2| = 2$$

We replace each element of A by its cofactor and get

$$B = \begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix} \qquad \ldots(1)$$

The transpose of the matrix B of cofactors obtained in (1) above is

$$B' = \begin{bmatrix} 7 & -1 \\ -5 & 2 \end{bmatrix} \qquad \ldots(2)$$

The matrix B' obtained above is called the adjoint of matrix A. It is denoted by Adj A.

Thus, adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix.

Working Rule: To find the Adj A of a matrix A:

(a) replace each element of A by its cofactor and obtain the matrix of cofactors; and

(b) take the transpose of the marix of cofactors, obtained in (a).

Example 22.6 Find the adjoint of

$$A = \begin{bmatrix} -4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution: Here, $|A| = \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$ Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\,(-3) = -3 \qquad A_{21} = (-1)^{2+1}\,(5) = -5$$

$$A_{12} = (-1)^{1+2}\,(2) = -2 \qquad A_{22} = (-1)^{2+2}\,(-4) = -4$$

We replace each element of A by its cofactor to obtain its matrix of cofators as

$$\begin{bmatrix} -3 & -2 \\ -5 & -4 \end{bmatrix} \qquad \ldots(1)$$

Transpose of matrix in (1) is Adj A.

Thus, Adj $A = \begin{pmatrix} -3 & -5 \\ -2 & -4 \end{pmatrix}$

Example 22.7 Find the adjoint of $A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{pmatrix}$

Solution: Here,

$$A = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{vmatrix}$$

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$

Then $A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = (-4-2) = -6$; $A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = -(3-5) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} = (-6-20) = -26$; $A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (-1-4) = 3$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = (-1-10) = -11$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = -(2+5) = -7$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = (-1-8) = -9$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = -(1+6) = -7$

and $A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} = (4-3) = 1$

Replacing the elements of A by their cofactors, we get the matrix of cofactors as

$$\begin{pmatrix} -6 & 2 & -26 \\ 3 & -11 & -7 \\ -9 & -7 & 1 \end{pmatrix} \qquad \text{Thus, Adj } A = \begin{pmatrix} -6 & 3 & -9 \\ 2 & -11 & -7 \\ -26 & -7 & 1 \end{pmatrix}$$

If A is any square matrix of order n, then $A(\text{Adj } A) = (\text{Adj } A)\, A = |A|\, I_n$

where I_n is the unit matrix of order n.

Verification:

(1) Consider $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$ or $|A| = 2 \times 3 - (-1) \times (4) = 10$

Here, $A_{11}=3$; $A_{12}=1$; $A_{21}= -4$ and $A_{22}=2$

Therefore, $\text{Adj } A = \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$

Now, $A\,(\text{Adj}A) = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| I_2$

(2) Consider, $A = \begin{pmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Then, $|A| = 3(-6-1) - 5\,(4-1) + 7\,(2+3) = -1$

Here, $A_{11} = -7$; $A_{12}= -3$; $A_{13}=5$

$A_{21}= -3$; $A_{22}= -1$; $A_{23}=2$

$A_{31}=26$; $A_{32}=11$; $A_{33}= -19$

Therefore, $\text{Adj } A = \begin{pmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{pmatrix}$

Now $(A)\,(\text{Adj } A) = \begin{pmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = |A| I_3$$

Also, $(\text{Adj A})\,A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}\begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3$$

Note : If A is a singular matrix, i.e. $|A|=0$, then $A\,(\text{Adj A}) = O$

CHECK YOUR PROGRESS 22.2

1. Find adjoint of the following matrices:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

2. Find adjoint of the following matrices :

(a) $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$

Also verify in each case that $A(\text{Adj A}) = (\text{Adj A})A = |A|I_2$.

3. Verify that

$A(\text{Adj A}) = (\text{Adj A})\,A = |A|\,I_3$, where A is given by

(a) $\begin{bmatrix} 6 & 8 & -1 \\ 0 & 5 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 7 & 9 \\ 0 & -1 & 2 \\ 3 & -7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

22.4 INVERSE OF A MATRIX

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will find, if possible, a matrix

$$B = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \text{ such that } AB = BA = I$$

i.e., $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or $$\begin{pmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On comparing both sides, we get

$$ax + bu = 1 \quad ay + bv = 0$$

$$cx + du = 0 \qquad cy + dv = 1$$

Solving for x, y, u and v, we get

$$x = \frac{d}{ad - bc},\ y = \frac{-b}{ad - bc},\ u = \frac{-c}{ad - bc},\ v = \frac{a}{ad - bc}$$

provided $ad - bc \neq 0$, i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Thus, $$B = \begin{pmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\[2ex] \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{pmatrix}$$

or $$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It may be verified that $BA = I$.

It may be noted from above that, we have been able to find a matrix.

$$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|} Adj\ A \qquad ...(1)$$

This matrix B, is called the inverse of A and is denoted by A^{-1}.

For a given matrix A, if there exists a matrix B such that $AB = BA = I$, then B is called the multiplicative inverse of A. We write this as $B = A^{-1}$.

Note: Observe that if $ad - bc = 0$, i.e., $|A| = 0$, the R.H.S. of (1) does not exist and $B\ (=A^{-1})$ is not defined. This is the reason why we need the matrix A to be non-singular in order that A possesses multiplicative inverse. Hence only non-singular matrices possess multiplicative inverse. Also B is non-singular and $A=B^{-1}$.

Example 22.8 Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution : $$A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$$

Therefore, $|A| = -12 - 10 = -22 \neq 0$

$\therefore A$ is non-singular. It means A has an inverse. i.e. A^{-1} exists.

$$\text{Now, Adj } A = \begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \frac{1}{-22}\begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & \frac{5}{22} \\ \frac{1}{11} & -\frac{2}{11} \end{bmatrix}$$

Note : Verify that $AA^{-1} = A^{-1}A = I$

Example 22.9 Find the inverse of matrix

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$$

Solution : Here, $$A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$$

$$\begin{aligned} \therefore \qquad |A| &= 3(5 - 24) - 2(-5 - 30) - 2(4 + 5) \\ &= 3(-19) - 2(-35) - 2(9) \\ &= -57 + 70 - 18 \\ &= -5 \neq 0 \end{aligned}$$

$\therefore \qquad A^{-1}$ exists.

Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} = 5 - 24 = -19,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 6 \\ 5 & -5 \end{vmatrix} = -(-5-30) = 35.$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix} = 4 - 5 = 9,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ 4 & -5 \end{vmatrix} = -(-10+8) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 5 & -5 \end{vmatrix} = -15 + 10 = -5,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = -(12 - 10) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ -1 & 6 \end{vmatrix} = 12 - 2 = 10,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} = -(18+2) = -20$$

and $$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -3 - 2 = -5$$

Matrix of cofactors $= \begin{bmatrix} -19 & 35 & 9 \\ 2 & -5 & -2 \\ 10 & -20 & -5 \end{bmatrix}$. Hence Adj$A$= $\begin{bmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}.Adj\ A = \frac{1}{-5}\begin{bmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{bmatrix} = \begin{bmatrix} \frac{19}{5} & \frac{-2}{5} & -2 \\ -7 & 1 & 4 \\ \frac{-9}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

Note : Verify that $A^{-1}A = AA^{-1} = I_3$

Example 22.10 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$; find

(i) $(AB)^{-1}$ (ii) $B^{-1}A^{-1}$ (iii) Is $(AB)^{-1} = B^{-1}A^{-1}$?

Solution : (i) Here, $AB = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2+0 & 1+0 \\ -4+0 & 2+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

$$\therefore \qquad |AB| = \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -6+4 = -2 \neq 0.$$

Thus, $(AB)^{-1}$ exists.

Let us denote AB by C_{ij}

Let C_{ij} be the cofactor of the element C_{ij} of $|C|$.

Then, $C_{11} = (-1)^{1+1}(3) = 3$ $\qquad C_{21} = (-1)^{2+1}(1) = -1$

$C_{12} = (-1)^{1+2}(-4) = 4$ $\qquad C_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj}(C) = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

$$C^{-1} = \frac{1}{|C|}\text{Adj}(C) = \frac{1}{-2}\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(ii) To find $B^{-1}A^{-1}$, first we will find B^{-1}.

Now. $B = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$ $\therefore$ $|B| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2-0 = 2 \neq 0$

$\therefore$ B^{-1} exists.

Let B_{ij} be the cofactor of the element bij of |B|

then $B_{11} = (-1)^{1+1}(-1) = -1$ $\qquad B_{21} = (-1)^{2+1}(1) = -1$

$B_{12} = (-1)^{1+2}(0) = 0$ and $\qquad B_{22} = (-1)^{2+2}(-2) = -2$

Hence, Adj $B = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$

$$\therefore \quad B^{-1} = \frac{1}{|B|}.AdjB = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}$$

Also, $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ Therefore, $|A| = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 1-0 = -1 \quad \neq 0$

Therefore, A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij} of |A|

then $A_{11} = (-1)^{1+1}(-1) = -1$ $\qquad A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(2) = -2$ and $\qquad A_{22} = (-1)^{2+2}(1) = 1$

Hence, Adj $A = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} Adj\ A = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

Thus, $$B^{-1}A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}-1 & 0+\frac{1}{2} \\ 0-2 & 0+1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(iii) From (i) and (ii), we find that

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

Hecne, $(AB)^{-1} = B^{-1}A^{-1}$

CHECK YOUR PROGRESS 22.3

1. Find, if possible, the inverse of each of the following matrices:

(a) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

2. Find, if possible, the inverse of each of the following matrices :

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 2 & 4 \\ 1 & -3 & -2 \end{bmatrix}$

Verify that $A^{-1}A = AA^{-1} = \text{I}$ for (a) and (b).

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

4. Find $(A')^{-1}$ if $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

show that ABA^{-1} is a diagonal matrix.

6. If $\phi(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $[\phi(x)]^{-1} = \phi(-x)$.

7. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

8. If $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$, show that $aA^{-1} = (a^2 + bc + 1)\, I - aA$

9. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$

10. If $A = \dfrac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & -4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, show that $A^{-1} = A'$

22.5 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

In earlier classes, you have learnt how to solve linear equations in two or three unknowns (simultaneous equations). In solving such systems of equations, you used the process of elimination of variables. When the number of variables invovled is large, such elimination process becomes tedious.

You have already learnt an alternative method, called Cramer's Rule for solving such systems of linear equations.

We will now illustrate another method called the matrix method, which can be used to solve the system of equations in large number of unknowns. For simplicity the illustrations will be for system of equations in two or three unknowns.

22.5.1 MATRIX METHOD

In this method, we first express the given system of equation in the matrix form $AX = B$, where A is called the co-efficient matrix.

For example, if the given system of equation is $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, we express them in the matrix equation form as :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

If the given system of equations is $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, then this system is expressed in the matrix equation form as:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Where, $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Before proceding to find the solution, we check whether the coefficient matrix A is non-singular or not.

Note: If A is singular, then $|A|=0$. Hence, A^{-1} does not exist and so, this method does not work.

Consider equation $AX = B$, where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

When $|A| \neq 0$, i.e. when $a_1b_2 - a_2b_1 \neq 0$, we multiply the equation $AX = B$ with A^{-1} on both side and get

$$A^{-1}(AX) = A^{-1}B$$

$\Rightarrow \quad (A^{-1}A)\, X = A^{-1}B$

$\Rightarrow \quad IX = A^{-1}B \qquad (\because A^{-1}A = I)$

$\Rightarrow \quad X = A^{-1}B$

Since $A^{-1} = \dfrac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$, we get

$$X = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2c_1 - b_1c_2 \\ -a_2c_1 + a_1c_2 \end{pmatrix} = \begin{bmatrix} \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\[2ex] \dfrac{-a_2c_1 + a_1c_2}{a_1b_2 - a_2b_1} \end{bmatrix}$$

Hence, $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Example 22.11 Using matrix method, solve the given system of linear equations.

$$\left.\begin{aligned} 4x - 3y &= 11 \\ 3x + 7y &= -1 \end{aligned}\right\} \qquad \text{......(i)}$$

Solution: This system can be expressed in the matrix equation form as

$$\begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix} \qquad \text{......(ii)}$$

Here, $A = \begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $B = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$

so, (ii) reduces to

$$AX = B \qquad \text{......(iii)}$$

Now, $|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 28 + 9 = 37 \neq 0$

Since $|A| \neq 0$, A^{-1} exists.

Now, on multiplying the equation $AX = B$ with A^{-1} on both sides, we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

i.e. $IX = A^{-1}B$

$$X = A^{-1}B$$

Hence, $X = \dfrac{1}{|A|}(Adj\ A)\ B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{37}\begin{pmatrix} 7 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 11 \\ -1 \end{pmatrix} \qquad = \frac{1}{37}\begin{pmatrix} 77-3 \\ -33-4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{37}\begin{pmatrix} 74 \\ -37 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So, $x = 2$, $y = -1$ is unique solution of the system of equations.

Example 22.12 Solve the following system of equations, using matrix method.

$$\left.\begin{aligned} x + 2y + 3z &= 14 \\ x - 2y + z &= 0 \\ 2x + 3y - z &= 5 \end{aligned}\right\}$$

Solution : The given equations expressed in the matrix equation form as :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix} \qquad \text{... (i)}$$

which is in the form $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix}$$

$\therefore \quad X = A^{-1} B \qquad$ ... (ii)

Here, $\quad |A| = 1\,(2-3)\ -2\,(-1-2)\ +3\,(3+4)$

$\qquad = 26 \neq 0$

$\therefore \quad$ A^{-1} exists.

Also, Adj $\quad A = \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

Hence, from (ii), we have $\quad X = A^{-1}B = \dfrac{1}{|A|} Adj A.B$

$$X = \frac{1}{26}\begin{pmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{pmatrix}\begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{26}\begin{pmatrix} 26 \\ 52 \\ 78 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ or, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus, $x = 1$, $y = 2$ and $z = 3$ is the solution of the given system of equations.

22.6 CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let $AX = B$ be a system of two or three linear equations.

Then, we have the following criteria :

(1) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution, given by $X = A^{-1}B$.

(2) If $|A| = 0$, then the system may or may not be consistent and if consistent, it does not have a unique solution. If in addition,

(a) (Adj A) B $\neq$ O, then the system is inconsistent.

(b) (Adj A) B = O, then the system is consistent and has infinitely many solutions.

Note : These criteria are true for a system of '*n*' equations in '*n*' variables as well.

We now, verify these with the help of the examples and find their solutions wherever possible.

(a)
$$5x + 7y = 1$$
$$2x - 3y = 3$$

This system is consistent and has a unique solution, because $\begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} \neq 0$ Here, the matrix equation is $\begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. $AX = B$ .... (i)

where, $A = \begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Here, $|A| = 5 \times (-3) - 2 \times 7 = -15 - 14 = -29 \neq 0$

$$\textit{and } A^{-1} = \frac{1}{|A|} \textit{Adj } A = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix} \quad \text{.... (ii)}$$

From (i), we have $X = A^{-1}B$

i.e., $$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{24}{29} \\ -\frac{13}{29} \end{pmatrix}$$ [From (i) and (ii)]

Thus, $x = \frac{24}{29}$, and $y = \frac{-13}{29}$ is the unique solution of the given system of equations.

(b) $\begin{array}{l} 3x + 2y = 7 \\ 6x + 4y = 8 \end{array}$

In the matrix form the system can be written as

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

or, $AX = B$

where $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Here, $|A| = 3 \times 4 - 6 \times 2 = 12 - 12 = 0$

$$\text{Adj } A = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}$$

Also, $(\text{Adj } A)\, B = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -20 \\ -18 \end{bmatrix} \neq 0$

Thus, the given system of equations is inconsistent.

(c) $\left.\begin{array}{r} 3x - y = 7 \\ 9x - 3y = 21 \end{array}\right\}$

In the matrix form the system can be written as

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

or, $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 9 & -3 \end{vmatrix} = 3 \times (-3) - 9 \times (-1) = -9 + 9 = 0$

$$\text{Adj A} = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}$$

Also, $(\text{Adj A})B = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$\therefore$ The given system has an infinite number of solutions.

Let us now consider another system of linear equations, where $|A| = 0$ and $(\text{Adj A})\,B \neq O$.
Consider the following system of equations

$$x + 2y + z = 5$$

$$2x + y + 2z = -1$$

$$x - 3y + z = 6$$

In matrix equation form, the above system of equations can be written as

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$$

i.e., $AX = B$

where $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0 \qquad (\because C_1 = C_3)$

Also, $(\text{Adj A})\,B = \begin{pmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$ [Verify (Adj A) yourself]

$$= \begin{pmatrix} 58 \\ 0 \\ -58 \end{pmatrix} \neq O$$

Since $|A| = 0$ and $(\text{Adj A})\,B \neq O$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{|A|}(Adj \;\; A)B$$

$$= \frac{1}{0}\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \text{ which is undefined.}$$

The given system of linear equation will have no solution.

Thus, we find that if $|A| = 0$ and (Adj A) $B \neq$ O then the system of equations will have no solution.

We can summarise the above findings as:

(i) If $|A| \neq 0$ and (Adj A) $B \neq$ O then the system of equations will have a non-zero, unique solution.

(ii) If $|A| \neq 0$ and (Adj A) B = O, then the system of equations will have trivial solutions.

(iii) If $|A| = 0$ and (Adj A) B = O, then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and (Adj A) $B \neq$ O, then the system of equations will have no solution Inconsistent.

CHECK YOUR PROGRESS 22.4

1. Solve the following system of equations, using the matrix inversion method:

(a) $2x + 3y = 4$
$x - 2y = 5$

(b) $x + y = 7$
$3x - 7y = 11$

2. Solve the following system of equations using matrix inversion method:

(a) $x + 2y + z = 3$
$2x - y + 3z = 5$
$x + y - z = 7$

(b) $2x + 3y + z = 13$
$3x + 2y - z = 12$
$x + y + 2z = 5$

(c) $-x + 2y + 5z = 2$
$2x - 3y + z = 15$
$-x + y + z = -3$

(d) $2x + y - z = 2$
$x + 2y - 3z = -1$
$5x - y - 2z = -1$

3. Determine whether the following system of equations are consistent or not. If consistent, find the solution:

(a) $2x - 3y = 5$
$x + y = 7$

(b) $2x - 3y = 5$
$4x - 6y = 10$

(c) $3x + y + 2z = 3$
$-2y - z = 7$
$x + 15y + 3z = 11$

LET US SUM UP

- A square matrix is said to be non-singular if its corresponding determinant is non-zero.
- The determinant of the matrix A obtained by deleting the i[th] row and j[th] column of A, is called the minor of a_{ij}. It is usually denoted by M_{ij}.
- The cofactor of a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$
- Adjoint of a matrix A is the transpose of the matrix whose elements are the cofactors of the elements of the determinat of given matrix. It is usually denoted by Adj A.
- If A is any square matrix of order n, then

  $A\,(\text{Adj}\,A) = (\text{Adj}\,A)\,A = |A|\;I_n$ where I_n is the unit matrix of order n.
- For a given non-singular square matrix A, if there exists a non-singular square matrix B such that $AB = BA = I$, then B is called the multiplicative inverse of A. It is written as $B = A^{-1}$.
- Only non-singular square matrices have multiplicative inverse.
- If $a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$, then we can express the system in the matrix equation form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, if $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$X = A^{-1}B = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- A system of equations, given by $AX = B$, is said to be consistent and has a unique solution, if $|A| \neq 0$.
- A system of equations, given by $AX = B$, is said to be inconsistent, if $|A| = 0$ and $(\text{Adj } A)\, B \neq O$.
- A system of equations, given by $AX = B$, is said be be consistent and has infinitely many solutions, if $|A| = 0$, and $(\text{Adj } A)\, B = O$.

SUPPORTIVE WEB SITES

http://www.mathsisfun.com/algebra/matrix-inverse.html

http://www.sosmath.com/matrix/coding/coding.html

TERMINAL EXERCISE

1. Find $|A|$, if

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & -1 & 0 \\ -2 & 5 & 4 \end{pmatrix}$ (b) $A = \begin{pmatrix} -1 & 3 & 4 \\ 7 & 5 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

2. Find the adjoint of A, if

(a) $A = \begin{pmatrix} -2 & 3 & 7 \\ -1 & 4 & 5 \\ -1 & 0 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix}$

Also, verify that $A(\text{Adj } A) = |A| I_3 = (\text{Adj } A)\, A$, for (a) and (b)

3. Find A^{-1}, if exists, when

(a) $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$

Also, verify that $(A')^{-1} = (A^{-1})'$, for (a), (b) and (c)

4. Find the inverse of the matrix A, if

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

5. Solve, using matrix inversion method, the following systems of linear equations

(a) $$\begin{aligned} x+2y &= 4 \\ 2x+5y &= 9 \end{aligned}$$

(b) $$\begin{aligned} 6x+4y &= 2 \\ 9x+6y &= 3 \end{aligned}$$

(c) $$\begin{aligned} 2x+y+z &= 1 \\ x-2y-z &= \frac{3}{2} \\ 3y-5z &= 9 \end{aligned}$$

(d) $$\begin{aligned} x-y+z &= 4 \\ 2x+y-3z &= 0 \\ x+y+z &= 2 \end{aligned}$$

(e) $$\begin{aligned} x+y-2z &= -1 \\ 3x-2y+z &= 3 \\ 2x+y-z &= 0 \end{aligned}$$

6. Solve, using matrix inversion method

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4;\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1;\ \frac{8}{x}+\frac{9}{y}-\frac{20}{z}=3$$

7. Find the value of λ for which the following system of equation becomes consistent

$$\begin{aligned} 2x-3y+4 &= 0 \\ 5x-2y-1 &= 0 \\ 21x-8y+\lambda &= 0 \end{aligned}$$

ANSWERS

CHECK YOUR PROGRESS 22.1

1. (a) –12 (b) 10
2. (a) singular (b) non-singular
3. (a) $M_{11} = 4; M_{12} = 7; M_{21} = -1; M_{22} = 3$ (b) $M_{11} = 5; M_{12} = 2; M_{21} = 6; M_{22} = 0$
4. (a) $M_{21} = 11; M_{22} = 7; M_{23} = 1$ (b) $M_{31} = -13; M_{32} = -13; M_{33} = 13$
5. (a) $C_{11} = 7; C_{12} = -9; C_{21} = 2; C_{22} = 3$ (b) $C_{11} = 6; C_{12} = 5; C_{21} = -4; C_{22} = 0$
6. (a) $C_{21} = 1; C_{22} = -8; C_{23} = -2$ (b) $C_{11} = -6; C_{12} = 10; C_{33} = 2$

CHECK YOUR PROGRESS 22.2

1. (a) $\begin{pmatrix} 6 & 1 \\ -3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (c) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

2. (a) $\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$ (b) $\begin{pmatrix} i & i \\ -i & i \end{pmatrix}$

CHECK YOUR PROGRESS 22.3

1. (a) $\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$ (b) $\begin{bmatrix} -\frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

2. (a) $\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ -\frac{8}{5} & \frac{6}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$ (b) $\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{17}{24} & -\frac{1}{3} & -\frac{11}{24} \end{pmatrix}$

4. $(A')^{-1} = \begin{pmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{pmatrix}$

CHECK YOUR PROGRESS 22.4

1. (a) $x = \frac{23}{7},\ y = \frac{-6}{7}$ (b) $x = 6, y = 1$

2. (a) $x = \frac{58}{11},\ y = -\frac{2}{11},\ z = -\frac{21}{11}$ (b) $x = 2,\ y = 3,\ z = 0$

(c) $x = 2,\ y = -3,\ z = 2$ (d) $x = 1,\ y = 2,\ z = 2$

3. (a) Consistent; $x = \dfrac{26}{5},\ y = \dfrac{9}{5}$ (b) Consistent; infinitely many solutions

(c) Inconsistent

TERMINAL EXERCISE

1. (a) –31 (b) –24

2. (a) $\begin{bmatrix} 4 & -3 & -13 \\ -4 & 5 & 3 \\ 4 & -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 8 & -7 \\ -13 & 13 & 13 \\ 5 & 1 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} -\dfrac{1}{18} & \dfrac{1}{6} \\ \dfrac{7}{36} & -\dfrac{1}{12} \end{bmatrix}$ (b) $\begin{bmatrix} \dfrac{5}{7} & -\dfrac{1}{7} \\ -\dfrac{3}{7} & \dfrac{2}{7} \end{bmatrix}$ (c) $\begin{bmatrix} -\dfrac{1}{7} & -\dfrac{5}{14} \\ -\dfrac{2}{7} & -\dfrac{3}{14} \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \dfrac{1}{3} & 0 \\ 3 & \dfrac{2}{3} & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \dfrac{3}{2} & -1 & -\dfrac{1}{2} \\ -\dfrac{1}{4} & \dfrac{1}{2} & \dfrac{1}{4} \\ -\dfrac{3}{4} & \dfrac{1}{2} & \dfrac{3}{4} \end{bmatrix}$

5. (a) $x = 2,\ y = 1$ (b) $x = k,\ y = \dfrac{1}{2} - \dfrac{3}{2}k$

(c) $x = 1,\ y = \dfrac{1}{2},\ z = -\dfrac{3}{2}$ (d) $x = 2,\ y = -1,\ z = 1$

(e) $x = \dfrac{1}{2},\ y = -\dfrac{1}{2},\ z = \dfrac{1}{2}$

6. $x = 2,\ y = 3,\ z = 5$

7. $\lambda = -5$

23

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RELATIONS AND FUNCTIONS-II

We have learnt about the basic concept of Relations and Functions. We know about the ordered pair, the cartesian product of sets, relation, functions, their domain, Co-doman and range. Now we will extend our knowledge to types of relations and functions, composition of functions, invertible functions and binary operations.

OBJECTIVES

After studying this lesson, you will be able to :

- verify the equivalence relation in a set
- verify that the given function is one-one, many one, onto/ into or one one onto
- find the inverse of a given function
- determine whether a given operation is binary or not.
- check the commutativity and associativity of a binary operation.
- find the inverse of an element and identity element in a set with respest to a binary operation.

EXPECTED BACKGROUND KNOWLEDGE

Before studying this lesson, you should know :

- Concept of set, types of sets, operations on sets
- Concept of ordered pair and cartesian product of set.
- Domain, co-domain and range of a relation and a function

23.1 RELATION

23.1.1 Relation :

Let A and B be two sets. Then a relation R from Set A into Set B is a subset of A × B.

Thus, R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A × B

- If $(a, b) \in$R then we write aRb which is read as 'a' is related to b by the relation R, if $(a, b) \notin$R, then we write $a \not{R}\, b$ and we say that a is not related to b by the relation R.
- If $n(A) = m$ and $n(B) = n$, then A × B has mn ordered pairs, therefore, total number of relations form A to B is 2^{mn}.

23.1.2 Types of Relations

(i) Reflexive Relation :

A relation R on a set A is said to be reflexive if every element of A is related to itself.
Thus, R is reflexive $\Leftrightarrow$ $(a, a) \in R$ for all $a \in A$
A relation R is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Let A = {1, 2, 3} be a set. Then

R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is a reflexive relation on A.

but R_1 = {(1, 1), (3, 3) (2, 1) (3, 2)} is not a reflexive relation on A, because $2 \in A$ but $(2,2) \notin R$.

(ii) Symmetric Relation

A relation R on a set A is said to be symmetric relation if

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $(a, b) \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Let A = {1, 2, 3, 4} and R_1 and R_2 be relations on A given by

R_1 = {(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)

and R_2 = {(1, 1), (2, 2), (3, 3), (1, 3)}

- R_1 is symmetric relation on A because $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$

or $aR_1b \Rightarrow bR_1$ a for all $a,b \in A$

but R_2 is not symmetric because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.

A reflexive relation on a set A is not necessarily symmetric. For example, the relation R ={(1, 1), (2, 2), (3, 3), (1, 3)} is a reflexive relation on set A = {1, 2, 3} but it is not symmetric.

(iii) Transitive Relation:

Let A be any set. A relation R on A is said to be transitive relation if

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$

For example :

On the set N of natural numbers, the relation R defined by xRy

$\Rightarrow$ 'x is less than y', is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z$

i.e. xRy and $yRz \Rightarrow xRz$

Take another example

Let A be the set of all straight lines in a plane. Then the relation 'is parallel to' on A is a transitive relation, because for any $l_1, l_2, l_3 \in A$

$l_1 \| l_2$ and $l_2 \| l_3 \Rightarrow l_1 \| l_3$

Example 23.1 Check the relation R for reflexivity, symmetry and transitivity, where R is defined as $l_1 R l_2$ iff $l_1 \perp l_2$ for all $l_1, l_2 \in A$

Solution : Let A be the set of all lines in a plane. Given that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in A$

Reflexivity : R is not reflexive because a line cannot be perpendicular to itself i.e. $l \perp l$ is not true.

Symmetry : Let $l_1, l_2 \in A$ such that $l_1 R l_2$

Then $l_1 R l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow l_2 R l_1$

So, R is symmetric on A

Transitive

R is not transitive, because $l_1 \perp l_2$ and $l_2 \perp l_3$ does not impty that $l_1 \perp l_3$

23.2 EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

For example the relation 'is congruent to' is an equivalence relation because

(i) it is reflexive as $\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S$ where S is a set of triangles.

(ii) it is symmetric as $\Rightarrow \Delta_1 R \Delta_2 \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1$

$\Rightarrow \Delta_2 R \Delta_1$

(iii) it is transitive as $\Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3$

it means $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow (\Delta_1, \Delta_3) \in R$

Example 23.2 Show that the relation R defined on the set A of all triangles in a plane as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$ is an equivalence relation.

Solution : We observe the following properties of relation R;

Reflexivity we know that every triangle is similar to itself. Therefore, $(T, T) \in R$ for all $T \in A \Rightarrow R$ is reflexive.

Symmetricity Let $(T_1, T_2) \in R$, then

$(T_1, T_2) \in R \Rightarrow T_1$ is similar to T_2

$\Rightarrow T_2$ is similar to T_1

$\Rightarrow (T_2, T_1) \in R$, So, R is symmetric.

Transitivity : Let $T_1, T_2, T_3 \in A$ such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$.

Then $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3

$\Rightarrow T_1$ is similar to T_3

$\Rightarrow (T_1, T_3) \in R$

Hence, R is an equivalence relation.

CHECK YOUR PROGRESS 23.1

1. Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R \Rightarrow$ line l_1 is parallel to l_2. Show that R is an equivalence relation.
2. Show that the relation R on the set A of points in a plane, given by

   R = {(P, Q) : Distance of the point P from the origin is same as the distance of the point Q from the origin} is an equivalence relation.
3. Show that each of the relation R in the set $A = \{x \in z : 0 \le x \le 12\}$, given by

   (i) $R = \{(a,b) : |a-b| \, is \, multiple \, of \; 4\}$

   (ii) $R = \{(a,b) : a = b\}$ is an equivalence relation
4. Prove that the relation 'is a factor of' from R to R is reflexive and transitive but not symmetric.
5. If R and S are two equivalence relations on a set A then $R \cap S$ is also an equivalence relation.
6. Prove that the relation R on set $N \times N$ defined by (a,b) R (c,d) $\Leftrightarrow$ a+d = b + c for all (a,b), (c,d) $\in N \times N$ is an equivalence relation.

23.3 CLASSIFICATION OF FUNCTIONS

Let *f* be a function from *A* to *B*. If every element of the set *B* is the image of at least one element of the set *A* i.e. if there is no unpaired element in the set *B* then we say that the **function f maps the set *A* onto the set *B*. Otherwise we say that the function maps the set *A* into the set *B*.**

Functions for which each element of the set *A* is mapped to a different element of the set *B* are said to be ***one-to-one.***

One-to-one function

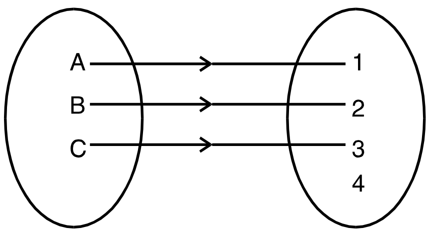


Fig.23.27

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 2, 3\}$

A function can map more than one element of the set A to the same element of the set B. Such a type of function is said to be ***many-to-one.***

Many-to-one function

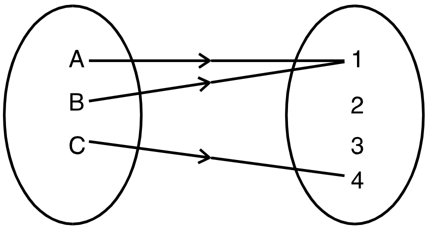


Fig. 23.2

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 4\}$

A function which is both one-to-one and onto is said to be a bijective function.

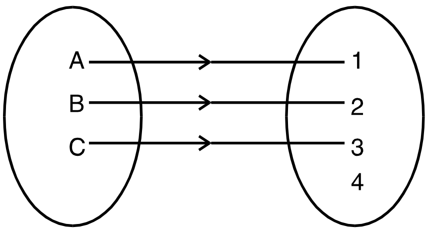


Fig. 23.3

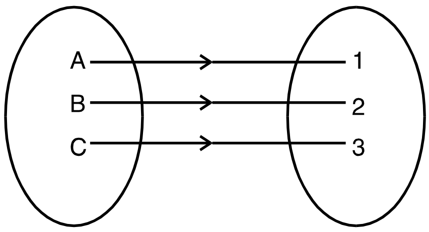


Fig. 23.4

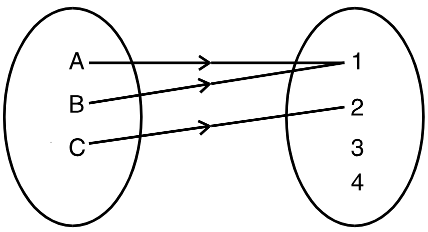


Fig. 23.5

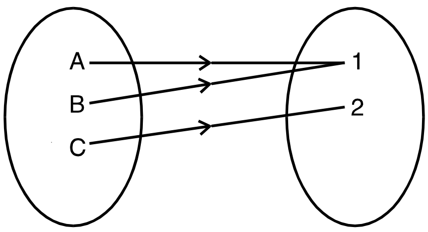


Fig. 23.6

Fig. 23.3 shows a one-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.4 shows a one-to-one function mapping $\{A, B, C\}$ onto $\{1, 2, 3\}$.

Fig. 23.5shows a many-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.6 shows a many-to-one function mapping $\{A, B, C\}$ onto $\{1, 2\}$.

Function shown in Fig. 23.4 is also a bijective Function.

Note : Relations which are one-to-many can occur, but they are not functions. The following figure illustrates this fact.

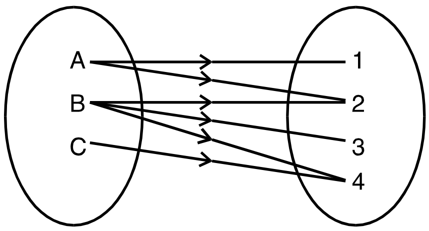


Fig. 23.7

Example 23.3 Without using graph prove that the function

$f : R \to R$ defiend by $f(x) = 4 + 3x$ is ***one-to-one***.

Solution : For a function to be one-one function

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall \quad x_1, x_2 \in \text{domain}$$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$$4 + 3x_1 = 4 + 3x_2 \quad \text{or} \quad x_1 = x_2$$

$\therefore$ f is a ***one-one function***.

Example 23.4 Prove that

$f : R \to R$ defined by $f(x) = 4x^3 - 5$ is a bijection

Solution : Now $f(x_1) = f(x_2) \quad \forall \quad x_1, x_2 \in$ Domain

$$\therefore \quad 4x_1^3 - 5 = 4x_2^3 - 5$$

$$\Rightarrow \quad x_1^3 = x_2^3$$

$$\Rightarrow \quad x_1^3 - x_2^3 = 0 \Rightarrow \quad (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2\right) = 0$$

$$\Rightarrow \quad x_1 = x_2 \text{ or}$$

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2.

$\therefore$ f is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\frac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists\, x \in$ domain such that $f(x) = y$.

Thus f is ***onto function***.

$\therefore$ f is a bijection.

Example 23.5 Prove that $f : R \to R$ defined by $f(x) = x^2 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $f(x_1) = f(x_2)\ \forall\ \ x_1, x_2 \in$ domain giving

$$x_1^2 + 3 = x_2^2 + 3 \Rightarrow x_1^2 = x_2^2$$

or $\quad x_1^2 - x_2^2 = 0 \Rightarrow x_1 = x_2 \quad$ or $\quad x_1 = -x_2$

or $\quad$ f is not ***one-one function***.

Again let $y = f(x)$ where $y \in$ codomain

$x \in$ domain.

$\Rightarrow \quad y = x^2 + 3 \quad \Rightarrow \quad x = \pm\sqrt{y-3}$

$\Rightarrow \quad \forall\ y < 3\ \exists$ no real value of x in the domain.

$\therefore \quad$ f is not an ***onto fiction***.

23.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$$y = x^2$$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

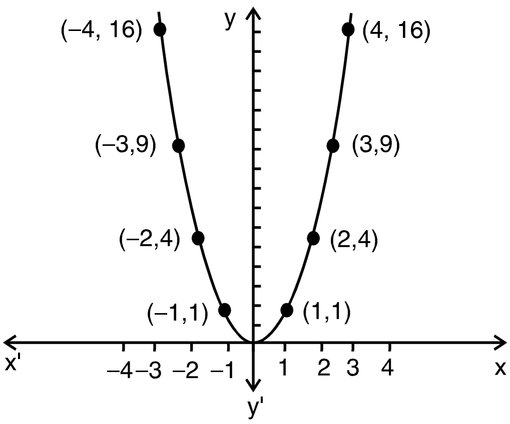


Fig. 23.8

Does this represent a function?

Yes, this represent a function because corresponding to each value of $x\ \exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

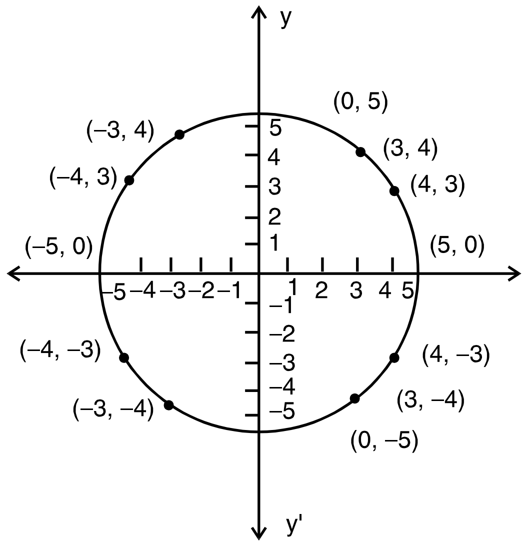


Fig. 23.9

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 23.2

1. (i) Does the graph represent a function?

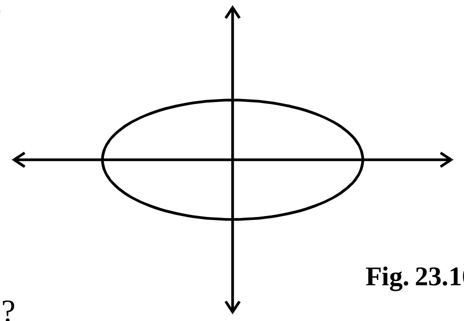


Fig. 23.10

(ii) Does the graph represent a function ?

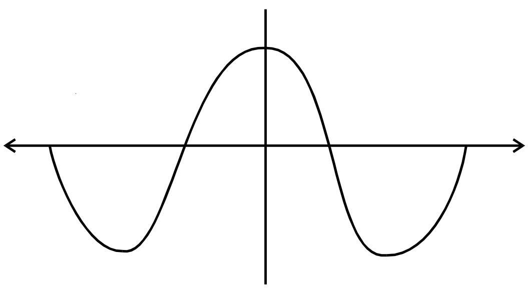


Fig. 23.11

2. Which of the following functions are into function ?

(a)

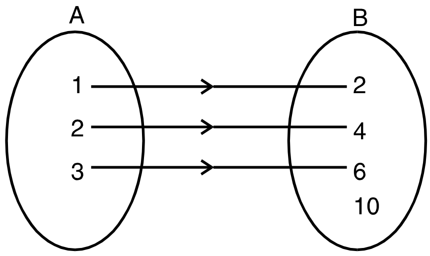


Fig.23.12

(b) $f : N \to N$, defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : N \to N$, defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$

(a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20, 21, 22\} \to \{40, 42, 44\}$ defined as $f(x) = 2x$

(b) $f : \{7, 8, 9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2, -1, 1, 2\} \to \{2, 5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0, 1, 2\} \to \{1\}$ defined as $f(x) = 1$

(c)

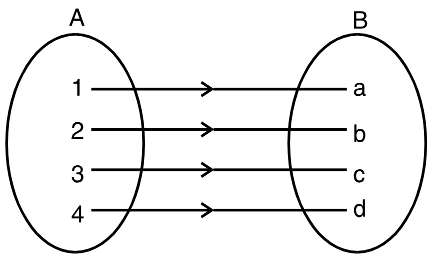


Fig.23.13

(d) $f : N \to N$ defined as $f(x) = 5x + 7$

23.5 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \qquad x \in \{1, 2, 3\}$$

$$z = y + 1, \qquad y \in \{3, 5, 7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Graphically one can represent this as given below :

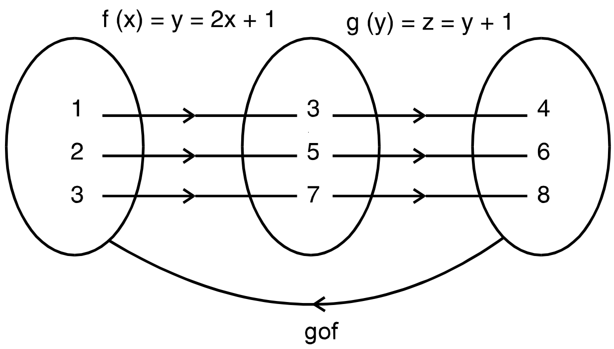


Fig. 23.18

The composition, say, gof of function g and f is defined as function g of function f.

If $\quad f : A \to B$ and $g : B \to C$

then $\quad g\,o\,f : A$ to C

Let $\quad f(x) = 3x + 1$ and $\quad g(x) = x^2 + 2$

Then $\quad fog(x) = f(g(x)) = f(x^2 + 2)$

$$= 3(x^2 + 2) + 1 = 3x^2 + 7 \qquad \text{(i)}$$

and $\quad (gof)(x) = g(f(x)) = g(3x + 1)$

$$= (3x + 1)^2 + 2 = 9x^2 + 6x + 3 \qquad \text{(ii)}$$

Check from (i) and (ii), if

fog = gof

Evidently, $\quad fog \neq gof$

Similarly, $(fof)(x) = f(f(x)) = f(3x + 1)$ [Read as function of function f].

$$= 3(3x + 1) + 1 = 9x + 3 + 1 = 9x + 4$$

$(gog)(x) = g(g(x)) = g(x^2 + 2)$ [Read as function of function g]

$$= (x^2 + 2)^2 + 2 = x^4 + 4x^2 + 4 + 2 = x^4 + 4x^2 + 6$$

Example 23. 6 If $f(x)=\sqrt{x+1}$ and $g(x)=x^2+2$, calculate *fog* and *gof*.

Solution :
$$fog(x)=f(g(x))$$
$$=f\left(x^2+2\right)=\sqrt{x^2+2+1}=\sqrt{x^2+3}$$
$$(gof)(x)=g(f(x))$$
$$=g\left(\sqrt{x+1}\right)=\left(\sqrt{x+1}\right)^2+2=x+1+2=x+3.$$

Here again, we see that $(fog)\neq gof$

Example 23. 7 If $f(x)=x^3$, $f: R\rightarrow R$ and $g(x)=\frac{1}{x}$, $g: R-\{0\}\rightarrow R-\{0\}$

Find *fog* and *gof*.

Solution :
$$(fog)(x)=f(g(x))=f\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)^3=\frac{1}{x^3}$$
$$(gof)(x)=g(f(x))=g\left(x^3\right)=\frac{1}{x^3}$$

Here we see that $fog = gof$

CHECK YOUR PROGRESS 23.3

1. Find *fog, gof, fof* and *gog* for the following functions :

   $f(x)=x^2+2,\quad g(x)=1-\frac{1}{1-x},\ x\neq 1.$

2. For each of the following functions write *fog, gof, fof* and *gog.*

   (a) $f(x)=x^2-4,\ g(x)=2x+5$

   (b) $f(x)=x^2,\ g(x)=3$

   (c) $f(x)=3x-7,\ g(x)=\frac{2}{x},\ x\neq 0$

3. Let $f(x)=|x|,\ g(x)=[x]$. Verify that $fog\neq gof$.

4. Let $f(x)=x^2+3,\ g(x)=x-2$

   Prove that $fog\neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right)=g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x)=x^2,\ g(x)=\sqrt{x}$. Show that $fog=gof$.

6. Let $f(x)=|x|,\ g(x)=(x)^{\frac{1}{3}},\ h(x)=\frac{1}{x};\ x\neq 0.$

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

23.6 INVERSE OF A FUNCTION

(A) Consider the relation

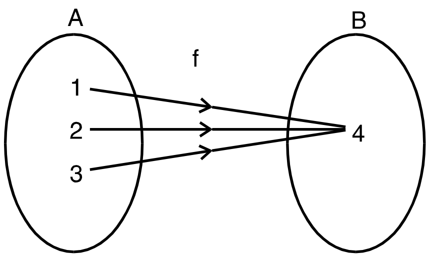


Fig. 23.19

This is a many-to-one function. Now let us find the inverse of this relation. Pictorially, it can be represented as

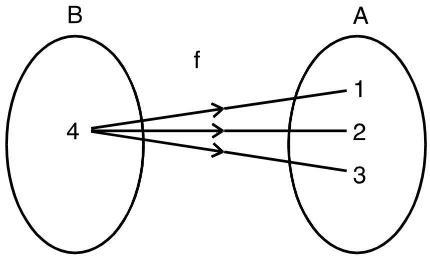


Fig 23.20

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

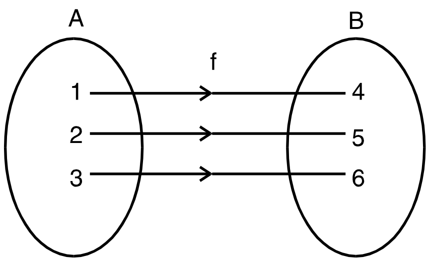


Fig.23.21

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

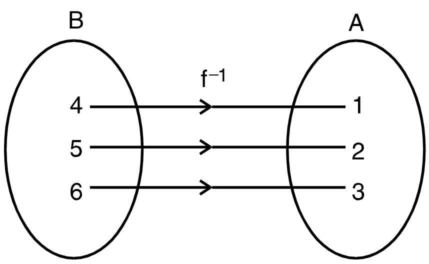


Fig. 23.22

This represents a function. (C) Consider the relation

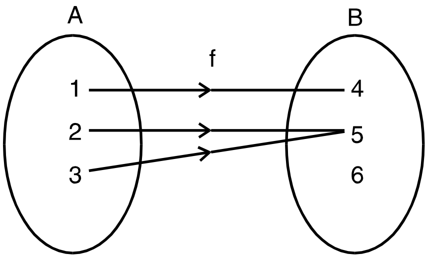


Fig. 23.23

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

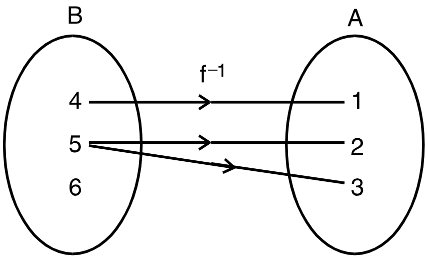


Fig. 23.24

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

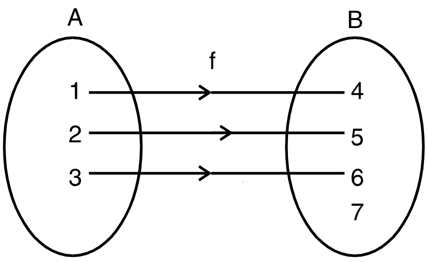


Fig. 23.25

It represent one-to-one into function. Find the inverse of the relation.

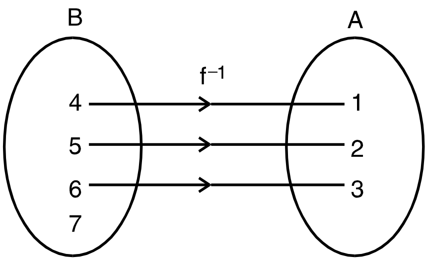


Fig. 23.26

It does not represent a function because the element 7 of B is not associated with any element of A. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

CHECK YOUR PROGRESS 23.4

1 (i) Show that the inverse of the function

$$y = 4x - 7 \text{ exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \quad \forall\ x \in R$

(b) $f(x) = 1 - 3x \quad \forall\ x \in R$

(c) $f(x) = x^2 \quad \forall\ x \in R$

(d) $f(x) = \dfrac{x+1}{x}, \quad x \neq 0 \quad x \in R$

23.7 BINARY OPERATIONS :

Let A, B be two non-empty sets, then a function from A × A to A is called a binary operation on A.

If a binary operation on A is denoted by '*', the unique element of A associated with the ordered pair (a, b) of A × A is denoted by $a * b$.

The order of the elements is taken into consideration, i.e. the elements associated with the pairs (a, b) and (b, a) may be different i.e. $a * b$ may not be equal to $b * a$.

Let A be a non-empty set and '*' be an operation on A, then

1. A is said to be closed under the operation * iff for all $a, b \in A$ implies $a * b \in A$.
2. The operation is said to be commutative iff $a * b = b * a$ for all $a, b \in A$.
3. The operation is said to be associative iff $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.
4. An element $e \in A$ is said to be an identity element iff $e * a = a = a * e$
5. An element $a \in A$ is called invertible iff these exists some $b \in A$ such that $a * b = e = b * a$, b is called inverse of a.

Note : If a non empty set A is closed under the operation *, then operation * is called a binary operation on A.

For example, let A be the set of all positive real numbers and '*' be an operation on A defined by $a * b = \frac{ab}{3}$ for all $a, b \in A$

For all $a, b, c \in A$, we have

(i) $a * b = \frac{ab}{3}$ is a positive real number $\Rightarrow$ A is closed under the given operation.

$\therefore$ * is a binary operation on A.

(ii) $a*b = \frac{ab}{3} = \frac{ba}{3} = b*a \Rightarrow$ the operation * is commutative.

(iii) $(a*b)*c = \frac{ab}{3}*c = \frac{\frac{ab}{3}.c}{3} = \frac{abc}{9}$ and $a*(b*c) = a*\frac{bc}{3} = \frac{a}{3}.\frac{bc}{3} = \frac{abc}{9}$ –

$\Rightarrow (a*b)*c = a*(b*c) \Rightarrow$ the operation * is associative.

(iv) There exists $3 \in A$ such that $3 * a = 3.\frac{a}{3} = a = \frac{a}{3}.3 = a*3$

$\Rightarrow$ 3 is an identity element.

(v) For every $a \in A$, there exists $\frac{9}{a} \in A$ such that $a*\frac{9}{a} = \frac{a.\frac{9}{a}}{3} = 3$ and $\frac{9}{a}*a = \frac{\frac{9}{a}.a}{3} = 3$

$\Rightarrow a*\frac{9}{a} = 3 = \frac{9}{a}*a \Rightarrow$ every element of A is invertible, and inverse of a is $\frac{9}{a}$

CHECK YOUR PROGRESS 23.5

1. Determine whether or not each of operation * defined below is a binary operation.

(i) $a*b = \frac{a+b}{2}, \forall a,b \in Z$

(ii) $a*b = a^b, \forall a,b \in Z$

(iii) $a*b = a^2 + 3b^2, \forall a,b \in R$

2. If $A = \{1,2\}$ find total number of binary operations on A.

3. Let a binary operation '*' on Q (set of all rational numbers) be defined as $a * b = a + 2b$ for all $a, b \in$Q.

Prove that

(i) The given operation is not commutative.

(ii) The given operation is not associative.

4. Let * be the binary operation difined on $Q^{+} by\, a * b = \frac{ab}{3}$ for all $a, b \in Q^{+}$ then find the inwrse of 4*6.

5. Let $A = N \times N$ and * be the binary operation on A defined by (a,b)*(c,d)=(a+c, b+d). Show that * is commutative and associative. Find the identity element of on A if any

6. A binary operation * on Q - {-1} is defined by a * b = a+b+ab; for all $a, b \in Q - \{-1\}$. Find identity element on Q. Also find the inverse of an element in Q-{-1}.

LET US SUM UP

- Reflexive relation R in X is a relation with $(a, a) \in$R $\forall$ $a \in$X.
- Symmetric relation R in X is a relation satisfying $(a, b) \in$R implies $(b, a) \in$R.
- Transitive relation R in X is a relation satisfying $(a, b) \in$R and $(b, c) \in$R implies that $(a, c) \in$R.
- Equivalence relation R in X is a relation which is reflexive, symmetric and transitive.
- If range is a subset of co-domain that function is called on into function.
- If f: $A \rightarrow B$, and f (x) = f (y) $\rightarrow$ x = y that function is called one-one function.
- Any function is inuertible if it is one-one-onto or bijective.
- If more than one element of A has only one image in to than function is called many one function .
- A binary operation * on a set A is a function * from A × A to A.
- If $a * b = b * a$ for all $a, b \in$A, then the operation is said to be commutative.
- If $(a * b) * c = a * (b * c)$ for all $a, b, \in$A, then the operation is said to be associative.
- If $e * a = a = a * e$ for all $a \in$A, then element $e \in$A is said to be an identity element.
- If $a * b = e = b * a$ then a and b are inverse of each other
- A pair of elements grouped together in a particular order is called an a ordered pair.
- If n(A) = p, n(B) = q then n(A × B) = pq
- R × R = $\{(x, y) : x, y \in$ R} and R × R × R = $\{(x, y, z) : x, y, z \in$R}

- In a function $f : A \to B$, B is the codomain of f.
- $f, g : X \to R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$

  $(f \cdot g)x = f(x) \cdot g(x)$, $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
- A real function has the set of real number or one of its subsets both as its domain and as its range.

SUPPORTIVE WEBSITES

http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/ index.shtml
http://mathworld.wolfram.com/Composition.html
http://www.cut-the-knot.org/Curriculum/Algebra/BinaryColorDevice.shtml
http://mathworld.wolfram.com/BinaryOperation.html

TERMINAL EXERCISE

1. Write for each of the following functions fog, gof, fof, gog.

   (a) $f(x) = x^3$ $\qquad g(x) = 4x - 1$

   (b) $f(x) = \dfrac{1}{x^2}, x \neq 0$ $\qquad g(x) = x^2 - 2x + 3$

   (c) $f(x) = \sqrt{x - 4}, x \geq 4$ $\qquad g(x) = x - 4$

   (d) $f(x) = x^2 - 1$ $\qquad g(x) = x^2 + 1$

2. (a) Let $f(x) = |x|$, $g(x) = \dfrac{1}{x}, x \neq 0$, $h(x) = x^{\frac{1}{3}}$. Find fogoh

   (b) $f(x) = x^2 + 3$, $g(x) = 2x^2 + 1$

   Find fog (3) and gof (3).

3. Which of the following equations describe a function whose inverse exists :

   (a) $f(x) = |x|$ $\qquad$ (b) $f(x) = \sqrt{x}, x \geq 0$

   (c) $f(x) = x^2 - 1, x \geq 0$ $\qquad$ (d) $f(x) = \dfrac{3x - 5}{4}$ $\qquad$ (e) $f(x) = \dfrac{3x + 1}{x - 1}$ $x \neq 1$.

4. If $gof(x) = |\sin x|$ and $gof(x) = \left(\sin \sqrt{x}\right)^2$ then find f (x) and g (x)

5. Let $*$ be a binary operation on Q defined by $a * b = \frac{a+b}{3}$ for all a,b $\in Q$, prove that $*$ is commutative on Q.

6. Let $*$ be a binary operation on on the set Q of rational numbers define by $a * b = \frac{ab}{5}$ for all a,b $\in Q$, show that $*$ is associative on Q.

7. Show that the relation R in the set of real numbers, defined as R = $\{(a, b)\} : a \leq b^2\}$ is neither reflexive, nor symmetric nor transitive.

8. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric and transitive.

9. Show that the relation R in the set A defined as R = $\{(a, b) \ \forall : a = b\}$ $a, b \in$A, is equivalence relation.

10. Let A = N × N, N being the set of natural numbers. Let * : A × A → A be defined as $(a, b) * (c, d) = \{ad + bc, bd)$ for all $(a, b), (c, d) \in$A. Show that

    (i) * is commutative

    (ii) * is associative

    (iii) identity element w.r.t * does not exist.

11. Let * be a binary operation on the set N of natural numbers defined by the rule $a * b = ab$ for all $a, b \in$N

    (i) Is * commutative? (ii) Is * associative?

ANSWERS

CHECK YOUR PROGRESS 23.2

1. (i) No (ii) Yes
2. (a), (b)
3. (a),
4. (a), (c),(e)
5. (a), (b)

CHECK YOUR PROGRESS 23.3

1. $\text{fog} = \dfrac{x^2}{(1-x)^2} + 2, \ \text{gof} = \dfrac{x^2+2}{x^2+1}$

   $\text{fof} = x^4 + 4x^2 + 6, \ \text{gog} = x$

2. (a) $\text{fog} = 4x^2 + 20x + 21, \ \text{gof} = 2x^2 - 3$

   $\text{fof} = x^4 - 8x^2 + 12, \ \text{gog} = 4x + 15$

   (b) $\text{fog} = 9, \ \text{gof} = 3, \ \text{fof} = x^4, \ \text{gog} = 3$

   (c) $\text{fog} = \dfrac{6-7x}{x}, \ \text{gof} = \dfrac{2}{3x-7}, \ \text{fof} = 9x - 28, \quad \text{gog} = x$

6. (a) $\text{fog} = \left| x^{\frac{1}{3}} \right|$ (b) $\text{goh} = \dfrac{1}{x^{\frac{1}{3}}}$ (c) $\text{foh} = \left| \dfrac{1}{x} \right|$

   (d) $\text{hog} = \dfrac{1}{x^{\frac{1}{3}}}$ (e) fogoh(1) = 1

CHECK YOUR PROGRESS 23.4

1. (ii) Domain is B. Range is A.

2. (a) $f^{-1}(x) = x - 3$ (b) $f^{-1}(x) = \dfrac{1-x}{3}$

   (c) Inverse does not exist. (d) $f^{-1}(x) = \dfrac{1}{x-1}$

CHECK YOUR PROGRESS 23.5

1. (i) No (ii) Yes (iii) Yes
2. 16
4. $\frac{9}{8}$
5. (0,0)
6. $\text{identity} = 0,\ a^{-1} = \frac{-a}{a+1}$

TERMINAL EXERCISE

1. (a) $fog = (4x-1)^3,\ gof = 4x^3 - 1,\ fog = x^9,\ gog = 16x - 5$

   (b) $fog = \frac{1}{(x^2 - 2x + 3)^2},\ gof = \frac{3x^4 - 2x^2 + 1}{x^4},\ fof = x^4, gog - x^4 - 4x^3 + 4x^2$

   (c) $fog = \sqrt{x-8},\ gof = \sqrt{x-4} - 4,\ fof = \sqrt{\sqrt{x-4} - 4} gog = x - 8$

   (d) $fog = x^4 + 2x^2,\ gof = x^4 - 2x^2 + 2,\ fof = x^4 - 2x^2,\ gog = x^4 + 2x^2 + 2,$

2. (a) $\left|\frac{1}{x^{1/3}}\right|$, (b) $(fog)(3) = 364,\ (gof)(3) = 289$
3. (c), (d), (e),
4. $f(x) = \sin^2 x,\ g(x) = \sqrt{x}$
8. Neither reflexive, nor symmetric, nor transitive
9. Yes, R is an equivalence relation
11. (i) Not commutative

24

311en24

INVERSE TRIGONOMETRIC FUNCTIONS

In the previous lesson, you have studied the definition of a function and different kinds of functions. We have defined inverse function.

Let us briefly recall :

Let f be a one-one onto function from A to B.

Let y be an arbitary element of B. Then, f being onto, $\exists$ an element $x \in A$ such that $f(x) = y$. Also, f being one-one, then x must be unique. Thus for each $y \in B$, $\exists$ a unique element $x \in A$ such that $f(x) = y$. So we may define a function, denoted by f^{-1} as $f^{-1} : B \to A$

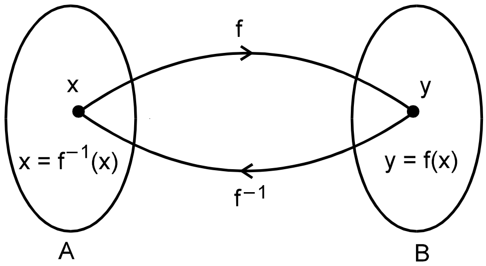


Fig. 24.1

$$\therefore \qquad f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f. A function is invertiable if and only if f is one-one onto.

In this case the domain of f^{-1} is the range of f and the range of f^{-1} is the domain f.

Let us take another example.

We define a function : f : Car $\to$ Registration No.

If we write, g : Registration No. $\to$ Car, we see that the domain of f is range of g and the range of f is domain of g.

So, we say g is an **inverse function** of f, i.e., $g = f^{-1}$.

In this lesson, we will learn more about inverse trigonometric function, its domain and range, and simplify expressions involving inverse trigonometric functions.

OBJECTIVES

After studying this lesson, you will be able to :

- define inverse trigonometric functions;
- state the condition for the inverse of trigonometric functions to exist;
- define the principal value of inverse trigonometric functions;
- find domain and range of inverse trigonometric functions;
- state the properties of inverse trigonometric functions; and
- simplify expressions involving inverse trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of function and their types, domain and range of a function
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

24.1 IS INVERSE OF EVERY FUNCTION POSSIBLE ?

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function :

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right) \text{ and } \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

f : Student $\rightarrow$ Score in Mathematics

Do you think f^{-1} will exist ?

It may or may not be because the moment two students have the same score, f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

Example 24.1 If $f : R \rightarrow R$ defined by $f(x) = x^3 + 4$. What will be f^{-1}?

Solution : In this case f is one-to-one and onto both.

$\Rightarrow$ f is invertible.

Let $\qquad y = x^3 + 4$

$\therefore \qquad y - 4 = x^3 \;\Rightarrow\; x = \sqrt[3]{y-4}$

So f^{-1}, inverse function of f i.e., $f^{-1}(y) = \sqrt[3]{y-4}$

The functions that are one-to-one and onto will be invertible.

Let us extend this to trigonometry :

Take $y = \sin x$. Here domain is the set of all real numbers. Range is the set of all real numbers lying between –1 and 1, including –1 and 1 i.e. $-1 \le y \le 1$.

We know that there is a unique value of y for each given number x.

In inverse process we wish to know a number corresponding to a particular value of the sine.

Suppose $\quad y = \sin x = \frac{1}{2}$

$$\Rightarrow \qquad \sin x = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = ...$$

x may have the values as $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6} = ...$

Thus there are infinite number of values of x.

$y = \sin x$ can be represented as

$$\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), ...$$

The inverse relation will be

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), ...$$

It is evident that it is not a function as first element of all the ordered pairs is $\frac{1}{2}$, which contradicts the definition of a function.

Consider $y = \sin x$, where $x \in R$ (domain) and $y \in [-1, 1]$ or $-1 \le y \le 1$ which is called range. This is many-to-one and onto function, therefore it is not invertible.

Can $y = \sin x$ be made invertible and how? Yes, if we restrict its domain in such a way that it becomes one-to-one and onto taking x as

(i) $\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \qquad y \in [-1, 1] \qquad$ or

(ii) $\quad \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \qquad y \in [-1, 1] \qquad$ or

(iii) $\quad -\frac{5\pi}{2} \le x \le -\frac{3\pi}{2} \qquad y \in [-1, 1] \qquad$ etc.

Now consider the inverse function $y = \sin^{-1} x$.

We know the domain and range of the function. We interchange domain and range for the inverse of the function. Therefore,

(i) $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \qquad x \in [-1, 1] \qquad$ or

(ii) $\quad \frac{3\pi}{2} \le y \le \frac{5\pi}{2} \qquad x \in [-1, 1] \qquad$ or

(iii) $-\frac{5\pi}{2} \le y \le -\frac{3\pi}{2}$ $\quad x \in [-1, 1]$ $\quad$ etc.

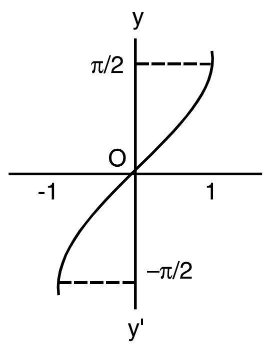
Here we take the least numerical value among all the values of the real number whose sine is x which is called the principle value of $\sin^{-1} x$.

For this the only case is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore, for principal value of $y = \sin^{-1} x$, the domain is $[-1, 1]$ i.e. $x \in [-1, 1]$ and range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

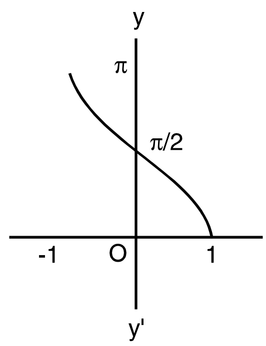
Similarly, we can discuss the other inverse trigonometric functions.

	Function	Domain	Range (Principal value)
1.	$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \mathrm{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

24.2 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS



$y = \sin^{-1} x$



$y = \cos^{-1} x$

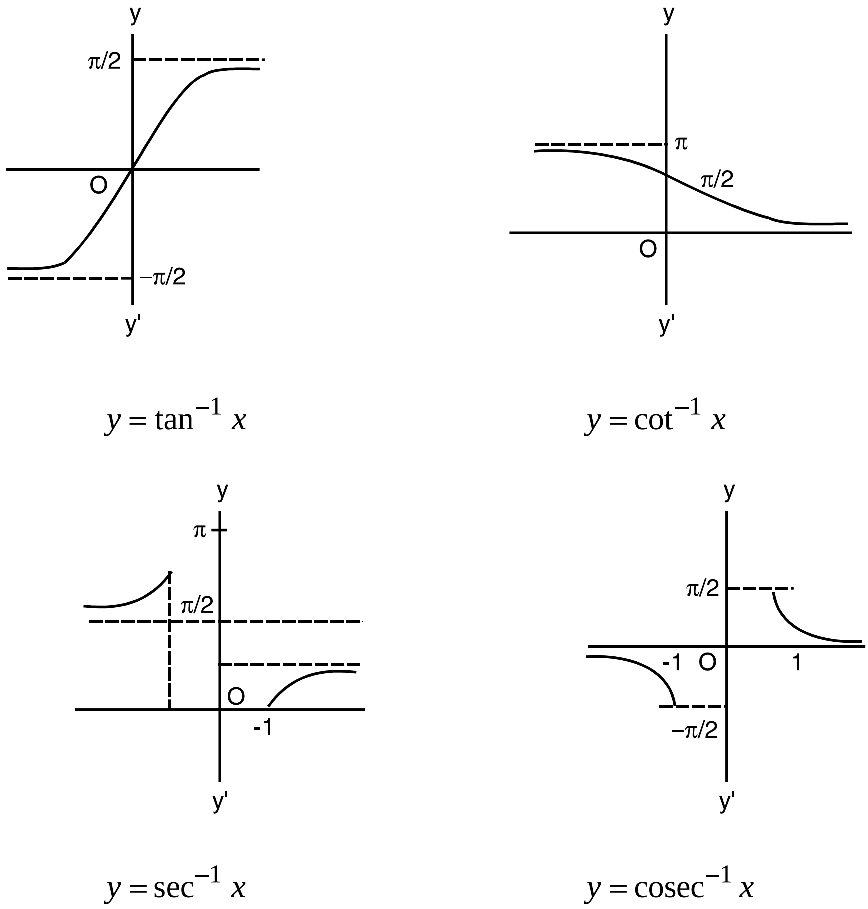


Fig. 24.2

Example 24.2 Find the principal value of each of the following :

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution : (i) Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

or $\sin\theta = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$ or $\theta = \frac{\pi}{4}$

(ii) Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$ or $\theta = \frac{2\pi}{3}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$ or $-\frac{1}{\sqrt{3}} = \tan\theta$ or $\tan\theta = \tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = -\frac{\pi}{6}$

Example 24.3 Find the principal value of each of the following :

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}(-1)$

(b) Find the value of the following using the principal value :

$$\sec\left[\cos^{-1}\frac{\sqrt{3}}{2}\right]$$

Solution : (a) (i) Let $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\theta$, then

$$\frac{1}{\sqrt{2}}=\cos\theta \quad \text{or} \quad \cos\theta=\cos\frac{\pi}{4}$$

$$\Rightarrow \quad \theta=\frac{\pi}{4}$$

(ii) Let $\tan^{-1}(-1)=\theta$, then

$$-1=\tan\theta \quad \text{or} \quad \tan\theta=\tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta=-\frac{\pi}{4}$$

(b) Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$, then

$$\frac{\sqrt{3}}{2}=\cos\theta \quad \text{or} \quad \cos\theta=\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \theta=\frac{\pi}{6}$$

$$\therefore \quad \sec\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\sec\theta=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$

Example 24.4 Simplify the following :

(i) $\cos\left(\sin^{-1}x\right)$ (ii) $\cot\left(\operatorname{cosec}^{-1}x\right)$

Solution : (i) Let $\sin^{-1}x=\theta$

$$\Rightarrow \quad x=\sin\theta$$

$$\therefore \quad \cos\left[\sin^{-1}x\right]=\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$$

(ii) Let $\operatorname{cosec}^{-1}x=\theta$

$$\Rightarrow \quad x=\operatorname{cosec}\theta$$

Also $$\cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$$

$$= \sqrt{x^2 - 1}$$

CHECK YOUR PROGRESS 24.1

1. Find the principal value of each of the following :

(a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (b) $\operatorname{cosec}^{-1}(-\sqrt{2})$ (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1}(-\sqrt{3})$ (e) $\cot^{-1}(1)$

2. Evaluate each of the following :

(a) $\cos\left(\cos^{-1}\frac{1}{3}\right)$ (b) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$ (c) $\cos\left(\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}\right)$

(d) $\tan\left(\sec^{-1}\sqrt{2}\right)$ (e) $\operatorname{cosec}\left[\cot^{-1}(-\sqrt{3})\right]$

3. Simplify each of the following expressions :

(a) $\sec\left(\tan^{-1}x\right)$ (b) $\tan\left(\operatorname{cosec}^{-1}\frac{x}{2}\right)$ (c) $\cot\left(\operatorname{cosec}^{-1}x^2\right)$

(d) $\cos\left(\cot^{-1}x^2\right)$ (e) $\tan\left(\sin^{-1}\left(\sqrt{1-x}\right)\right)$

24.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property 1 $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution : Let $\sin\theta = x$

$\Rightarrow$ $$\theta = \sin^{-1}x$$

$$= \sin^{-1}(\sin\theta) = \theta$$

Also $\sin\left(\sin^{-1}x\right) = x$

Similarly, we can prove that

(i) $\cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(ii) $\tan^{-1}(\tan\theta) = \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Property 2 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ (ii) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution : (i) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \quad x = \operatorname{cosec}\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $\cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta \quad$ or $\quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Property 3 (i) $\sin^{-1}(-x) = -\sin^{-1} x$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution : (i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \sin\theta \quad$ or $\quad x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -\theta = \sin^{-1} x \quad$ or $\quad \theta = -\sin^{-1} x$

or $\quad \sin^{-1}(-x) = -\sin^{-1} x$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \tan\theta \quad$ or $\quad x = -\tan\theta = \tan(-\theta)$

$\therefore \quad \theta = -\tan^{-1} x \quad$ or $\quad \tan^{-1}(-x) = -\tan^{-1} x$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \cos\theta \quad$ or $\quad x = -\cos\theta = \cos(\pi - \theta)$

$\therefore \quad \cos^{-1} x = \pi - \theta$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

Property 4. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Soluton : (i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

Let $\sin^{-1} x = \theta \quad \Rightarrow \quad x = \sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$

or $\qquad \cos^{-1} x = \left(\dfrac{\pi}{2} - \theta\right)$

$\Rightarrow \qquad \theta + \cos^{-1} x = \dfrac{\pi}{2} \qquad$ or $\qquad \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

(ii) Let $\cot^{-1} x = \theta \quad \Rightarrow \quad x = \cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore \qquad \tan^{-1} x = \dfrac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \tan^{-1} x = \dfrac{\pi}{2}$

or $\qquad \cot^{-1} x + \tan^{-1} x = \dfrac{\pi}{2}$

(iii) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \qquad x = \operatorname{cosec}\theta = \sec\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore \qquad \sec^{-1} x = \dfrac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \sec^{-1} x = \dfrac{\pi}{2}$

$\Rightarrow \operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$

Property 5 (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Solution : (i) Let $\tan^{-1} x = \theta$, $\tan^{-1} y = \phi \;\Rightarrow\; x = \tan\theta, y = \tan\phi$

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

By substituting that above values on L.H.S. and R.H.S., we have

$$\text{L.H.S.} = \theta + \phi \text{ and } \text{R.H.S.} = \tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$$

$$= \tan^{-1}\left[\tan(\theta+\phi)\right] = \theta + \phi = \text{L.H.S.}$$

$\therefore$ The result holds.

Simiarly (ii) can be proved.

Property 6 $\quad 2\tan^{-1}x = \sin^{-1}\left[\frac{2x}{1+x^2}\right] = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

Let $x = \tan\theta$

Substituting in (i), (ii), (iii), and (iv) we get

$$2\tan^{-1}x = 2\tan^{-1}(\tan\theta) = 2\theta \qquad \text{......(i)}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta \qquad \text{.....(ii)}$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos^2\theta - \sin^2\theta)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta \qquad \text{......(iii)}$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta \qquad \text{.....(iv)}$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Property 7

(i) $$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \operatorname{cosec}^{-1}\left[\frac{1}{x}\right]$$

(ii) $$\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \operatorname{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \sec^{-1}\left[\frac{1}{x}\right]$$

Proof : Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$

(i) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, $\sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$ and $\operatorname{cosec}\theta = \frac{1}{x}$

$\therefore \quad \sin^{-1}x = \theta = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) Let $\cos^{-1} x = \theta \;\Rightarrow\; x = \cos\theta$

$$\therefore \quad \sin\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{\sqrt{1-x^2}}{x}, \quad \sec\theta = \frac{1}{x}, \quad \cot\theta = \frac{x}{\sqrt{1-x^2}}$$

and $\operatorname{cosec}\theta = \dfrac{1}{\sqrt{1-x^2}}$

$$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

Example 24.5 Prove that

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

Solution : Applying the formula :

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we have}$$

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right) = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

Example 24.6 Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

Solution : Let $\sqrt{x} = \tan\theta$ then

$$\text{L.H.S.} = \theta \text{ and R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2}\times 2\theta = \theta$$

$\therefore$ L.H.S. = R.H.S.

Example 24.7 Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, \; x > 0$$

Solution : Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{1}{2}\theta, \ \Rightarrow \ \frac{\pi}{4}-\theta = \frac{1}{2}\theta, \ \Rightarrow \ \theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}$$

$$\therefore \quad x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 24.8 Show that

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Solution : Let $x^2 = \cos 2\theta$, then

$$2\theta = \cos^{-1}(x^2), \ \Rightarrow \ \theta = \frac{1}{2}\cos^{-1}x^2$$

Substituting $x^2 = \cos 2\theta$ in L.H.S. of the given equation, we have

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

CHECK YOUR PROGRESS 24.2

1. Evaluate each of the following :

(a) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cot(\tan^{-1}\alpha+\cot^{-1}\alpha)$

(c) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$

(d) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ (e) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

2. If $\cos^{-1} x + \cos^{-1} y = \beta$, prove that $x^2 - 2xy\cos\beta + y^2 = \sin^2\beta$

3. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$

4. Prove each of the following :

   (a) $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{2}{\sqrt{5}} = \dfrac{\pi}{2}$ (b) $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$

   (c) $\cos^{-1}\dfrac{4}{5} + \tan^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{27}{11}$ (d) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$

5. Solve the equation $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

LET US SUM UP

- Inverse of a trigonometric function exists if we restrict the domain of it.

  (i) $\sin^{-1} x = y$ if $\sin y = x$ where $-1 \le x \le 1, -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

  (ii) $\cos^{-1} x = y$ if $\cos y = x$ where $-1 \le x \le 1, 0 \le y \le \pi$

  (iii) $\tan^{-1} x = y$ if $\tan y = x$ where $x \in R, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

  (iv) $\cot^{-1} x = y$ if $\cot y = x$ where $x \in R, 0 < y < \pi$

  (v) $\sec^{-1} x = y$ if $\sec y = x$ where $x \ge 1,\ 0 \le y < \dfrac{\pi}{2}$ or $x \le -1, \dfrac{\pi}{2} < y \le \pi$

  (vi) $\operatorname{cosec}^{-1} x = y$ if $\operatorname{cosec} y = x$ where $x \ge 1, 0 < y \le \dfrac{\pi}{2}$

  or $x \le -1, -\dfrac{\pi}{2} \le y < 0$

- Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of $y = \sin x$, etc.

- **Properties :**

  (i) $\sin^{-1}(\sin\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}\theta) = \theta$ and $\sin(\sin^{-1}\theta) = \theta$

  (ii) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\dfrac{1}{x}\right)$, $\cot^{-1} x = \tan^{-1}\left(\dfrac{1}{x}\right)$, $\sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right)$

  (iii) $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

  (iv) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$

  (v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

  (vi) $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

(vii) $\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Inverse_trigonometric_functions

http://mathworld.wolfram.com/InverseTrigonometricFunctions.html

TERMINAL EXERCISE

1. Prove each of the following :

(a) $\sin^{-1}\left(\dfrac{3}{5}\right) + \sin^{-1}\left(\dfrac{8}{17}\right) = \sin^{-1}\left(\dfrac{77}{85}\right)$

(b) $\tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{1}{9}\right) = \dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{5}\right)$

(c) $\cos^{-1}\left(\dfrac{4}{5}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\left(\dfrac{27}{11}\right)$

2. Prove each of the following :

(a) $2\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \tan^{-1}\left(\dfrac{23}{11}\right)$

(b) $\tan^{-1}\left(\dfrac{1}{2}\right) + 2\tan^{-1}\left(\dfrac{1}{3}\right) = \tan^{-1}2$

(c) $\tan^{-1}\left(\dfrac{1}{8}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \tan^{-1}\left(\dfrac{1}{3}\right)$

3. (a) Prove that $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(b) Prove that $2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right)$

(c) Prove that $\cos^{-1}x = 2\sin^{-1}\left(\sqrt{\dfrac{1-x}{2}}\right) = 2\cos^{-1}\left(\sqrt{\dfrac{1+x}{2}}\right)$

4. Prove the following :

(a) $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)=\dfrac{\pi}{4}-\dfrac{x}{2}$

(b) $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)=\dfrac{\pi}{4}-x$

(c) $\cot^{-1}\left(\dfrac{ab+1}{a-b}\right)+\cot^{-1}\left(\dfrac{bc+1}{b-c}\right)+\cot^{-1}\left(\dfrac{ca+1}{c-a}\right)=0$

5. Solve each of the following :

(a) $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$

(b) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

(c) $\cos^{-1}x+\sin^{-1}\left(\dfrac{1}{2}x\right)=\dfrac{\pi}{6}$

(d) $\cot^{-1}x-\cot^{-1}(x+2)=\dfrac{\pi}{12},\ x>0$

ANSWERS

CHECK YOUR PROGRESS 24.1

1. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$ (e) $\frac{\pi}{4}$

2. (a) $\frac{1}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1 (e) -2

3. (a) $\sqrt{1+x^2}$ (b) $\frac{2}{\sqrt{x^2-4}}$ (c) $\sqrt{x^4-1}$ (d) $\frac{x^2}{\sqrt{x^4+1}}$ (e) $\sqrt{\frac{1-x}{x}}$

CHECK YOUR PROGRESS 24.2

1. (a) 1 (b) 0 (c) $\frac{x+y}{1-xy}$ (d) $\frac{5}{12}$ (e) $-\frac{7}{17}$

5. $0, \pm\frac{1}{2}$

TERMINAL EXERCISE

5. (a) $\frac{1}{6}$ (b) $\frac{\pi}{4}$ (c) ± 1 (d) $\sqrt{3}$

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25

LIMIT AND CONTINUITY

Consider the function $f(x)=\dfrac{x^2-1}{x-1}$

You can see that the function f(x) is not defined at $x = 1$ as $x-1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x-1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x+1)(x-1)}{(x-1)}=x+1$, because $x-1\neq 0$ and so division by $(x-1)$ is possible.

Table -1

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

Table - 2

x	f(x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.
However, in this case f(x) is not defined at x = 1. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function f (x) =2x. Here, we are interested to see its behavior near the point 1 and at x = 1. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at x = 1 and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near $x = 2$ and at $x = 2$?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

OBJECTIVES

After studying this lesson, you will be able to :

- define limit of a function
- derive standard limits of a function
- evaluate limit using different methods and standard limits.
- define and interprete geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

25.1 LIMIT OF A FUNCTION

In the introduction, we considered the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. We have seen that as x approaches l, $f(x)$ approaches 2. In general, if a function $f(x)$ approaches L when x approaches 'a', we say that L is the limiting value of $f(x)$

Symbolically it is written as

$$\lim_{x \to a} f(x) = L$$

Now let us find the limiting value of the function $(5x - 3)$ when x approaches 0.

i.e. $$\lim_{x \to 0} (5x - 3)$$

For finding this limit, we assign values to x from left and also from right of 0.

x	-0.1	-0.01	-0.001	-0.0001..........
$5x-3$	-3.5	-3.05	-3.005	-3.0005..........

x	0.1	0.01	0.001	0.0001..........
$5x-3$	-2.5	-2.95	-2.995	-2.9995.........

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

i.e., $$\lim_{x \to 0} (5x-3) = -3$$

This is illustrated graphically in the Fig. 20.1

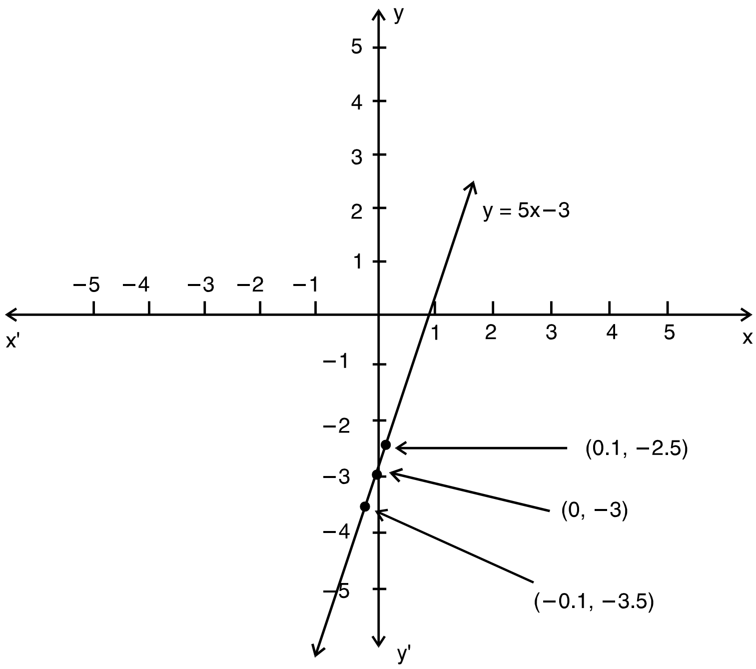


Fig. 25.1

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

Consider an example : Find $\lim_{x \to 3} f(x)$, where $f(x) = \dfrac{x^2-9}{x-3}$

We can solve it by the method of substitution. Steps of which are as follows :

Remarks : It may be noted that f (3) is not defined, however, in this case the limit of the

Step 1: We consider a value of x close to a say $x = a + h$, where h is a very small positive number. Clearly, as $x \to a$, $h \to 0$	For $f(x) = \dfrac{x^2 - 9}{x - 3}$ we write $x = 3 + h$, so that as $x \to 3, h \to 0$
Step 2 : Simplify $f(x) = f(a+h)$	Now $f(x) = f(3+h)$ $= \dfrac{(3+h)^2 - 9}{3 + h - 3}$ $= \dfrac{h^2 + 6h}{h}$ $= h + 6$
Step 3 : Put $h = 0$ and get the requried result	$\therefore \lim\limits_{x \to 3} f(x) = \lim\limits_{h \to 0} (6 + h)$ As $x \to 0$, $h \to 0$ Thus, $\lim\limits_{x \to 3} f(x) = 6 + 0 = 6$ by putting $h = 0$.

function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

Consider the example :

Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} \dfrac{x^3 - 1}{x^2 - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

Here, for $x \neq 1$, $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$

It shows that if f (x) is of the form $\dfrac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

Step 1. Factorise g (x) and h (x)	**Sol.** $f(x)=\dfrac{x^3-1}{x^2-1}$ $=\dfrac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}$ $(\because x\neq 1, \therefore x-1\neq 0$ and as such can be cancelled)
Step 2 : Simplify f (x)	$\therefore \quad f(x)=\dfrac{x^2+x+1}{x+1}$
Step 3 : Putting the value of x, we get the required limit.	$\therefore \lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{1+1+1}{1+1}=\dfrac{3}{2}$ Also $f(1)=1$(given) In this case, $\lim\limits_{x\to 1} f(x)\neq f(1)$

Thus, the limit of a function f (x) as $x \to a$ may be different from the value of the functioin at $x = a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate $\quad \lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps :

Step 1. Rationalise the factor containing square root.

Step 2. Simplify.

Step 3. Put the value of x and get the required result.

Solution :

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = 1$$

25.2 LEFT AND RIGHT HAND LIMITS

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^-$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^+$.

Thus, if a function f (x) approaches a limit ℓ_1, as x approaches 'a' from left, we say that the left hand limit of $f(x)$ as $x \to a$ is ℓ_1.

We denote it by writing

$$\lim_{x \to a^-} f(x) = \ell_1 \qquad \text{or} \qquad \lim_{h \to 0} f(a-h) = \ell_1, h > 0$$

Similarly, if f (x) approaches the limit ℓ_2, as x approaches 'a' from right we say, that the right hand limit of $f(x)$ as $x \to a$ is ℓ_2.

We denote it by writing

$$\lim_{x \to a^+} f(x) = \ell_2 \qquad \text{or} \qquad \lim_{h \to 0} f(a+h) = \ell_2, h > 0$$

Working Rules

Finding the right hand limit i.e.,

$$\lim_{x \to a^+} f(x)$$

Put $x = a + h$

Find $\lim_{h \to 0} f(a+h)$

Finding the left hand limit, i.e,

$$\lim_{x \to a^-} f(x)$$

Put $x = a - h$

Find $\lim_{h \to 0} f(a-h)$

Note : In both cases remember that h takes only positive values.

25.3 LIMIT OF FUNCTION y = f(x) AT x = a

Consider an example :

Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = x^2 + 5x + 3$

Here $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\left[(1+h)^2 + 5(1+h) + 3\right]$$

$$= \lim_{h \to 0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad .....(i)$$

and $$\lim_{x \to 1^-} f(x) = \lim_{h \to 0}\left[(1-h)^2 + 5(1-h) + 3\right]$$

$$= \lim_{x \to 0}\left[1 - 2h + h^2 + 5 - 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad .....(ii)$$

From (i) and (ii), $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

Now consider another example :

Evaluate : $$\lim_{x \to 3} \frac{|x-3|}{x-3}$$

Here $$\lim_{x \to 3^+} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3+h)-3|}{[(3+h)-3]}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} \text{(as h>0, so } |h| = h)$$

$$= 1 \qquad .....(iii)$$

and $$\lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3-h)-3|}{[(3-h)-3]}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} \quad \text{(as } h > 0, \text{ so } |-h| = h)$$

$$= -1 \qquad .....(iv)$$

$\therefore$ From (iii) and (iv), $\lim\limits_{x \to 3^+} \dfrac{|x-3|}{x-3} \neq \lim\limits_{x \to 3^-} \dfrac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit $\neq$ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x \to 1} (x^2 + 5x + 3)$ exists (which is equal to 9) and $\lim_{x \to 3} \frac{|x-3|}{x-3}$ does not exist.

Note :

I $\left.\begin{array}{l} \lim_{x \to a^+} f(x) = \ell \\ \text{and} \quad \lim_{x \to a^-} f(x) = \ell \end{array}\right\} \Rightarrow \lim_{x \to a} f(x) = \ell$

II $\left.\begin{array}{l} \lim_{x \to a^+} f(x) = \ell_1 \\ \text{and} \quad \lim_{x \to a^-} f(x) = \ell_2 \end{array}\right\} \Rightarrow \lim_{x \to a} f(x)$ does not exist.

III $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist $\Rightarrow \lim_{x \to a} f(x)$ does not exist.

25.4 BASIC THEOREMS ON LIMITS

1. $\lim_{x \to a} cx = c \lim_{x \to a} x$, c being a constant.

To verify this, consider the function $f(x) = 5x$.

We observe that in $\lim_{x \to 2} 5x$, 5 being a constant is not affected by the limit.

$$\therefore \quad \lim_{x \to 2} 5x = 5 \lim_{x \to 2} x$$
$$= 5 \times 2 = 10$$

2. $\lim_{x \to a} [g(x) + h(x) + p(x) + ...] = \lim_{x \to a} g(x) + \lim_{x \to a} h(x) + \lim_{x \to a} p(x) + ...$

where $g(x), h(x), p(x), ...$ are any function.

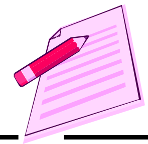
3. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

To verify this, consider $f(x) = 5x^2 + 2x + 3$

and $g(x) = x + 2$.

Then $\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x^2 + 2x + 3)$

$$= 5 \lim_{x \to 0} x^2 + 2 \lim_{x \to 0} x + 3 = 3$$



$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (x+2) = \lim_{x\to 0} x + 2 = 2$$

$\therefore$ $$\lim_{x\to 0} (5x^2+2x+3) \lim_{x\to 0} (x+2) = 6 \qquad .....(i)$$

Again $$\lim_{x\to 0} [f(x)\cdot g(x)] = \lim_{x\to 0} [(5x^2+2x+3)(x+2)]$$

$$= \lim_{x\to 0} (5x^3+12x^2+7x+6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6 \qquad .....(ii)$$

From (i) and (ii), $\lim\limits_{x\to 0} [(5x^2+2x+3)(x+2)] = \lim\limits_{x\to 0} (5x^2+2x+3) \lim\limits_{x\to 0} (x+2)$

4. $$\lim_{x\to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$ provided $\lim\limits_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2+5x+6}{x+2}$

we have $$\lim_{x\to -1} (x^2+5x+6) = (-1)^2 + 5(-1) + 6 = 1-5+6 = 2$$

and $$\lim_{x\to -1} (x+2) = -1+2 = 1$$

$$\frac{\lim\limits_{x\to -1} (x^2+5x+6)}{\lim\limits_{x\to -1} (x+2)} = \frac{2}{1} = 2 \qquad .....(i)$$

Also $$\lim_{x\to -1} \frac{(x^2+5x+6)}{x+2} = \lim_{x\to -1} \frac{(x+3)(x+2)}{x+2} \quad \left[\begin{array}{l} \because x^2+5x+6 \\ = x^2+3x+2x+6 \\ = x(x+3)+2(x+3) \\ = (x+3)(x+2) \end{array}\right]$$

$$= \lim_{x\to -1} (x+3)$$

$$= -1+3 = 2 \qquad .....(ii)$$

$\therefore$ From (i) and (ii),

Notes

$$\lim_{x \to -1} \frac{x^2+5x+6}{x+2} = \frac{\lim_{x \to -1}\left(x^2+5x+6\right)}{\lim_{x \to -1}(x+2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x \to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = m$, then

(i) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k\ell$ where k is a constant.

(ii) $\lim_{x \to a} \left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m$

(iii) $\lim_{x \to a} \left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = \ell \cdot m$

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\ell}{m}$, provided $\lim_{x \to a} g(x) \neq 0$

The above results can be easily extended in case of more than two functions.

Example 25.1 Find $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

Solution : $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = (x+1)$ $\quad [\because x \neq 1]$

$\therefore \quad \lim_{x \to 1} f(x) = \lim_{x \to 1}(x+1) = 1 + 1 = 2$

Note : $\dfrac{x^2-1}{x-1}$ is not defined at x=1. The value of $\lim_{x \to 1} f(x)$ is independent of the value of f(x) at x = 1.

Example 25.2 Evaluate : $\lim_{x \to 2} \dfrac{x^3-8}{x-2}$.

Solution : $\lim_{x \to 2} \dfrac{x^3-8}{x-2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x \to 2} \left(x^2+2x+4\right) \qquad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4 = 12$$

Example 25.3 Evaluate : $\lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x}$.

Solution : Rationalizing the numerator, we have

$$\frac{\sqrt{3-x}-1}{2-x} = \frac{\sqrt{3-x}-1}{2-x} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \frac{3-x-1}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$\therefore \qquad \lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \to 2} \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \lim_{x \to 2} \frac{1}{\left(\sqrt{3-x}+1\right)} = \frac{1}{\left(\sqrt{3-2}+1\right)} = \frac{1}{1+1} = \frac{1}{2}$$

Example 25.4 Evaluate : $\lim_{x \to 3} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$.

Solution : Rationalizing the numerator as well as the denominator, we get

$$\lim_{x \to 3} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3} = \lim_{x \to 3} \frac{\left(\sqrt{12-x}-x\right)\left(\sqrt{12-x}+x\right) \cdot \left(\sqrt{6+x}+3\right)}{\sqrt{6+x}-3\left(\sqrt{6+x}+3\right)\left(\sqrt{12-x}+x\right)}$$

$$= \lim_{x \to 3} \frac{\left(12-x-x^2\right)}{6+x-9} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

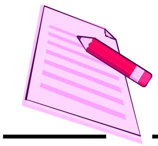
$$= \lim_{x \to 3} \frac{-(x+4)(x-3)}{(x-3)} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x} \qquad [\because x \neq 3]$$

$$= -(3+4) \cdot \frac{6}{6} = -7$$

Note : Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero.

However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :



Example 25.5 Find $\lim_{x \to 0} \frac{1}{x}$, if it exists.

Solution : We choose values of x that approach 0 from both the sides and tabulate the correspondling values of $\frac{1}{x}$.

x	−0.1	−.01	−.001	−.0001
$\frac{1}{x}$	−10	−100	−1000	−10000

x	0.1	.01	.001	.0001
$\frac{1}{x}$	10	100	1000	10000

We see that as $x \to 0$, the corresponding values of $\frac{1}{x}$ are not getting close to any number.

Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist. This is illustrated by the graph in Fig. 20.2

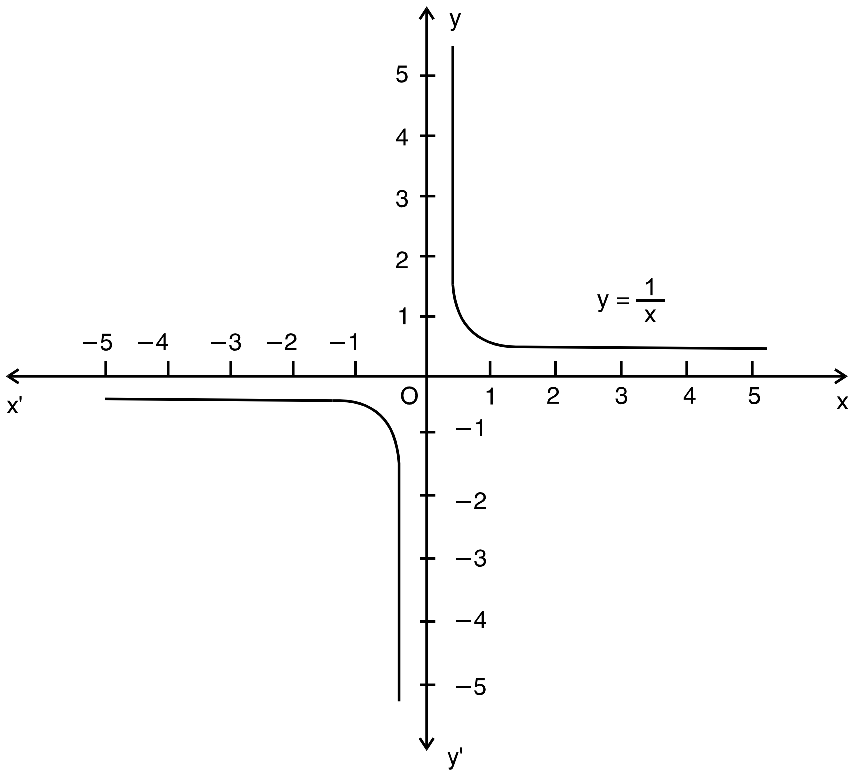


Fig. 25.2

Example 25.6 Evaluate : $\lim_{x \to 0} (|x| + |-x|)$

Solution : Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0} (|0-h| + |-(0-h)|)$$

$$= \lim_{h \to 0} (|-h| + |-(-h)|)$$

$$= \lim_{h \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad ...(i)$$

and
$$\lim_{x \to 0^+} (|x| + |-x|) = \lim_{h \to 0} (|0+h| + |-(0+h)|)$$

$$= \lim_{x \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad ...(ii)$$

From (i) and (ii),

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0^+} [|x| + |-x|]$$

Thus, $\lim_{h \to 0} [|x| + |-x|] = 0$

Note : We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

Example 25.7 Find the vlaue of 'a' so that

$\lim_{x \to 1} f(x)$ exist, where $f(x) = \begin{cases} 3x+5 & , x \leq 1 \\ 2x+a, & x > 1 \end{cases}$

Solution :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x+5) \qquad [\because f(x) = 3x+5 \text{ for } x \leq 1]$$

$$= \lim_{h \to 0} [3(1-h)+5]$$

$$= 3 + 5 = 8 \qquad .....(i)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x+a) \qquad [\because f(x) = 2x+a \text{ for } x > 1]$$

$$= \lim_{h \to 0} (2(1+h)+a)$$

$$= 2 + a \qquad .....(ii)$$

We are given that $\lim_{x \to 1} f(x)$ will exists provided

$$\Rightarrow \qquad \lim_{x \to 1^-} = \lim_{x \to 1^+} f(x)$$

$\therefore$ From (i) and (ii),

$$2 + a = 8$$

$\therefore$ or, $a = 6$

Example 25.8 If a function f (x) is defined as

Notes

$$f(x)=\begin{cases} x & , \; 0 \le x < \frac{1}{2} \\ 0 & , \; x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\lim\limits_{x \to \frac{1}{2}} f(x)$.

Solution : Here

$$f(x)=\begin{cases} x & , \; 0 \le x < \frac{1}{2} & \text{.....(i)} \\ 0 & , \; x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 & \text{.....(ii)} \end{cases}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}-h\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{2}-h\right) \quad \left[\because \frac{1}{2}-h < \frac{1}{2} \text{ and from (i)}, f\left(\frac{1}{2}-h\right) = \frac{1}{2}-h\right]$$

$$= \frac{1}{2} - 0 = \frac{1}{2} \qquad \text{.....(iii)}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}+h\right)$$

$$= \lim_{h \to 0}\left[\left(\frac{1}{2}+h\right)-1\right] \left[\because \frac{1}{2}+h > \frac{1}{2} \text{ and from (ii)}, f\left(\frac{1}{2}+h\right) = \left(\frac{1}{2}+h\right)-1\right]$$

$$= \frac{1}{2} + -1$$

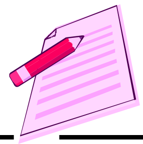
$$= -\frac{1}{2} \qquad \text{.....(iv)}$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore$ $\lim\limits_{x \to \frac{1}{2}} f(x)$ does not exist.

CHECK YOUR PROGRESS 25.1

1. Evaluate each of the following limits :



Notes

(a) $\lim\limits_{x \to 2} [2(x+3)+7]$ (b) $\lim\limits_{x \to 0} \left(x^2+3x+7\right)$ (c) $\lim\limits_{x \to 1} \left[(x+3)^2-16\right]$

(d) $\lim\limits_{x \to -1} \left[(x+1)^2+2\right]$ (e) $\lim\limits_{x \to 0} \left[(2x+1)^3-5\right]$ (f) $\lim\limits_{x \to 1} (3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim\limits_{x \to 5} \dfrac{x-5}{x+2}$ (b) $\lim\limits_{x \to 1} \dfrac{x+2}{x+1}$ (c) $\lim\limits_{x \to -1} \dfrac{3x+5}{x-10}$

(d) $\lim\limits_{x \to 0} \dfrac{px+q}{ax+b}$ (e) $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3}$ (f) $\lim\limits_{x \to -5} \dfrac{x^2-25}{x+5}$

(g) $\lim\limits_{x \to 2} \dfrac{x^2-x-2}{x^2-3x+2}$ (h) $\lim\limits_{x \to \frac{1}{3}} \dfrac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim\limits_{x \to 1} \dfrac{x^3-1}{x-1}$ (b) $\lim\limits_{x \to 0} \dfrac{x^3+7x}{x^2+2x}$ (c) $\lim\limits_{x \to 1} \dfrac{x^4-1}{x-1}$

(d) $\lim\limits_{x \to 1} \left[\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim\limits_{x \to 3} \dfrac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{1+x}-1}$ (e) $\lim\limits_{x \to 2} \dfrac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim\limits_{x \to 0} \dfrac{2}{x}$, if it exists. (b) Find $\lim\limits_{x \to 2} \dfrac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim\limits_{x \to 0} \dfrac{x}{5-|x|}$ (b) $\lim\limits_{x \to 2} \dfrac{1}{|x+2|}$ (c) $\lim\limits_{x \to 2} \dfrac{1}{|x-2|}$

(d) Show that $\lim\limits_{x \to 5} \dfrac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x) = \begin{cases} -2x+3, & x \le 1 \\ 3x-5, & x > 1 \end{cases} \text{ as } x \to 1$$

(b) If $f(x) = \begin{cases} x^2, & x \le 1 \\ 1, & x > 1 \end{cases}$, find $\lim\limits_{x \to 1} f(x)$

(c) Find $\lim_{x \to 4} f(x)$ if it exists, given that $f(x) = \begin{cases} 4x+3, & x < 4 \\ 3x+7, & x \geq 4 \end{cases}$

8. Find the value of 'a' such that $\lim_{x \to 2} f(x)$ exists, where $f(x) = \begin{cases} ax+5, & x < 2 \\ x-1, & x \geq 2 \end{cases}$

9. Let $f(x) = \begin{cases} x, & x < 1 \\ 1, & x = 1 \\ x^2, & x > 1 \end{cases}$

Establish the existence of $\lim_{x \to 1} f(x)$.

10. Find $\lim_{x \to 2} f(x)$ if it exists, where

$$f(x) = \begin{cases} x-1, & x < 2 \\ 1, & x = 2 \\ x+1, & x > 2 \end{cases}$$

25.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ where n is a positive integer.

Proof : $\lim_{x \to a} \dfrac{x^n - a^n}{x-a} = \lim_{h \to 0} \dfrac{(a+h)^n - a^n}{a+h-a}$

$$= \lim_{h \to 0} \frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots\ldots + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots\ldots + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + \ldots\ldots + 0$$

$$= n\,a^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

Note : However, the result is true for all n

(ii) Prove that (a) $\lim\limits_{x\to 0} \sin x = 0$ **and (b)** $\lim\limits_{x\to 0} \cos x = 1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0, A \to C$

or when $x \to 0, AC \to 0$

and $BC \to AB, \text{i.e.}, BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

Thus we have

$$\lim_{x\to 0} \sin x = 0 \quad \text{and} \quad \lim_{x\to 0} \cos x = 1$$

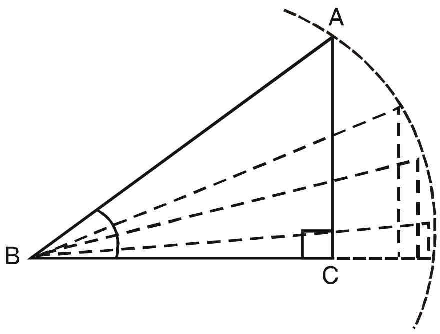


Fig. 25.3

(iii) Prove that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC \perp OX$.

Let $\angle AOX = x$ radians, where $0<x<\dfrac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of ΔOBD.

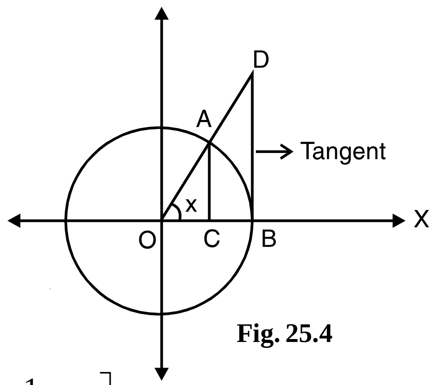


Fig. 25.4

$$\text{or} \frac{1}{2} OC \times AC < \frac{1}{2} x (1)^2 < \frac{1}{2} OB \times BD$$

$$\left[\because \text{area of triangle} = \frac{1}{2} \text{base} \times \text{height and area of sector} = \frac{1}{2} \theta\, r^2\right]$$

$$\therefore \qquad \frac{1}{2} \cos x \sin x < \frac{1}{2} x < \frac{1}{2} \cdot 1 \cdot \tan x$$

$$\left[\because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$$

i.e., $\qquad \cos x < \dfrac{x}{\sin x} < \dfrac{\tan x}{\sin x}$ [Dividing throughout by $\dfrac{1}{2}\sin x$]

or $$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or $$\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$$

i.e., $$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

Taking limit as $x \to 0$, we get

$$\lim_{x\to0} \cos x < \lim_{x\to0} \frac{\sin x}{x} < \lim_{x\to0} \frac{1}{\cos x}$$

or $$1 < \lim_{x\to0} \frac{\sin x}{x} < 1 \qquad \left[\because \lim_{x\to0} \cos x = 1 \text{ and } \lim_{x\to0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$$

Thus, $$\lim_{x\to0} \frac{\sin x}{x} = 1$$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\boldsymbol{\lim_{x\to0} (1+x)^{\frac{1}{x}} = e}$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + \text{..........}\infty\right]$$

$$= \left[1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \text{..........}\infty\right]$$

$$\therefore \lim_{x\to0} (1+x)^{\frac{1}{x}} = \lim_{x\to0} \left[1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + \text{..........}\infty\right]$$

$$= \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \text{..........}\infty\right]$$

$= e$ (By definition)

Thus $\boldsymbol{\lim_{x\to0} (1+x)^{\frac{1}{x}} = e}$

(v) Prove that

$$\boldsymbol{\lim_{x\to0} \frac{\log(1+x)}{x} = \lim_{x\to0} \frac{1}{x}\log(1+x) = \lim_{x\to0} \log(1+x)^{1/x}}$$

$$= \log e \left(\text{Using } \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \right)$$

$$= 1$$

(vi) Prove that $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Proof : We know that $\displaystyle e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\right)$

$$\therefore \quad e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots - 1\right) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$= \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots\dots\dots\right)}{x} = \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots\dots\dots\right)$$

$$\therefore \quad \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots\dots\dots\right)$$

$$= 1 + 0 + 0 + \dots\dots = 1$$

Thus, $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Example 25.9 Find the value of $\displaystyle \lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

Solution : We know that

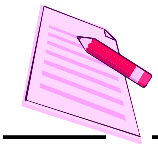
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \qquad \text{.....(i)}$$

$\therefore$ Putting $x = -x$ in (i), we get

$$\lim_{x\to 0} \frac{e^{-x} - 1}{-x} = 1 \qquad \text{.....(ii)}$$

Given limit can be written as

$$\lim_{x\to 0} \frac{e^x - 1 + 1 - e^{-x}}{x} \qquad \text{[Adding (i) and (ii)]}$$



$$= \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{1-e^{-x}}{x}\right] = \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right]$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{e^{-x}-1}{-x} = 1+1 = 2 \qquad \text{[Using (i) and (ii)]}$$

Thus $\quad \lim_{x\to 0}\dfrac{e^x-e^{-x}}{x}=2$

Example 25.10 Evaluate : $\lim_{x\to 1}\dfrac{e^x-e}{x-1}$.

Solution : Put $x = 1 + h$, where $h \to 0$

$$\lim_{x\to 1}\frac{e^x-e}{x-1} = \lim_{h\to 0}\frac{e^{1+h}-e}{h} = \lim_{h\to 0}\frac{e^1\cdot e^h-e}{h} = \lim_{h\to 0}\frac{e(e^h-1)}{h}$$

$$= e\lim_{h\to 0}\frac{e^h-1}{h} = e \times 1 = e.$$

Thus, $\quad \lim_{x\to 1}\dfrac{e^x-e}{x-1}=e$

Example 25.11 Evaluate : $\lim_{x\to 0}\dfrac{\sin 3x}{x}$.

Solution : $\quad \lim_{x\to 0}\dfrac{\sin 3x}{x} = \lim_{x\to 0}\dfrac{\sin 3x}{3x}\cdot 3 \qquad$ [Multiplying and dividing by 3]

$$= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} \qquad [\because \text{ when } x\to 0, 3x\to 0]$$

$$= 3.1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$= 3$$

Thus, $\quad \lim_{x\to 0}\dfrac{\sin 3x}{x}=3$

Example 25.12 Evaluate $\lim_{x\to 0}\dfrac{1-\cos x}{2x^2}$.

Solution : $\quad \lim_{x\to 0}\dfrac{1-\cos x}{2x^2} = \lim_{x\to 0}\dfrac{2\sin^2\frac{x}{2}}{2x^2} \qquad \left[\begin{array}{l}\because \cos 2x = 1-2\sin^2 x,\\ \therefore 1-\cos 2x = 2\sin^2 x\\ \text{or } 1-\cos x = 2\sin^2\frac{x}{2}\end{array}\right]$

$$= \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{2 \times \frac{x}{2}} \right)^2$$ [Multiplying and dividing the denominator by 2]

$$= \frac{1}{4} \lim_{\frac{x}{2} \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \frac{1}{4} \times 1 = \frac{1}{4}$$

$$\therefore \quad \lim_{x \to 0} \frac{1 - \cos x}{2x^2} = \frac{1}{4}$$

Example 25.13 Find the value of $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$.

Solution : Put $x = \frac{\pi}{2} + h$ $\quad \because$ when $x \to \frac{\pi}{2}, h \to 0$

$$\therefore \quad 2x = \pi + 2h$$

$$\therefore \quad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{[\pi - (\pi + 2h)]^2}$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2} = \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 h}{4h^2} = \frac{1}{2} \lim_{h \to 0} \left(\frac{\sin h}{h} \right)^2 = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\therefore \quad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \frac{1}{2}$$

$$= \frac{a}{b}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

CHECK YOUR PROGRESS 25.2

1. Evaluate each of the following :

(a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$ (b) $\lim\limits_{x\to 0}\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

2. Find the value of each of the following :

(a) $\lim\limits_{x\to 1}\dfrac{e^{-x}-e^{-1}}{x-1}$ (b) $\lim\limits_{x\to 1}\dfrac{e-e^{x}}{x-1}$

3. Evaluate the following :

(a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{2x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sin x^2}{5x^2}$ (c) $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x}$

(d) $\lim\limits_{x\to 0}\dfrac{\sin a x}{\sin b x}$

4. Evaluate each of the following :

(a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ (b) $\lim\limits_{x\to 0}\dfrac{1-\cos 8x}{x}$ (c) $\lim\limits_{x\to 0}\dfrac{\sin 2x(1-\cos 2x)}{x^3}$

(d) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{3\tan^2 x}$

5. Find the values of the following :

(a) $\lim\limits_{x\to 0}\dfrac{1-\cos a x}{1-\cos b x}$ (b) $\lim\limits_{x\to 0}\dfrac{x^3\cot x}{1-\cos x}$ (c) $\lim\limits_{x\to 0}\dfrac{\operatorname{cosec} x-\cot x}{x}$

6. Evaluate each of the following :

(a) $\lim\limits_{x\to \pi}\dfrac{\sin x}{\pi - x}$ (b) $\lim\limits_{x\to 1}\dfrac{\cos\frac{\pi}{2}x}{1-x}$ (c) $\lim\limits_{x\to \frac{\pi}{2}}\left(\sec x-\tan x\right)$

7. Evaluate the following :

(a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$ (b) $\lim\limits_{\theta\to 0}\dfrac{\tan 7\theta}{\sin 4\theta}$ (c) $\lim\limits_{x\to 0}\dfrac{\sin 2x+\tan 3x}{4x-\tan 5x}$

25.6 CONTINUITY OF A FUNCTION AT A POINT

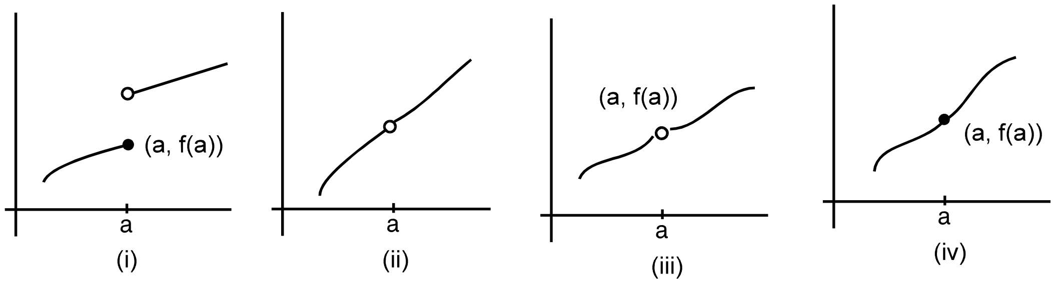


Fig. 25.5

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

Example 25.14 Examine the continuity of the function $f(x) = x - a$ at x = a.

Solution : $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} [(a+h) - a]$$

$$= 0 \qquad \text{.....(i)}$$

Also $\qquad f(a) = a - a = 0 \qquad$ .....(ii)

From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

Thus f (x) is continuous at x = a.

Example 25. 15 Show that f (x) = c is continuous.

Solution : The domain of constant function c is R.Let 'a' be any arbitrary real number.

$\therefore \qquad \lim\limits_{x \to a} f(x) = c$ and $f(a) = c$

$\therefore \qquad \lim\limits_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

MODULE - VIII
Calculus

Notes

Example 25.16 Show that $f(x) = cx + d$ is a continuous function.

Solution : The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to a}[c(a+h)+d]$$

$$= ca + d \qquad .....(i)$$

Also $\qquad f(a) = ca + d \qquad .....(ii)$

From (i) and (ii), $\lim_{x\to a} f(x) = f(a)$

$\therefore$ $\quad$ f (x) is continuous at x = a

and $\quad$ since a is any arbitrary , $\quad$ f (x) is a continuous function.

Example 25.17 Prove that $f(x) = \sin x$ is a continuous function.

Solution : Let f (x) = sin x

The domain of sin x is R. let 'a' be any arbitrary real number.

$$\therefore \qquad \lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} \sin(a+h)$$

$$= \lim_{h\to 0} [\sin a.\cos h + \cos a.\sin h]$$

$$= \sin a \lim_{h\to 0} \cos h + \cos a \lim_{h\to 0} \sin h \qquad \left[\because \lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x\to 0} \sin x = 0 \text{ and } \lim_{x\to 0} \cos x = 1\right]$$

= sin a $\qquad$ .....(i)

Also f (a) = sin a $\qquad$ .....(ii)

From (i) and (ii), $\lim_{x\to a} f(x) = f(a)$

$\therefore$ sin x is continuous at x = a

$\because$ sin x is continuous at x = a and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

Definition :

1. A function f (x) is said to be continuous in an open inteval]a,b[if it is continuous at every point of]a,b[*.

2. A function f (x) is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval]a,b[and is continuous at the point a from the right and continuous at b from the left.

i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$

and $\lim\limits_{x \to b^-} f(x) = f(b)$

* *In the open interval]a,b[we do not consider the end points a and b.*

CHECK YOUR PROGRESS 25.3

1. Examine the continuity of the functions given below :

   (a) $f(x) = x - 5$ at $x = 2$ (b) $f(x) = 2x + 7$ at $x = 0$

   (c) $f(x) = \frac{5}{3}x + 7$ at $x = 3$ (d) $f(x) = px + q$ at $x = -q$

2. Show that f (x)=2a+3b is continuous, where a and b are constants.

3. Show that $5x + 7$ is a continuous function

4. (a) Show that cos x is a continuous function.

   (b) Show that cot x is continuous at all points of its domain.

5. Find the value of the constants in the functions given below :

   (a) $f(x) = px - 5$ and $f(2) = 1$ such that f (x) is continuous at x = 2.

   (b) $f(x) = a + 5x$ and $f(0) = 4$ such that f (x) is continuous at x = 0.

   (c) $f(x) = 2x + 3b$ and $f(-2) = \frac{2}{3}$ such that f(x) is continuous at $x = -2$.

25.7 DISCONTINUITY OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example 25.18 Show that the function $f(x) = e^x$ is a continuous function.

Solution : Domain of e^x is R. Let $a \in R$. where 'a' is arbitrary.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)\text{, where h is a very small number.}$$

$$= \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h = e^a \lim_{h \to 0} e^h = e^a \times 1 \qquad \text{.....(i)}$$

$$= e^a \qquad \text{.....(ii)}$$

Also $f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim\limits_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

Example 25.19 By means of graph discuss the continuity of the function $f(x)=\dfrac{x^2-1}{x-1}$.

Solution : The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

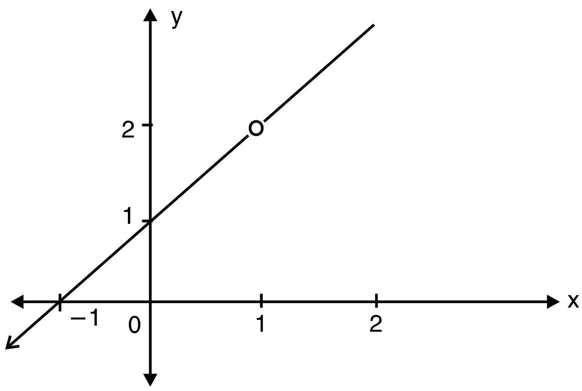


Fig. 25.6

CHECK YOUR PROGRESS 25.4

1. (a) Show that $f(x)=e^{5x}$ is a continuous function.

   (b) Show that $f(x)=e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x)=e^{3x+2}$ is a continuous function.

   (d) Show that $f(x)=e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) f (x) = x+1. (b) $f(x)=\dfrac{x+2}{x-2}$

   (c) $f(x)=\dfrac{x^2-9}{x+3}$ (d) $f(x)=\dfrac{x^2-16}{x-4}$

25.8 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function f (x) = 4. Graph of the function f (x) = 4 is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

(ii) If a function is continuous then the constant multiple of that function is also continuous.

Consider the function $f(x) = \frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \frac{7}{2}(a+h)$$

$$= \frac{7}{2}a \qquad .....(i)$$

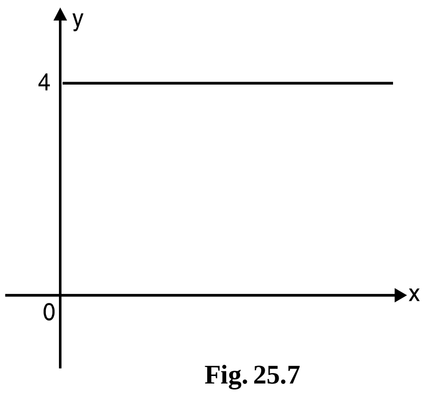


Fig. 25.7

Also $\qquad f(a) = \frac{7}{2}a \qquad .....(ii)$

$\therefore$ From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f(x) = \frac{7}{2}x$ is continuous at $x = a$.

As $\frac{7}{2}$ is constant, and x is continuous function at $x = a$, $\frac{7}{2}x$ is also a continuous function at $x = a$.

(iii) Consider the function $f(x) = x^2 + 2x$. We know that the function x^2 and 2x are continuous.

Now $\qquad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \left[(a+h)^2 + 2(a+h)\right]$$

$$= \lim_{h \to 0} \left[a^2 + 2ah + h^2 + 2a + 2ah\right]$$

$$= a^2 + 2a \qquad .....(i)$$

Also $\qquad f(a) = a^2 + 2a \qquad .....(ii)$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f(x) is continuous at $x = a$.

Thus we can say that if x^2 and 2x are two continuous functions at $x = a$ then $(x^2 + 2x)$ is also continuous at $x = a$.

(iv) Consider the function $f(x) = (x^2+1)(x+2)$. We know that (x^2+1) and $(x+2)$ are two continuous functions.

Also $\qquad f(x) = (x^2+1)(x+2)$

$$= x^3 + 2x^2 + x + 2$$

As $x^3, 2x^2, x$ and 2 are continuous functions, therefore.

$$x^3 + 2x^2 + x + 2 \text{ is also a continuous function.}$$

$\therefore$ We can say that if $\left(x^2 + 1\right)$ and (x+2) are two continuous functions then $\left(x^2 + 1\right)(x+2)$ is also a continuous function.

(v) Consider the function $f(x) = \dfrac{x^2 - 4}{x + 2}$ at $x = 2$. We know that $\left(x^2 - 4\right)$ is continuous at $x = 2$. Also $(x + 2)$ is continuous at $x = 2$.

Again
$$\lim_{x\to 2} \frac{x^2 - 4}{x + 2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x+2}$$

$$= \lim_{x\to 2} (x - 2)$$

$$= 2 - 2 = 0$$

Also
$$f(2) = \frac{(2)^2 - 4}{2 + 2}$$

$$= \frac{0}{4} = 0$$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$. Thus $f(x)$ is continuous at $x = 2$.

$\therefore$ If $x^2 - 4$ and $x + 2$ are two continuous functions at $x = 2$, then $\dfrac{x^2 - 4}{x + 2}$ is also continuous.

(vi) Consider the function $f(x) = |x - 2|$. The function can be written as

$$f(x) = \begin{cases} -(x-2), & x < 2 \\ (x-2), & x \geq 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2 - h), \; h > 0$$

$$= \lim_{h\to 0} \left[(2 - h) - 2\right]$$

$$= 2 - 2 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2 + h), \; h > 0 \qquad \text{......(i)}$$

$$= \lim_{x\to 2} \left[(2 + h) - 2\right]$$

$$= 2 - 2 = 0 \qquad \text{.....(ii)}$$

Also
$$f(2) = (2 - 2) = 0 \qquad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim_{x\to 2} f(x) = f(2)$

Thus, $|x - 2|$ is continuous at $x = 2$.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) $f(x) \cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

Note : Every constant function is continuous.

25.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$ (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} (px + q) = pa + q \quad [= \text{value of } p\ x + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + a_n x^n$

where $a_0, a_1, a_2 .....a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1 x, a_2 x^2, ...a_n x^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., (x – 5) is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\frac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at $x = 5$.

In general if $f(x) = \frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and $q(x) \neq 0$, then f (x) is continuous if p (x) and q (x) both are continuous.

Example 25.20 Examine the continuity of the following function at $x = 2$.

$$f(x) = \begin{cases} 3x-2 & \text{for } x < 2 \\ x+2 & \text{for } x \geq 2 \end{cases}$$

Solution : Since f (x) is defined as the polynomial function $3x-2$ on the left hand side of the point $x = 2$ and by another polynomial function $x+2$ on the right hand side of $x = 2$, we shall find the left hand limit and right hand limit of the function at $x = 2$ separately.

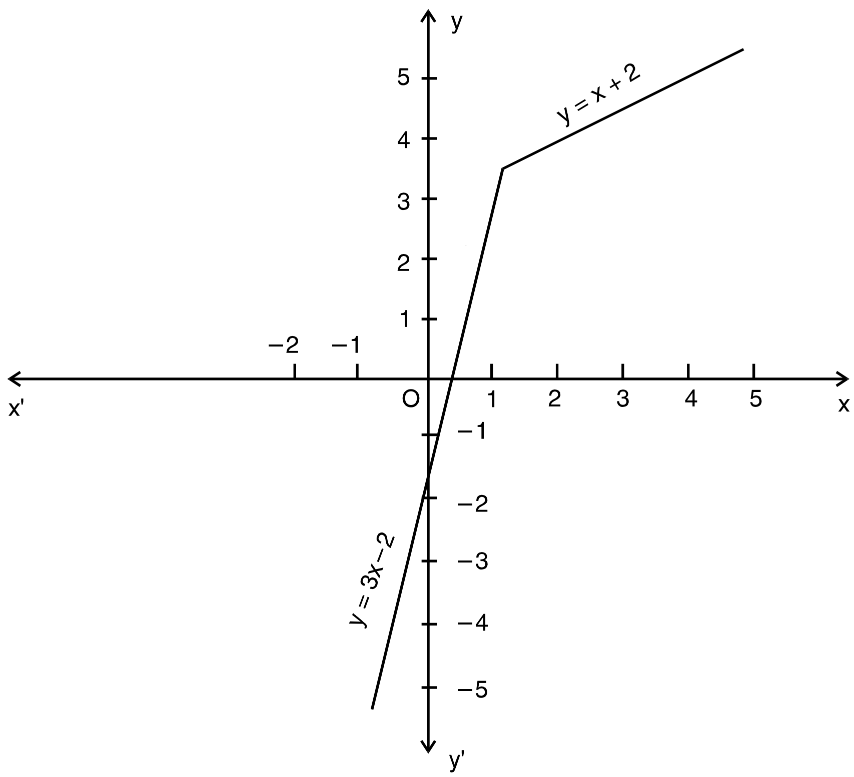


Fig. 25.8

Left hand limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2} (3x-2) = 3 \times 2 - 2 = 4$

Right hand limit at $x = 2$;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x+2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x \to 2} f(x) = 4$.

Also f (x) is defined by (x +2) at x = 2

$\therefore$ $$f(2) = 2 + 2 = 4.$$

Thus, $$\lim_{x \to 2} f(x) = f(2)$$

Hence f (x) is continuous at x = 2.

Example 25.21

(i) Draw the graph of f (x) = |x|.

(ii) Discusss the continuity of f (x) at x = 0.

Solution : We know that for $x \ge 0, |x| = x$ and for $x < 0, |x| = -x$. Hence f (x) can be written as.

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

(i) The graph of the function is given in Fig 20.9

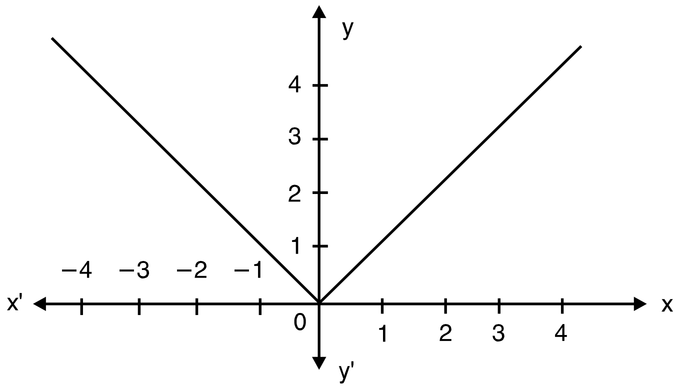


Fig. 25.9

(ii) Left hand limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

Thus, $$\lim_{x \to 0} f(x) = 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0} f(x) = f(0)$$

Notes

Hence the function $f(x)$ is continuous at $x = 0$.

Example 25.22 Examine the continuity of $f(x) = |x - b|$ at $x = b$.

Solution : We have $f(x) = |x - b|$. This function can be written as

$$f(x) = \begin{cases} -(x-b), & x < b \\ (x-b), & x \geq b \end{cases}$$

Left hand limit $= \lim\limits_{x \to b^-} f(x) = \lim\limits_{h \to 0} f(b-h)$

$= \lim\limits_{h \to 0} [-(b-h-b)]$

$= \lim\limits_{h \to 0} h = 0$ .....(i)

Right hand limit $= \lim\limits_{x \to b^+} f(x) = \lim\limits_{h \to 0} f(b+h)$

$= \lim\limits_{h \to 0} [(b+h)-b]$

$= \lim\limits_{h \to 0} h = 0$ .....(ii)

Also, $f(b) = b - b = 0$ .....(iii)

From (i), (ii) and (iii), $\lim\limits_{x \to b} f(x) = f(b)$

Thus, f(x) is continuous at $x = b$.

Example 25.23 If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

find whether f(x) is continuous at $x = 0$ or not.

Solution : Here $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \lim\limits_{x \to 0^-} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0-h)}{0-h} = \lim\limits_{h \to 0} \dfrac{-\sin 2h}{-h}$

$= \lim\limits_{h \to 0} \left(\dfrac{\sin 2h}{2h} \times \dfrac{2}{1} \right) = 1 \times 2 = 2$ .....(i)

Right hand limit $= \lim\limits_{x \to 0^+} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0+h)}{0+h} = \lim\limits_{h \to 0} \dfrac{\sin 2h}{2h} \times \dfrac{2}{1}$

$= 1 \times 2 = 2$ (ii)

Also $f(0) = 2$ (Given) (iii)

From (i) to (iii),

$$\lim_{x\to 0} f(x) = 2 = f(0)$$

Hence f(x) is continuous at x = 0.

Signum Function : The function f (x)=sgn(x) (read as signum x) is defined as

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Find the left hand limit and right hand limit of the function from its graph given below:

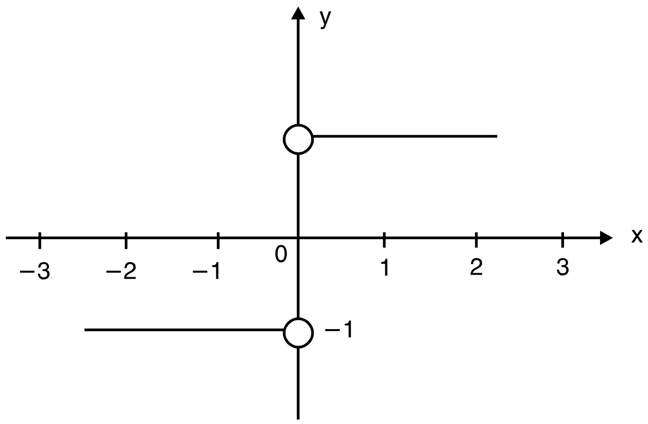


Fig. 25.11

From the graph, we see that as $x \to 0^+$, $f(x) \to 1$ and as $(x) \to 0^-$, $f(x) \to -1$

Hence, $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$

As these limits are not equal, $\lim_{x\to 0} f(x)$ does not exist. Hence f (x) is discontinuous at x =0.

Greatest Integer Function : Let us consider the function f (x)=[x] where [x] denotes the greatest integer less than or equal to x. Find whether f (x) is continuous at

(i) $x = \frac{1}{2}$ (ii) x =1

To solve this, let us take some arbitrary values of x say 1.3, 0.2,–0.2..... By the definition of greatest integer function,

$$[1.3] = 1, [1.99] = 1, [2] = 2, [0.2] = 0, [-0.2] = -1, [-3.1] = -4, \text{ etc.}$$

In general :

for $-3 \le x < -2$, $[x] = -3$

for $-2 \le x < -1$, $[x] = -2$

for $-1 \le x < 0$, $[x] = -1$

for $0 \le x < 1$, $[x] = 0$

for $1 \le x < 2$, $[x]=1$ and so on.

The graph of the function f (x) = [x] is given in Fig. 25.12

(i) From graph

$$\lim_{x\to\frac{1}{2}^-} f(x) = 0, \quad \lim_{x\to\frac{1}{2}^+} f(x) = 0,$$

$\therefore$ $$\lim_{x\to\frac{1}{2}} f(x) = 0$$

Also $$f\left(\frac{1}{2}\right) = [0.5] = 0$$

Thus $$\lim_{x\to\frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at

$$x = \frac{1}{2}$$

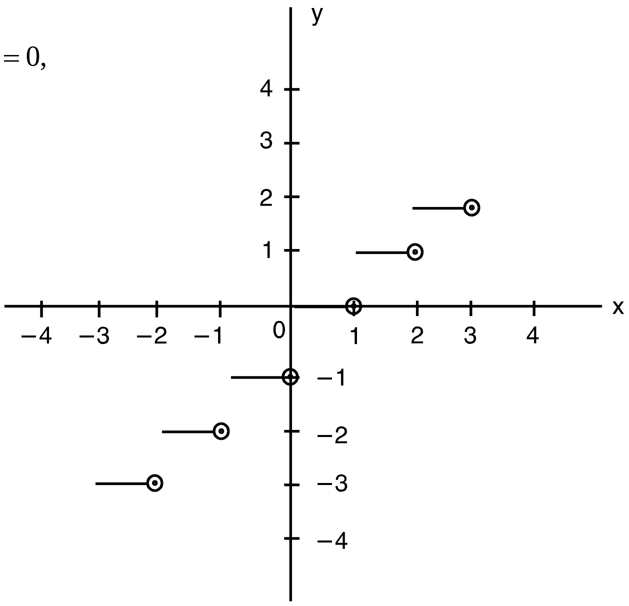


Fig. 25.12

(ii) $$\lim_{x\to 1^-} f(x) = 0, \quad \lim_{x\to 1^+} f(x) = 1$$

Thus $\lim_{x\to 1} f(x)$ does not exist.

Hence, f (x) is discontinuous at x = 1.

Note : The function f (x) = [x] is also known as Step Function.

Example 25.24 At what points is the function $\frac{x-1}{(x+4)(x-5)}$ continuous?

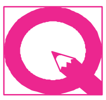
Solution : Here $f(x) = \dfrac{x-1}{(x+4)(x-5)}$

The function in the numerator i.e., x – 1 is continuous. The function in the demoninator is (x+4)(x – 5) which is also continuous.

But f (x) is not defined at the points – 4 and 5.

$\therefore$ The function f (x) is continuous at all points except – 4 and 5 at which it is not defined.

In other words, f (x) is continuous at all points of its domain.



CHECK YOUR PROGRESS 25.5

1. (a) If $f(x) = 2x + 1$, when $x \neq 1$ and f(x)=3 when x = 1, show that the function f (x) continuous at x =1.

(b) If $f(x)=\begin{cases}4x+3, & x\neq 2\\ 3x+5, & x=2\end{cases}$, find whether the function f is continuous at x = 2.

(c) Determine whether f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}4x+3, & x\leq 2\\ 8-x, & x>2\end{cases}$$

(d) Examine the continuity of f (x) at x = 1, where

$$f(x)=\begin{cases}x^2, & x\leq 1\\ x+5, & x>1\end{cases}$$

(e) Determine the values of k so that the function

$$f(x)=\begin{cases}kx^2, & x\leq 2\\ 3, & x>2\end{cases} \text{ is con tinuous at } x=2.$$

2. Examine the continuity of the following functions :

(a) $f(x)=|x-2|$ at $x=2$ (b) $f(x)=|x+5|$ at $x=-5$

(c) $f(x)=|a-x|$ at $x=a$

(d) $f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$ at x = 2

(e) $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x\neq a\\ 1, & x=a\end{cases}$ at x = a

3. (a) If $f(x)=\begin{cases}\sin 4x, & x\neq 0\\ 2, & x=0\end{cases}$, at x = 0

(b) If $f(x)=\begin{cases}\dfrac{\sin 7x}{x}, & x\neq 0\\ 7, & x=0\end{cases}$, at x = 0

(c) For what value of a is the function

$$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & x\neq 0\\ a, & x=0\end{cases} \text{ continuous at } x=0\ ?$$

4. (a) Show that the function f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, & \text{for } x\neq 2\\ 3, & \text{for } x=2\end{cases}$$

(b) Test the continuity of the function f (x) at x = 1, where

$$f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-1} & \text{for } x\neq 1\\ -2 & \text{for } x=1\end{cases}$$

(c) For what value of k is the following function continuous at x = 1?

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{when } x\neq 1\\ k & \text{when } x=1\end{cases}$$

(d) Discuss the continuity of the function f (x) at x = 2, when

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 7, & x=2\end{cases}$$

5. (a) If $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, find whether f is continuous at x = 0.

(b) Test the continuity of the function f (x) at the origin.

where $$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$$

6. Find whether the function f (x)=[x] is continuous at

(a) $x=\dfrac{4}{3}$ (b) x = 3 (c) $x=-1$ (d) $x=\dfrac{2}{3}$

7. At what points is the function f (x) continuous in each of the following cases ?

(a) $f(x)=\dfrac{x+2}{(x-1)(x-4)}$ (b) $f(x)=\dfrac{x-5}{(x+2)(x-3)}$ (c) $f(x)=\dfrac{x-3}{x^2+5x-6}$

(d) $f(x)=\dfrac{x^2+2x+5}{x^2-8x+16}$

LET US SUM UP

- If a function f(x) approaches l when x approches a, we say that l is the limit of f (x). Symbolically, it is written as

$$\lim_{x\to a} f(x)=\ell$$

- If $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$, then

(i) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k\ell$

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m$

(iii) $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = \ell m$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{\ell}{m}$, provided $\lim_{x \to a} g(x) \neq 0$

- **LIMIT OF IMPORTANT FUNCTIONS**

(i) $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \sin x = 0$

(iii) $\lim_{x \to 0} \cos x = 1$

(iv) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

(v) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

(vi) $\lim_{x \to 0} \dfrac{\log(1 + x)}{x} = 1$

(vii) $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=HB8CzZEd4xw

http://www.zweigmedia.com/RealWorld/Calcsumm3a.html

http://www.intuitive-calculus.com/limits-and-continuity.html

TERMINAL EXERCISE

Evaluate the following limits :

1. $\lim_{x \to 1} 5$

2. $\lim_{x \to 0} \sqrt{2}$

3. $\lim_{x \to 1} \dfrac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$

4. $\lim_{x \to -2} \dfrac{x^2 + 2x}{x^3 + x^2 - 2x}$

5. $\lim_{x \to 0} \dfrac{(x + k)^4 - x^4}{k(k + 2x)}$

6. $\lim_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$

7. $\lim_{x\to -1}\left[\frac{1}{x+1}+\frac{2}{x^2-1}\right]$

8. $\lim_{x\to 1}\frac{(2x-3)\sqrt{x}-1}{(2x+3)(x-1)}$

9. $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

10. $\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{x^2-1}\right]$

11. $\lim_{x\to \pi}\frac{\sin x}{\pi - x}$

12. $\lim_{x\to a}\frac{x^2-(a+1)x+a^2}{x^2-a^2}$

Find the left hand and right hand limits of the following functions :

13. $f(x)=\begin{cases}-2x+3 \text{ if } x\le 1\\ 3x-5 \quad \text{if } x>1\end{cases}$ as $x\to 1$

14. $f(x)=\frac{x^2-1}{|x+1|}$ as $x\to 1$

Evaluate the following limits :

15. $\lim_{x\to 1^-}\frac{|x+1|}{x+1}$

16. $\lim_{x\to 2^+}\frac{|x-2|}{x-2}$

17. $\lim_{x\to 2^-}\frac{x-2}{|x-2|}$

18. If $f(x)=\frac{(x+2)^2-4}{x}$, prove that $\lim_{x\to 0} f(x)=4$ though $f(0)$ is not defined.

19. Find k so that $\lim_{x\to 2} f(x)$ may exist where $f(x)=\begin{cases}5x+2, x\le 2\\ 2x+k, x>2\end{cases}$

20. Evaluate $\lim_{x\to 0}\frac{\sin 7x}{2x}$

21. Evauate $\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2}{x^2}\right]$

22. Evaluate $\lim_{x\to 0}\frac{1-\cos 3x}{x^2}$

23. Find the value of $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\sin 3x}$

24. Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

25. Evaluate $\lim_{\theta \to 0} \frac{\sin 5\theta}{\tan 8\theta}$

Examine the continuity of the following :

26. $f(x)\begin{cases} 1+3x \text{ if } x > -1 \\ 2 \text{ if } x \le -1 \end{cases}$

at $x = -1$

27. $f(x) = \begin{cases} \frac{1}{x} - x, 0 < x < \frac{1}{2} \\ \frac{1}{2}, x = \frac{1}{2} \\ \frac{3}{2} - x, \frac{1}{2} < x < 1 \end{cases}$

at $x = \frac{1}{2}$

28. For what value of k, will the function

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} \text{ if } x \ne 4 \\ k \text{ if } x = 4 \end{cases}$$

be continuous at x = 4 ?

29. Determine the points of discontinuity, if any, of the following functions :

(a) $\frac{x^2+3}{x^2+x+1}$ (b) $\frac{4x^2+3x+5}{x^2-2x+1}$

(b) $\frac{x^2+x+1}{x^2-3x+1}$ (d) $f(x) = \begin{cases} x^4 - 16, & x \ne 2 \\ 16, & x = 2 \end{cases}$

30. Show that the function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos, & x \ne 0 \\ 2, & x = 0 \end{cases}$ is continuous at x = 0

31. Determine the value of 'a', so that the function f (x) defined by

$$f(x) = \begin{cases} \frac{a \cos x}{\pi - 2x}, x \ne \frac{\pi}{2} \\ 5, \quad x = \frac{\pi}{2} \end{cases}$$ is continuous.

ANSWERS

CHECK YOUR PROGRESS 25.1

1. (a) 17 (b) 7 (c) 0 (d) 2
(e)- 4 (f) 8

2. (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{2}{11}$ (d) $\frac{q}{b}$ (e) 6
(f) -10 (g) 3 (h) 2

3. (a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{1}{2}$

4. (a) $\frac{1}{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2\sqrt{6}}$ (d) 2 (e) -1

5. (a) Does not exist (b) Does not exist

6. (a) 0 (b) $\frac{1}{4}$ (c) does not exist

7. (a) $1, -2$ (b) 1 (c) 19

8. $a = -2$

10. limit does not exist

CHECK YOUR PROGRESS 25.2

1. (a) 2 (b) $\frac{e^2-1}{e^2+1}$

2. (a) $-\frac{1}{e}$ (b) $-e$

3. (a) 2 (b) $\frac{1}{5}$ (c) 0 (d) $\frac{a}{b}$

4. (a) $\frac{1}{2}$ (b) 0 (c) 4 (d) $\frac{2}{3}$

5. (a) $\frac{a^2}{b^2}$ (b) 2 (c) $\frac{1}{2}$

6. (a) 1 (b) $\frac{\pi}{2}$ (c) 0

7. (a) $\frac{5}{3}$ (b) $\frac{7}{4}$ (c) -5

CHECK YOUR PROGRESS 25.3

1. (a) Continuous (b) Continuous

   (c) Continuous (d) Continuous

5. (a) p =3 (b) a = 4

   (c) $b = \frac{14}{9}$

CHECK YOUR PROGRESS 25.4

2. (a) Continuous

   (b) Discontinuous at x = 2

   (c) Discontinuous at $x = -3$

   (d) Discontinuous at x = 4

CHECK YOUR PROGRESS 25.5

1. (b) Continuous (c) Discontinuous

   (d) Discontinuous (e) $k = \frac{3}{4}$

2 (a) Continuous (c) Continuous,

   (d) Discontinuous (e) Discontinuous

3 (a) Discontinuous (b) Continuous (c) $\frac{5}{3}$

4 (b) Continuous (c) k = 2

   (d) Discontinuous

5. (a) Discontinuous (b) Discontinuous

6 (a) Continuous (b) Discontinuous

   (c) Discontinuous (d) Continuous

7. (a) All real number except 1 and 4

   (b) All real numbers except –2 and 3

   (c) All real number except –6 and 1

   (d) All real numbers except 4

TERMINAL EXERCISE

1. 5 2. $\sqrt{2}$ 3. 4 4. $-\frac{1}{3}$

5. $2x^2$ 6. 1 7. $-\frac{1}{2}$ 8. $-\frac{1}{10}$

9.	-8	10.	$\frac{1}{2}$
11.	1	12.	$\frac{a-1}{2a}$
13.	$1, -2$	14.	$-2, 2$
15.	-1	16.	1
17.	-1	19.	$k = 8$
20.	$\frac{7}{2}$	21.	1
22.	$\frac{9}{2}$	23.	1
24.	$\frac{2}{\pi}$	25.	$\frac{5}{8}$

26. Discontinuous

27. Discontinuous

28. $k = 8$

29. (a) No (b) $x = 1$

(c) $x = 1, x = 2$ (d) $x = 2$

31. 10

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26

DIFFERENTIATION

The differential calculus was introduced sometime during 1665 or 1666, when Isaac Newton first concieved the process we now know as differentiation (a mathematical process and it yields a result called derivative). Among the discoveries of Newton and Leibnitz are rules for finding derivatives of sums, products and quotients of composite functions together with many other results. In this lesson we define derivative of a function, give its geometrical and physical interpretations, discuss various laws of derivatives and introduce notion of second order derivative of a function.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the derivative of a function $y = f(x)$ at $x = a$;
- prove that the derivative of a constant function $f(x) = c$, is zero;
- find the derivative of $f(x) = x^n, n \in Q$ from first principle and apply to find the derivatives of various functions;
- find the derivatives of the functions of the form $cf(x)$, $[f(x) \pm g(x)]$ and polynomial functions;
- state and apply the results concerning derivatives of the product and quotient of two functions;
- state and apply the chain rule for the derivative of a function;
- find the derivative of algebraic functions (including rational functions); and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Binomial Theorem
- Functions and their graphs
- Notion of limit of a function

26.1 DERIVATIVE OF A FUNCTION

Consider a function and a point say (5,25) on its graph. If x changes from 5 to 5.1, 5.01, 5.001..... etc., then correspondingly, y changes from 25 to 26.01, 25.1001, 25.010001,....A small change in x causes some small change in the value of y. We denote this change in the value of x by a symbol δx and the corresponding change caused in y by δy and call these respectively as an increment in x and increment in y, irrespective of sign of increment. The ratio $\frac{\delta x}{\delta y}$ of increment is termed as incrementary ratio. Here, observing the following table for $y = x^2$ at (5,25), we have for $\delta x = 0.1, 0.01, 0.001, 0.0001, ...$ $\delta y = 1.01, .1001, .010001, .00100001, ...$

x	5.1	5.01	5.001	5.0001
δx	.1	.01	.001	.0001
y	26.01	25.1001	25.010001	25.00100001
δy	1.01	.1001	.010001	.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

We make the following observations from the above table :

(i) δy varies when δx varies.

(ii) $\delta y \to 0$ when $\delta x \to 0$.

(iii) The ratio $\frac{\delta y}{\delta x}$ tends to a number which is 10.

Hence, this example illustrates that $\delta y \to 0$ when $\delta x \to 0$ but $\frac{\delta y}{\delta x}$ tends to a finite number, not necessarily zero. The limit, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is equivalently represented by $\frac{dy}{dx}$. $\frac{dy}{dx}$ is called the derivative of y with respect to x and is read as differential coefficient of y with respect to x.

That is, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = 10$ in the above example and note that while δx and δy are small numbers (increments), the ratio $\frac{\delta y}{\delta x}$ of these small numbers approaches a definite value 10.

In general, let us consider a function

$$y = f(x) \qquad ...(i)$$

To find its derivative, consider δx to be a small change in the value of x, so $x + \delta x$ will be the new value of x where f (x) is defined. There shall be a corresponding change in the value of y. Denoting this change by δy ; y+ δy will be the resultant value of y, thus,

$$y + \delta y = f(x + \delta x) \qquad ...(ii)$$

Subtracting (i) from (ii), we have,

$$(y + \delta y) - y = f(x + \delta x) - f(x)$$

or
$$\delta y = f(x + \delta x) - f(x) \qquad ...(iii)$$

To find the rate of change, we divide (iii) by δx

$$\therefore \qquad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \qquad ...(iv)$$

Lastly, we consider the limit of the ratio $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

If $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \text{...(v)}$$

is a finite quantity, then f(x) is called derivable and the limit is called derivative of f(x) with respect to (w.r.t.) x and is denoted by the symbol $f'(x)$ or by $\frac{d}{dx}$ of f(x)

i.e. $\frac{d}{dx} f(x)$ or $\frac{dy}{dx}$ (read as $\frac{d}{dx}$ of y).

Thus, $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x)$$

Remarks

(1) The limiting process indicated by equation (v) is a mathematical operation. This mathematical process is known as differentiation and it yields a result called a derivative.

(2) A function whose derivative exists at a point is said to be derivable at that point.

(3) It may be verified that if $f(x)$ is derivabale at a point $x = a$, then, it must be continuous at that point. However, the converse is not necessarily true.

(4) The symbols Δx and h are also used in place of δx i.e.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(5) If $y = f(x)$, then $\frac{dy}{dx}$ is also denoted by y_1 or y'.

26.2 VELOCITY AS LIMIT

Let a particle initially at rest at 0 moves along a strainght line OP., The distance s

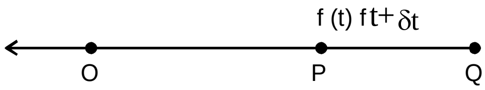

Fig. 26.1

covered by it in reaching P is a function of time t, We may write distance

$$OP = s = f(t) \qquad \text{...(i)}$$

In the same way in reaching a point Q close to P covering PQ

i.e., δs is a fraction of time δt so that

$$OQ = OP + PQ$$

$$= s + \delta s$$

$$= f(t + \delta t) \qquad \text{...(ii)}$$

The average velocity of the particle in the interval δt is given by

Notes

$$= \frac{\text{Change in distance}}{\text{Change in time}}$$

$$= \frac{(s+\delta s) - s}{(t+\delta t) - t}, \qquad \text{[From (i) and (ii)]}$$

$$= \frac{f\,(t+\delta t) - f(t)}{\delta t}$$

(average rate at which distance is travelled in the interval δt).

Now we make δt smaller to obtain average velocity in smaller interval near P. The limit of average velocity as $\delta t \to 0$ is the instantaneous velocity of the particle at time t (at the point P).

$\therefore$ Velocity at time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

It is denoted by $\dfrac{ds}{dt}$.

Thus, if f (t) gives the distance of a moving particle at time t, then the derivative of 'f' at $t = t_0$ represents the instantaneous speed of the particle at the point P i.e. at time $t = t_0$.

This is also referred to as the physical interpretation of a derivative of a function at a point.

Note : The derivative $\dfrac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Example 26.1 The distance 's' meters travelled in time t seconds by a car is given by the relation

$$s = 3t^2$$

Find the velocity of car at time t =4 seconds.

Solution : Here, $f\,(t) = s = 3t^2$

$\therefore$ $$f\,(t+\delta t) = s + \delta s = 3(t+\delta t)^2$$

Velocity of car at any time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{3(t+\delta t)^2 - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{3(t^2 + 2t.\delta t + \delta t^2) - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} (6t + 3\delta t)$$

$$= 6t$$

$\therefore$ Velocity of the car at t = 4 sec = (6 × 4) m/sec = 24 m/sec.

CHECK YOUR PROGRESS 26.1

1. Find the velocity of particles moving along a straight line for the given time-distance relations at the indicated values of time t :

   (a) $s = 2 + 3t; t = \frac{1}{3}.$ (b) $s = 8t - 7; t = 4.$

   (c) $s = t^2 + 3t; t = \frac{3}{2}.$ (d) $s = 7t^2 - 4t + 1; t = \frac{5}{2}.$

2. The distance s metres travelled in t seconds by a particle moving in a straight line is given by $s = t^4 - 18t^2$. Find its speed at t = 10 seconds.

3. A particle is moving along a horizontal line. Its distance s meters from a fixed point O at t seconds is given by $s = 10 - t^2 + t^3$. Determine its instantaneous speed at the end of 3 seconds.

26.3 GEOMETRICAL INTERPRETATION OF dy/dx

Let y = f (x) be a continuous function of x, draw its graph and denote it by APQB.

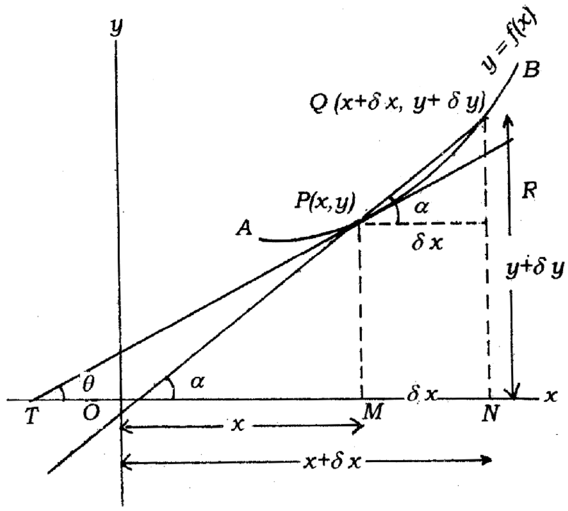

Fig. 26.2

Let P (x,y) be any point on the graph of y = f(x) or curve represented by y = f(x). Let $Q(x + \delta x, y + \delta y)$ be another point on the same curve in the neighbourhood of point P.

Draw PM and QN perpendiculars to x-axis and PR parallel to x-axis such that PR meets QN at R.Join QP and produce the secant line to any point S. Secant line QPS makes angle say α with the positive direction of x-axis. Draw PT tangent to the curve at the point P, making angle θ with the x-axis.

Now, In $\Delta QPR, \angle QPR = \alpha$

$$\tan\alpha = \frac{QR}{PR} = \frac{QN - RN}{MN} = \frac{QN - PM}{ON - OM} = \frac{(y + \delta y) - y}{(x + \delta x) - x} = \frac{\delta y}{\delta x} \qquad \text{(i)}$$

Now, let the point Q move along the curve towards P so that Q approaches nearer and nearer the point P.

Thus, when $Q \to P$, $\delta x \to 0$, $\delta y \to 0$, $\alpha \to 0$, $(\tan\alpha \to \tan\theta)$ and consequently, the secant QPS tends to coincide with the tangent PT.

From (i).
$$\tan\alpha = \frac{\delta y}{\delta x}$$

In the limiting case,
$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \alpha \to 0}} \tan\alpha = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

or
$$\tan\theta = \frac{dy}{dx} \qquad \text{...(ii)}$$

Thus the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at any point P (x,y) on the curve represents the slope or gradient of the tangent at the point P.

This is called the geometrical interpretation of $\frac{dy}{dx}$.

It should be noted that $\frac{dy}{dx}$ has different values at different points of the curve.

Therefore, in order to find the gradient of the curve at a particular point, find $\frac{dy}{dx}$ from the equation of the curve $y = f(x)$ and substitute the coordinates of the point in $\frac{dy}{dx}$.

Corollary 1

If tangent to the curve at P is parallel to x-axis, then $\theta = 0^\circ$ or 180°, i.e., $\frac{dy}{dx} = \tan 0^\circ$ or $\tan 180^\circ$ i.e., $\frac{dy}{dx} = 0$.

That is tangent to the curve represented by $y = f(x)$ at P is parallel to x-axis.

Corollary 2

If tangent to the curve at P is perpendicular to x-axis, $\theta = 90^\circ$ or $\frac{dy}{dx} = \tan 90^\circ = \infty$.

That is, the tangent to the curve represented by $y = f(x)$ at P is parallel to y-axis.

26.4 DERIVATIVE OF CONSTANT FUNCTION

Statement : The derivative of a constant is zero.

Proof : Let $y = c$ be a constant function. Then $y = c$ can be written as

$$y = cx^0 \qquad [\because x^0 = 1] \qquad ..(i)$$

Let δx be a small increment in x. Corresponding to this increment, let δy be the increment in the value of y so that

$$y + \delta y = c(x + \delta x)^0 \qquad ..(ii)$$

Subtracting (i) from (ii),

$$(y + \delta y) - y = c(x + \delta x)^0 - cx^0, \qquad (\because x^0 = 1)$$

or $\qquad \delta y = c - c \quad$ or $\quad \delta y = 0$

Dividing by δx, $\quad \dfrac{\delta y}{\delta x} = \dfrac{0}{\delta x} \quad$ or $\quad \dfrac{\delta y}{\delta x} = 0$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 0 \quad \text{or} \quad \frac{dy}{dx} = 0$$

or $\qquad \dfrac{dc}{dx} = 0 \qquad [\, y = c \text{ from (i)}]$

This proves that rate of change of constant quantity is zero. Therefore, derivative of a constant quantity is zero.

26.5 DERIVATIVE OF A FUNCTION FROM FIRST PRINCIPLE

Recalling the definition of derivative of a function at a point, we have the following working rule for finding the derivative of a function from first principle:

Step I. Write down the given function in the form of $y = f(x)$ ....(i)

Step II. Let dx be an increment in x, δy be the corresponding increment in y so that

$$y + \delta y = f(x + \delta x) \qquad ....(ii)$$

Step III. Subtracting (i) from (ii), we get

$$\delta y = f(x + \delta x) - f(x) \qquad ..(iii)$$

Step IV. Dividing the result obtained in step (iii) by δx, we get,

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step V. Proceeding to limit as $\delta x \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Note : The method of finding derivative of function from first principle is also called **delta** or **ab-ininitio** method.

Next, we find derivatives of some standard and simple functions by first principle.

$$\text{Let } y = x^n \qquad \text{.....(i)}$$

For a small increment δx in x, let the corresponding increment in y be δy .

Then $$y + \delta y = (x + \delta x)^n . \qquad \text{...(ii)}$$

Subtracing (i) from (ii) we have,

$$(y + \delta y) - y = (x + \delta x)^n - x^n$$

$\therefore$ $$\delta y = x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]$$

Since $\frac{\delta x}{x} < 1$, as δx is a small quantity compared to x, we can expand $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem for any index.

Expanding $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\delta y = x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots - 1\right]$$

$$= x^n (\delta x)\left[\frac{n}{x} + \frac{n(n-1)}{2}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Dividing by δx , we have

$$\frac{\delta y}{\delta x} = x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Proceeding to limit when $\delta x \to 0$, $(\delta x)^2$ and higher powers of δx will also tend to zero.

$\therefore$ $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

or $$\operatorname*{lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n \left[\frac{n}{x} + 0 + 0 + \ldots\right]$$

or $$\frac{dy}{dx} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

or $$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1},} \qquad \left[\because\ y = x^n\right]$$

This is known as Newton's Power Formula or Power Rule

Note : We can apply the above formula to find derivative of functions like $x, x^2, x^3, \ldots$

i.e. when n = 1,2,3,...

e.g. $$\frac{d}{dx}x = \frac{d}{dx}x^1 = 1x^{1-1} = 1x^0 = 1.1 = 1$$

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2, \text{ and so on.}$$

Example 26.2 Find the derivative of each of the following :

(i) x^{10} (ii) x^{50} (iii) x^{91}

Solution :

(i)
$$\frac{d}{dx}\left(x^{10}\right) = 10x^{10-1} = 10x^9$$

(ii)
$$\frac{d}{dx}\left(x^{50}\right) = 50x^{50-1} = 50x^{49}$$

(iii)
$$\frac{d}{dx}\left(x^{91}\right) = 91x^{91-1} = 91x^{90}$$

We shall now find the derivatives of some simple functions from definition or first principles.

Example 26.3 Find the derivative of x^2 from the first principles.

Solution : Let $y = x^2$ (i)

For a small increment δx in x let the corresponding increment in y be δy.

$$y + \delta y = (x + \delta x)^2 \qquad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$(y + \delta y) - y = (x + \delta x)^2 - x^2$$

or $$\delta y = x^2 + 2x(\delta x) + (\delta x)^2 - x^2$$

or $$\delta y = 2x(\delta x) + (\delta x)^2$$

Divide by δx, we have

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

Proceeding to limit when $\delta x \to 0$, we have

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(2x + \delta x)$$

or $$\frac{dy}{dx} = 2x + \lim_{\delta x \to 0}(\delta x)$$

$$= 2x + 0$$
$$= 2x$$

or $\quad \frac{dy}{dx} = 2x \quad$ or $\quad \frac{d}{dx}\left(x^2\right) = 2x$

or $\quad \frac{dy}{dx} = \frac{-1}{x(x+0)} \quad$ or $\quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

Example 26.4 Find the derivative of $\sqrt{x}$ by ab-initio method.

Solution : Let $\quad y = \sqrt{x}$ ..(i)

For a small increment δx in x, let δy be the corresponding increment in y.

$\therefore \quad y + \delta y = \sqrt{x + \delta x}$ ...(ii)

Subtracting (i) from (ii), we have

$$(y+\delta y) - y = \sqrt{x+\partial x} - \sqrt{x} \qquad \text{...(iii)}$$

or $\quad \delta y = \sqrt{x + \delta x} - \sqrt{x}$

Rationalising the numerator of the right hand side of (iii), we have

$$\delta y = \frac{\sqrt{x+\delta x} - \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}\left(\sqrt{x+\delta x} + \sqrt{x}\right)$$

$$= \frac{(x+\delta x) - x}{\sqrt{x+\delta x} + \sqrt{x}} \quad \text{or} \quad \delta y = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}}$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\frac{1}{\sqrt{x+\delta} + \sqrt{x}}\right]$$

or $\quad \frac{dy}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} \quad$ or $\quad \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

Example 26.5 If f (x) is a differentiable function and c is a constant, find the derivative of

$$\phi(x) = cf(x)$$

Solution : We have to find derivative of function

$$\phi(x) = cf(x) \qquad \text{...(i)}$$

For a small increment δx in x, let the values of the functions $\phi(x)$ be $\phi(x+\delta x)$ and that of f (x)be $f(x+\delta x)$

$\therefore \quad \phi(x+\delta x) = cf(x+\delta x)$ ...(ii)

Subtracting (i) from (ii), we have

$$\phi(x+\delta x)-\phi(x)=c[f(x+\delta x)-f(x)]$$

Dividing by δx , we have

$$\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\lim_{\delta x\to 0} c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=c\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=cf'(x)$$

Thus, $$\frac{d}{dx}[cf(x)]=c\frac{df}{dx}$$

CHECK YOUR PROGRESS 26.2

1. Find the derivative of each of the following functions by delta method :

   (a) $10x$ (b) $2x+3$ (c) $3x^2$

   (d) x^2+5x (e) $7x^3$

2. Find the derivative of each of the following functions using ab-initio method:

   (a) $\frac{1}{x}, x \neq 0$ (b) $\frac{1}{ax}, x \neq 0$ (c) $x+\frac{1}{x}, x \neq 0$

   (d) $\frac{1}{ax+b}, x \neq \frac{-b}{a}$ (e) $\frac{ax+b}{cx+d}, x \neq \frac{-d}{c}$ (f) $\frac{x+2}{3x+5}, x \neq \frac{-5}{3}$

3. Find the derivative of each of the following functions from first principles :

   (a) $\frac{1}{\sqrt{x}}, x \neq 0$ (b) $\frac{1}{\sqrt{ax+b}}, x \neq \frac{-b}{a}$ (c) $\sqrt{x}+\frac{1}{\sqrt{x}}, x \neq 0$

   (d) $\frac{1+x}{1-x}, x \neq 1$

4. Find the derivative of each of the following functions by using delta method :

   (a) $f(x)=3\sqrt{x}$. Also find $f'(2)$. (b) $f(r)=\pi r^2$. Also find $f'(2)$.

   (c) $f(r)=\frac{4}{3}\pi r^3$. Also find $f'(3)$.

26.7 ALGEBRA OF DERIVATIVES

Many functions arise as combinations of other functions. The combination could be sum, difference, product or quotient of functions. We also come across situations where a given function can be expressed as a function of a function.

In order to make derivative as an effective tool in such cases, we need to establish rules for finding derivatives of sum, difference, product, quotient and function of a function. These, in turn, will enable one to find derivatives of polynomials and algebraic (including rational) functions.

26.7 DERIVATIVES OF SUM AND DIFFERENCE OF FUNCTIONS

If f (x) and g (x) are both derivable functions and h (x) = f (x) + g (x), then what is h' (x) ?

Here $$h(x) = f(x) + g(x)$$

Let δx be the increment in x and δy be the correponding increment in y.

$$\therefore \quad h(x+\delta x) = f(x+\delta x) + g(x+\delta x)$$

Hence
$$h'(x) = \lim_{\delta x \to 0} \frac{[f(x+\delta x) + g(x+\delta x)] - [f(x) + g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)] + [g(x+\delta x) - g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} + \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

or $$h'(x) = f'(x) + g'(x)$$

Thus we see that the *derivative of sum of two functions is sum of their derivatives*.

This is called the **SUM RULE**.

e.g. $$y = x^2 + x^3$$

Then $$y' = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$$

$$= 2x + 3x^2$$

Thus $$y' = 2x + 3x^2$$

This sum rule can easily give us the difference rule as well, because

if $$h(x) = f(x) - g(x)$$

then $$h(x) = f(x) + [-g(x)]$$

$$\therefore \quad h'(x) = f'(x) + [-g'(x)]$$
$$= f'(x) - g'(x)$$

i.e. *the derivative of difference of two functions is the difference of their derivatives*.

This is called **DIFFERENCE RULE.**

Thus we have

Sum rule : $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}[f(x)] + \dfrac{d}{dx}[g(x)]$

Difference rule : $\dfrac{d}{dx}[f(x)-g(x)] = \dfrac{d}{dx}[f(x)] - \dfrac{d}{dx}[g(x)]$

Example 26.6 Find the derivative of each of the following functions :

(i) $y = 10t^2 + 20t^3$

(ii) $y = x^3 + \dfrac{1}{x^2} - \dfrac{1}{x}, \quad x \neq 0$

Solution :

(i) We have, $y = 10t^2 + 20t^3$

$$\therefore \quad \frac{dy}{dt} = 10(2t) + 20(3t^2)$$
$$= 20t + 60t^2$$

(ii) $y = x^3 + \dfrac{1}{x^2} - \dfrac{1}{x} \qquad x \neq 0$

$$= x^3 + x^{-2} - x^{-1}$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 + (-2)x^{-3} - (-1)x^{-2} = 3x^2 - \frac{2}{x^3} + \frac{1}{x^2}$$

Example 26.7 Evaluate the derivative of

$$y = x^3 + 3x^2 + 4x + 5, \quad x = 1$$

Solution :

(ii) We have $y = x^3 + 3x^2 + 4x + 5$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4x + 5\right] = 3x^2 + 6x + 4$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=1} = 3(1)^2 + 6(1) + 4 = 13$$

CHECK YOUR PROGRESS 26.3

1. Find y' when :
   (a) $y = 12$ (b) $y = 12x$ (c) $y = 12x + 12$

2. Find the derivatives of each of the following functions :

   (a) $f(x) = 20x^9 + 5x$ (b) $f(x) = -50x^4 - 20x^2 + 4$

   (c) $f(x) = 4x^3 - 9 - 6x^2$ (d) $f(x) = \frac{5}{9}x^9 + 3x$

   (e) $f(x) = x^3 - 3x^2 + 3x - \frac{2}{5}$ (f) $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$

   (g) $f(x) = \frac{2}{5}x^{\frac{2}{3}} - x^{\frac{-4}{5}} + \frac{3}{x^2}$ (h) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

3. (a) If $f(x) = 16x + 2$, find $f'(0), f'(3), f'(8)$

   (b) If $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 16$, find $f'(-1), f'(0), f'(1)$

   (c) If $f(x) = \frac{x^4}{4} + \frac{3}{7}x^7 + 2x - 5$, find $f'(-2)$

   (d) Given that $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ and hence $\left.\frac{dV}{dr}\right]_{r=2}$

26.8 DERIVATIVE OF PRODUCT OF FUNCTIONS

You are all familiar with the four fundamental operations of Arithmetic : addition, subtraction, multiplication and division. Having dealt with the sum and the difference rules, we now consider the derivative of product of two functions.

Consider $\qquad y = (x^2 + 1)^2$

This is same as $\qquad y = (x^2 + 1)(x^2 + 1)$

So we need now to derive the way to find the derivative in such situation.

We write $\qquad y = (x^2 + 1)(x^2 + 1)$

Let δx be the increment in x and δy the correrponding increment in y. Then

$$y + \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)]$$

$$\Rightarrow \qquad \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)] - (x^2 + 1)(x^2 + 1)$$

$$= [(x + \delta x)^2 + 1][(x + \delta x)^2 - x^2)] + (x^2 + 1)[(x + \delta x)^2 + 1] - (x^2 + 1)(x^2 + 1)$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[x+\delta x)^2+1-(x^2+1)]$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[(x+\delta x)^2-x^2]$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]$$

$$=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]$$

$$=[(x+\delta x)^2+1](2x+\delta x)+(x^2+1)(2x+\delta x)$$

$$\therefore \qquad \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[(x+\delta x)^2+1]\cdot[2x+\delta x]+\lim_{\delta x\to 0}(x^2+1)(2x+\delta x)$$

or $$\frac{dy}{dx}=\left(x^2+1\right)(2x)+\left(x^2+1\right)\cdot(2x)$$

$$=4x\left(x^2+1\right)$$

Let us analyse : $$\frac{dy}{dx}=\left(x^2+1\right)\underset{\text{derivative of } x^2+1}{(2x)}+\left(x^2+1\right)\underset{\text{derivative of } x^2+1}{(2x)}$$

Consider $$y=x^3\cdot x^2$$

Is $$\frac{dy}{dx}=x^3\cdot(2x)+x^2\cdot\left(3x^2\right)\ ?$$

Let us check $$x^3(2x)+x^2\left(3x^2\right)$$

$$=2x^4+3x^4$$

$$=5x^4$$

We have $$y=x^3\cdot x^2$$

$$=x^5$$

$$\therefore \qquad \frac{dy}{dx}=5x^4$$

In general, if f (x) and g (x) are two functions of x then the derivative of their product is defined by

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

$$=\left[\text{Ist function}\right]\left[\frac{d}{dx}(\text{Second function})\right]+\left[\text{Second function}\right]\left[\frac{d}{dx}(\text{Ist function})\right]$$

which is read as derivative of product of two functions is equal to

= [Ist function] [Derivative of Second function] +

[Second function] [Derivative of Ist function]

This is called the **PRODUCT RULE.**

Example 26.8 Find $\frac{dy}{dx}$, if $y = 5x^6\left(7x^2 + 4x\right)$

Method I. Here y is a product of two functions.

$$\therefore \quad \frac{dy}{dx} = \left(5x^6\right)\cdot\frac{d}{dx}\left(7x^2 + 4x\right) + \left(7x^2 + 4x\right)\frac{d}{dx}\left(5x^6\right)$$

$$= \left(5x^6\right)(14x + 4) + \left(7x^2 + 4x\right)\left(30x^5\right)$$

$$= 70x^7 + 20x^6 + 210x^7 + 120x^6$$

$$= 280x^7 + 140x^6$$

Method II $\quad y = 5x^6\left(7x^2 + 4x\right)$

$$= 35x^8 + 20x^7$$

$$\therefore \quad \frac{dy}{dx} = 35 \times 8x^7 + 20 \times 7x^6 = 280x^7 + 140x^6$$

which is the same as in Method I.

This rule can be extended to find the derivative of two or more than two functions.

Remark : If f (x), g (x) and h(x) are three given functions of x, then

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = f(x)g(x)\frac{d}{dx}h(x) + g(x)h(x)\frac{d}{dx}f(x) + h(x)f(x)\frac{d}{dx}g(x)$$

Example 26.9 Find the derivative of $\left[f(x)g(x)h(x)\right]$ if

$$f(x) = x,\ g(x) = (x - 3), \text{ and } \quad h(x) = x^2 + x$$

Solution : Let $y = x(x - 3)\left(x^2 + x\right)$

To find the derivative of y, we can combine any two functions, given on the R.H.S. and apply the product rule or use result mentioned in the above remark.

In other words, we can write

$$y = \left[x(x - 3)\right]\left(x^2 + x\right)$$

Let $\quad u(x) = f(x)g(x) = x(x - 3) = x^2 - 3x$

Also $\quad h(x) = x^2 + x$

$\therefore \quad y = u(x) \times h(x)$

Hence $\quad \frac{dy}{dx} = x(x - 3)\frac{d}{dx}\left(x^2 + x\right) + \left(x^2 + x\right)\frac{d}{dx}\left(x^2 - 3x\right)$

$$= x(x-3)(2x+1)+\left(x^2+x\right)(2x-3)$$

$$= x(x-3)(2x+1)+\left(x^2+x\right)(x-3)+x\left(x^2+x\right)$$

$$= [f(x)g(x)]\cdot h'(x)+[g(x)h(x)]f'(x)+[h(x)f(x)].g'(x)$$

Hence $$\frac{d}{dx}[f(x)g(x)h(x)]=[f(x)g(x)]\cdot\frac{d}{dx}[h(x)]$$

$$+[g(x)h(x)]\frac{d}{dx}[f(x)]+h(x)f(x)\frac{d}{dx}[g(x)]$$

Alternatively, we can directly find the derivative of product of the given three functions.

$$\frac{dy}{dx}=[x(x-3)]\frac{d}{dx}(x^2+x)+[(x-3)(x^2+x)]\frac{d}{dx}(x)+[(x^2+x)\cdot x]\frac{d}{dx}(x-3)$$

$$= x(x-3)(2x+1)+(x-3)(x^2+x)\cdot 1+(x^2+x)\cdot x\cdot 1$$

$$= 4x^3-6x^2-6x$$

CHECK YOUR PROGRESS 26.4

1. Find the derivative of each of the following functions by product rule :

(a) $f(x)=(3x+1)(2x-7)$ (b) $f(x)=(x+1)(-3x-2)$

(c) $f(x)=(x+1)(-2x-9)$ (d) $y=(x-1)(x-2)$

(e) $y=x^2(2x^2+3x+8)$ (f) $y=(2x+3)(5x^2-7x+1)$

(g) $u(x)=(x^2-4x+5)(x^3-2)$

2. Find the derivative of each of the functions given below :

(a) $f(r)=r(1-r)(\pi r^2+r)$ (b) $f(x)=(x-1)(x-2)(x-3)$

(c) $f(x)=(x^2+2)(x^3-3x^2+4)(x^4-1)$

(d) $f(x)=(3x^2+7)(5x-1)\left(3x^2+9x+8\right)$

26.9 QUOTIENT RULE

You have learnt sum Rule, Difference Rule and Product Rule to find derivative of a function expressed respectively as either the sum or difference or product of two functions. Let us now take a step further and learn the "Quotient Rule for finding derivative of a function which is the quotient of two functions.

Let $g(x)=\dfrac{1}{r(x)}$, $[r(x)\neq 0]$

Let us find the derivative of g (x) by first principles

$$g(x) = \frac{1}{r(x)}$$

$$\therefore \qquad g'(x) = \lim_{\delta x \to 0} \left[\frac{\frac{1}{r(x+\delta x)} - \frac{1}{r(x)}}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} \left[\frac{r(x) - r(x+\delta x)}{\delta(x) r(x) r(x+\delta x)} \right]$$

$$= \lim_{\delta x \to 0} \left[\frac{r(x) - r(x+\delta x)}{\delta x} \right] \lim_{\delta x \to 0} \frac{1}{r(x).r(x+\delta x)}$$

$$= -r'(x) \cdot \frac{1}{[r(x)]^2} = -\frac{r'(x)}{[r(x)]^2}$$

Consider any two functions f (x) and g (x) such that $\phi(x) = \frac{f(x)}{g(x)}$, $\qquad g(x) \neq 0$

We can write $\phi(x) = f(x) \cdot \frac{1}{g(x)}$

$$\therefore \qquad \phi(x) = f'(x) \cdot \frac{1}{g(x)} + f(x) \frac{d}{dx}\left[\frac{1}{g(x)} \right]$$

$$= \frac{f'(x)}{g(x)} + f(x) \left[\frac{-g'(x)}{[g(x)]^2} \right]$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$$

$$= \frac{(\text{Denominator})(\text{Derivative of Numerator}) - (\text{Numerator})(\text{Derivative of Denominator})}{(\text{Denominator})^2}$$

Hence $\qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)} \right] = \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}$

This is called the quotient Rule.

Example 26.10 Find $f'(x)$ if $f(x) = \frac{4x+3}{2x-1}, \quad x \neq \frac{1}{2}$

Solution :

$$f'(x) = \frac{(2x-1)\frac{d}{dx}(4x+3) - (4x+3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$=\frac{(2x-1).4-(4x+3).2}{(2x-1)^2}$$

$$=\frac{-10}{(2x-1)^2}$$

Let us consider the following example:

Let $\quad f(x)=\frac{1}{2x-1}, \qquad x\neq\frac{1}{2}$

$$\frac{d}{dx}\left[\frac{1}{2x-1}\right]=\frac{(2x-1)\frac{d}{dx}(1)-1\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$=\frac{(2x-1)\times 0-2}{(2x-1)^2} \qquad \left[\because \frac{d}{dx}(1)=0\right]$$

i.e. $\quad \frac{d}{dx}\left[\frac{1}{2x-1}\right]=-\frac{2}{(2x-1)^2}$

CHECK YOUR PROGRESS 26.5

1. Find the derivative of each of the following :

(a) $y=\frac{2}{5x-7}, \; x\neq\frac{7}{5}$ (b) $y=\frac{3x-2}{x^2+x-1}$ (c) $y=\frac{x^2-1}{x^2+1}$

(d) $f(x)=\frac{x^4}{x^2-3}$ (e) $f(x)=\frac{x^5-2x}{x^7}$ (f) $f(x)=\frac{x}{x^2+x+1}$

(g) $f(x)=\frac{\sqrt{x}}{x^3+4}$

2. Find $f'(x)$ if

(a) $f(x)=\frac{x(x^2+3)}{x-2}, \qquad [x\neq 2]$

(b) $f(x)=\frac{(x-1)(x-2)}{(x-3)(x-4)}, \qquad [x\neq 3, \quad x\neq 4]$

26.10 CHAIN RULE

Earlier, we have come across functions of the type $\sqrt{x^4+8x^2+1}$. This function can not be expressed as a sum, difference, product or a quotient of two functions. Therefore, the techniques developed so far do not help us find the derivative of such a function. Thus, we need to develop a rule to find the derivative of such a function.

Let us write : $y=\sqrt{x^4+8x^2+1}$ or $y=\sqrt{t}$ where $t=x^4+8x^2+1$

That is, y is a function of t and t is a function of x. Thus y is a function of a function. We proceed to find the derivative of a function of a function.

Let δt be the increment in t and δy, the corresponding increment in y.

Then $\delta y \to 0$ as $\delta t \to 0$

$$\therefore \qquad \frac{dy}{dt}=\lim_{\delta t\to 0}\frac{\delta y}{\delta t} \qquad \text{(i)}$$

Similarly t is a function of x.

$$\therefore \qquad \delta t \to 0 \quad \text{as} \quad \delta x \to 0$$

$$\therefore \qquad \frac{dt}{dx}=\lim_{\delta x\to 0}\frac{\delta t}{\delta x} \qquad \text{(ii)}$$

Here y is a function of t and t is a function of x. Therefore $\delta y \to 0$ as $\delta x \to 0$

From (i) and (ii), we get

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\left[\lim_{\delta t\to 0}\frac{\delta y}{\delta t}\right]\left[\lim_{\delta x\to 0}\frac{\delta t}{\delta x}\right]$$

$$=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

Thus $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

This is called the **Chain Rule.**

Example 26.11 If $y=\sqrt{x^4+8x^2+1}$, find $\frac{dy}{dx}$

Solution : We are given that

$$y=\sqrt{x^4+8x^2+1}$$

which we may write as

$$y=\sqrt{t}\text{, where } t=x^4+8x^2+1 \qquad \text{(i)}$$

$$\therefore \qquad \frac{dy}{dt}=\frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx}=4x^3+16x$$

Here $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{1}{2\sqrt{t}}\cdot(4x^3+16x)$$

$$=\frac{4x^3+16x}{2\sqrt{x^4+8x^2+1}}=\frac{2x^3+8x}{\sqrt{x^4+8x^2+1}} \qquad \text{(Using (i))}$$

Example 26.12 Find the derivative of the function $y = \dfrac{5}{(x^2-3)^7}$

Solution :
$$\frac{dy}{dx} = \frac{d}{dx}\left\{5(x^2-3)^{-7}\right\}$$

$$= 5[(-7)(x^2-3)^{-8}]\cdot\frac{d}{dx}(x^2-3) \qquad \text{(Using chain Rule)}$$

$$= -35(x^2-3)^{-8}\cdot(2x)$$

$$= \frac{-70x}{(x^2-3)^8}$$

Example 26.13 Find $\dfrac{dy}{dx}$ where $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

Solution : We have $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

$$\frac{dy}{dv} = \frac{1}{4}(4v^3) = v^3 = \left(\frac{2}{3}x^3+5\right)^3 \qquad \text{...(i)}$$

and
$$\frac{dv}{dx} = \frac{2}{3}(3x^2) = 2x^2 \qquad \text{..(ii)}$$

Thus
$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$$

$$= \left(\frac{2}{3}x^3+5\right)^3(2x^2) \qquad \text{[Using (i) and (ii)]}$$

Remark

We have seen in the previous examples that by using various rules of derivatives we can find derivatives of algebraic functions.

CHECK YOUR PROGRESS 26.6

1. Find the derivative of each of the following functions :

(a) $f(x) = (5x-3)^7$ (b) $f(x) = (3x^2-15)^{35}$

(c) $f(x) = (1-x^2)^{17}$ (d) $f(x) = \dfrac{(3-x)^5}{7}$

(e) $y = \dfrac{1}{x^2+3x+1}$ (f) $y = \sqrt[3]{(x^2+1)^5}$

(g) $y = \dfrac{1}{\sqrt{7-3x^2}}$ (h) $y = \left[\dfrac{1}{6}x^6 + \dfrac{1}{2}x^4 + \dfrac{1}{16}\right]^5$

(i) $y = (2x^2 + 5x - 3)^{-4}$ (j) $y = x + \sqrt{x^2 + 8}$

2. Find $\dfrac{dy}{dx}$ if

(a) $y = \dfrac{3-v}{2+v}, v = \dfrac{4x}{1-x^2}$ (b) $y = at^2, t = \dfrac{x}{2a}$

Second Order Derivative : Given y is a function of x, say f (x). If the derivative $\dfrac{dy}{dx}$ is a derivable function of x, then the derivative of $\dfrac{dy}{dx}$ is known as the second derivative of $y = f(x)$ with respect to x and is denoted by $\dfrac{d^2y}{dx^2}$. Other symbols used for the second derivative of y are D^2, f'', y'', y_2 etc.

Remark

Thus the value of f '' at x is given by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(h)}{h}$$

The derivatives of third, fourth, ...orders can be similarly defined.

Thus the second derivative, or second order derivative of y with respect to x is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 26.14 Find the second order derivative of

(i) x^2 (ii) $x^3 + 1$ (iii) $(x^2+1)(x-1)$ (iv) $\dfrac{x+1}{x-1}$

Solution : (i) Let $y = x^2$, then $\dfrac{dy}{dx} = 2x$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x) = 2 \cdot \frac{d(x)}{dx}$$

$$= 2.1 = 2$$

$\therefore$ $$\frac{d^2y}{dx^2} = 2$$

(ii) Let $y = x^3 + 1$, then

$$\frac{dy}{dx} = 3x^2 \text{ (by sum rule and derivative of a constant is zero)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 3.2x = 6x$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x$$

(iii) Let $y = (x^2 + 1)(x - 1)$, then

$$\frac{dy}{dx} = (x^2 + 1)\frac{d}{dx}(x - 1) + (x - 1),\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)\cdot 1 + (x - 1)\cdot 2x \quad \text{or} \quad \frac{dy}{dx} = x^2 + 1 + 2x^2 - 2x = 3x^2 - 2x + 1$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 - 2x + 1\right) = 6x - 2$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x - 2$$

(iv) Let $y = \frac{x+1}{x-1}$, then

$$\frac{dy}{dx} = \frac{(x-1)\cdot 1 - (x+1).1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-2}{(x-1)^2}\right] = -2.-2.\frac{1}{(x-1)^3} = \frac{4}{(x-1)^3}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3}$$

CHECK YOUR PROGRESS 26.7

Find the derivatives of second order for each of the following functions :

(a) x^3 (b) $x^4 + 3x^3 + 9x^2 + 10x + 1$

(c) $\frac{x^2+1}{x+1}$ (d) $\sqrt{x^2+1}$

LET US SUM UP

- The derivative of a function f (x) with respect to x is defined as

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}, \ \delta x > 0$$

- The derivative of a constant is zero i.e., $\frac{dc}{dx} = 0$, where c a is constant.
- Newton's Power Formula

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- Geometrically, the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at point $P(x, y)$ is the slope or gradient of the tangent on the curve represented by $y = f(x)$ at the point P.
- The derivative of y with respect to x is the instantaneous rate of change of y with respect to x.
- If f (x) is a derivable function and c is a constant, then

$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$, where $f'(x)$ denotes the derivative of $f(x)$.

- 'Sum or difference rule' of functions :

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Derivative of the sum or difference of two functions is equal to the sum or diference of their derivatives respectively.

- **Product rule :**

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= (\text{Ist function})\left(\frac{d}{dx}\text{IInd function}\right) + (\text{IInd function})\left(\frac{d}{dx}\text{Ist function}\right)$$

- **Quotient rule :** If $\phi(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0$, then

$$\phi'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$= \frac{(\text{Denominator})\left(\frac{d}{dx}(\text{Numerator})\right) - \text{Numerator}\left(\frac{d}{dx}(\text{Denominator})\right)}{(\text{Denominator})^2}$$

- **Chain Rule :** $\frac{d}{dx}\left[f\{g(x)\}\right] = f'[g(x)] \cdot \frac{d}{dx}[g(x)]$

  $=$derivative of $f(x)$ w.r.t $g(x) \times$ derivative of $g(x)$ w.r.t.x

- The derivative of second order of y w.r.t. to x is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=MKWBx78L7Qg

http://www.youtube.com/watch?v=IiBC4ngwH6E

http://www.youtube.com/watch?v=1015d63VKh4

http://www.youtube.com/watch?v=Bkkk0RLSEy8

http://www.youtube.com/watch?v=ho87DN9wO70

http://www.youtube.com/watch?v=UXQGzgPf1LE

http://www.youtube.com/watch?v=4bZyfvKazzQ

http://www.bbc.co.uk/education/asguru/maths/12methods/03differentiation/index.shtml

TERMINAL EXERCISE

1. The distance s meters travelled in time t seconds by a car is given by the relation $s = t^2$. Caclulate.

   (a) the rate of change of distance with respect to time t.

   (b) the speed of car at time t = 3 seconds.

2. Given $f(t) = 3 - 4t^2$. Use delta method to find $f'(t), f'\left(\frac{1}{3}\right)$.

3. Find the derivative of $f(x) = x^4$ from the first principles. Hence find

   $$f'(0), f'\left(-\frac{1}{2}\right)$$

4. Find the derivative of the function $\sqrt{2x+1}$ from the first principles.

5. Find the derivatives of each of the following functions by the first principles :

   (a) ax + b, where a and b are constants (b) $2x^2 + 5$

   (c) $x^3 + 3x^2 + 5$ (d) $(x-1)^2$

6. Find the derivative of each of the following functions :

(a) $f(x) = px^4 + qx^2 + 7x - 11$ (b) $f(x) = x^3 - 3x^2 + 5x - 8$

(c) $f(x) = x + \dfrac{1}{x}$ (d) $f(x) = \dfrac{x^2 - a}{a - 2}, a \neq 2$

7. Find the derivative of each of the functions given below by two ways, first by product rule, and then by expanding the product. Verify that the two answers are the same.

(a) $y = \sqrt{x}\left(1 + \dfrac{1}{\sqrt{x}}\right)$ (b) $y = x^{\frac{3}{2}}\left(2 + 5x + \dfrac{1}{x}\right)$

8. Find the derivative of the following functions :

(a) $f(x) = \dfrac{x}{x^2 - 1}$ (b) $f(x) = \dfrac{3}{(x-1)^2} + \dfrac{10}{x^3}$

(c) $f(x) = \dfrac{1}{(1 + x^4)}$ (d) $f(x) = \dfrac{(x+1)(x-2)}{\sqrt{x}}$

(e) $f(x) = \dfrac{3x^2 + 4x - 5}{x}$ (f) $f(x) = \dfrac{x - 4}{2\sqrt{x}}$

(g) $f(x) = \dfrac{(x^3 + 1)(x - 2)}{x^2}$

9. Use chain rule, to find the derivative of each of the functions given below :

(a) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ (b) $\sqrt{\dfrac{1+x}{1-x}}$ (c) $\sqrt[3]{x^2(x^2 + 3)}$

10. Find the derivatives of second order for each of the following :

(a) $\sqrt{x+1}$ (b) $x \cdot \sqrt{x - 1}$

ANSWERS

CHECK YOUR PROGRESS 26.1

1. (a) 3 (b) 8 (c) 6 (d) 31

2. 3640 m/s 3. 21 m/s

CHECK YOUR PROGRESS 26.2

1. (a) 10 (b) 2 (c) 6x (d) 2x+5 (e) $21x^2$

2. (a) $-\frac{1}{x^2}$ (b) $-\frac{1}{ax^2}$ (c) $1-\frac{1}{x^2}$ (d) $\frac{-a}{(ax+b)^2}$ (e) $\frac{ad-bc}{(cx+d)^2}$ (f) $-\frac{1}{(3x+5)^2}$

3. (a) $-\frac{1}{2x\sqrt{x}}$ (b) $\frac{-a}{2(ax+b)(\sqrt{ax+b})}$ (c) $\frac{1}{2\sqrt{x}}\left(1-\frac{1}{x}\right)$ (d) $\frac{2}{(1-x)^2}$

4. (a) $\frac{3}{2\sqrt{x}};\frac{3}{2\sqrt{2}}$ (b) $2\pi r$; 4π (c) $2\pi r^2$; 36π

CHECK YOUR PROGRESS 26.3

1. (a) 0 (b) 12 (c) 12

2. (a) $180x^8+5$ (b) $-200x^3-40x$ (c) $12x^2-12x$ (d) $5x^8+3$ (e) $3x^2-6x+3$

   (f) $x^7-x^5+x^3$ (g) $\frac{4}{15}x^{\frac{-1}{3}}+\frac{4}{5}x^{\frac{-9}{5}}-6x^{-3}$ (h) $\frac{1}{2\sqrt{x}}+\frac{1}{2x^{\frac{3}{2}}}$

3. (a) 16, 16, 16 (b) 3,1 ,1 (c) 186 (d) $4\pi r^2, 16\pi$

CHECK YOUR PROGRESS 26.4

1. (a) $12x-19$ (b) $-6x-5$ (c) $4x-11$ (d) $2x-3$

   (e) $8x^3+9x^2+16x$ (f) $30x^2+2x-19$ (g) $5x^4-16x^3+15x^2-4x+8$

2. (a) $-4\pi r^3+3(\pi-1)r^2+2r$ (b) $3x^2-12x+11$

   (c) $9x^8-28x^7+14x^6-12x^5-5x^4+44x^3-6x^2+4x$

   (d) $(5x-1)(3x^2+9x+8).6x+5(3x^2+7)(3x^2+9x+8)+(3x^2+7)(5x-1)(6x+9)$

CHECK YOUR PROGRESS 26.5

1. (a) $\frac{-10}{(5x-7)^2}$ (b) $\frac{-3x^2+4x-1}{(x^2+x+1)^2}$ (c) $\frac{4x}{(x^2+1)^2}$

   (d) $\frac{2x^5-12x^3}{(x^2-3)^2}$ (e) $\frac{-2x^4+12}{x^7}$ (f) $\frac{1-x^2}{(x^2+x+1)^2}$ (g) $\frac{4-5x^3}{2\sqrt{x}(x^3+4)^2}$

2. (a) $\dfrac{2x^3-6x^2-6}{(x-2)^2}$ (b) $\dfrac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$

CHECK YOUR PROGRESS 26.6

1. (a) $35(5x-6)^6$ (b) $210x(3x^2-15)^{34}$ (c) $-34x(1-x^2)^{16}$

(d) $\dfrac{-5}{7}(3-x)^4$ (e) $-(2x+3)(x^2+3x+1)^{-2}$ (f) $\dfrac{10x}{3}\left(x^2+1\right)^{\frac{2}{3}}$

(g) $3x(7-3x^2)^{-3/2}$ (h) $5(x^5+2x^3)\left(\dfrac{x^6}{6}+\dfrac{x^4}{2}+\dfrac{1}{16}\right)^4$

(i) $-4(4x+5)(2x^2+5x-3)^{-5}$ (j) $1+\dfrac{x}{\sqrt{x^2+8}}$

2. (a) $\dfrac{-5(1+x^2)}{(1+2x-x^2)^2}$ (b) $\dfrac{x}{2a}$

CHECK YOUR PROGRESS 26.7

1. (a) $6x$ (b) $12x^2+18x+18$ (c) $\dfrac{4}{(x+1)^3}$ (d) $\dfrac{1}{(1+x^2)^{3/2}}$

TERMINAL EXERCISE

1. (a) $2t$ (b) 6 seconds

2 $-8t, -\dfrac{8}{3}$ 3. $0, \dfrac{-1}{2}$ 4. $\dfrac{1}{\sqrt{2x+1}}$

5. (a) a (b) $4x$ (c) $3x^2+6x$ (d) $2(x-1)$

6. (a) $4px^3+2qx+7$ (b) $3x^2-6x+5$ (c) $1-\dfrac{1}{x^2}$ (d) $\dfrac{2x}{a-2}$

7. (a) $\dfrac{1}{2\sqrt{x}}$ (b) $3\sqrt{x}+\dfrac{25}{2}x\sqrt{x}+\dfrac{1}{2\sqrt{x}}$

8. (a) $\dfrac{-(x^2+1)}{(x^2-1)^2}$ (b) $\dfrac{-6}{(x-1)^3}-\dfrac{30}{x^4}$ (c) $\dfrac{-4x^3}{(1+x^4)^2}$ (d) $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{2\sqrt{x}}+\dfrac{1}{x^{3/2}}$

(e) $3+\dfrac{5}{x^2}$ (f) $\dfrac{1}{4\sqrt{x}}+\dfrac{1}{x\sqrt{x}}$ (g) $3x^2-2-\dfrac{1}{x^2}+\dfrac{4}{x^3}$

9. (a) $1-\dfrac{1}{x^2}$ (b) $\dfrac{1}{\sqrt{1+x}\cdot(1-x)^{\frac{3}{2}}}$ (c) $\dfrac{4x^3+6x}{3(x^4+3x^2)^{\frac{2}{3}}}$

10. (a) $-\dfrac{1}{4(x+1)^{\frac{3}{2}}}$ (b) $\dfrac{2+x-x^2}{4(x-1)^{\frac{1}{2}}}$

27

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Trigonometry is the branch of Mathematics that has made itself indispensable for other branches of higher Mathematics may it be calculus, vectors, three dimensional geometry, functions-harmonic and simple and otherwise just can not be processed without encountering trigonometric functions. Further within the specific limit, trigonometric functions give us the inverses as well.

The question now arises: Are all the rules of finding the derivative studied by us so far appliacable to trigonometric functions?

This is what we propose to explore in this lesson and in the process, develop the fornulae or results for finding the derivatives of trigonometric functions and their inverses. In all discussions involving the trignometric functions and their inverses, radian measure is used, unless otherwise specifically mentioned.

OBJECTIVES

After studying this lesson, you will be able to:

- find the derivative of trigonometric functions from first principle;
- find the derivative of inverse trigomometric functions from first principle;
- apply product, quotient and chain rule in finding derivatives of trigonometric and inverse trigonometric functions; and
- find second order derivative of a functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of trigonometric ratios as functions of angles.
- Standard limits of trigonometric functions
- Definition of derivative, and rules of finding derivatives of function.

27.1 DERIVATIVE OF TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

(i) Let $y = \sin x$

For a small increment δ x in x, let the corresponding increment in y be δ y.

$$\therefore \quad y+\delta y=\sin\left(x+\delta x\right)$$

and $$\delta y=\sin\left(x+\delta x\right)-\sin x$$

$$=2\cos\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2} \qquad \left[\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C+D}{2}\right]$$

$$\therefore \quad \frac{\delta y}{\delta x}=2\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=\cos x.1 \qquad \left[\therefore \lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1\right]$$

Thus $$\frac{dy}{dx}=\cos x$$

i.e., $$\frac{d}{dx}(\sin x)=\cos x$$

(ii) Let $y=\cos x$

For a small increment δ x, let the corresponding increment in y be δ y.

$$\therefore \quad y+\delta y=\cos\left(x+\delta x\right)$$

and $$\delta y=\cos\left(x+\delta x\right)-\cos x$$

$$=-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\therefore \quad \frac{\delta y}{\delta x}=-2\sin\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\sin\left(x+\frac{dx}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$=-\sin x\cdot 1$$

Thus, $$\frac{dy}{dx} = -\sin x$$

i.e, $$\frac{d}{dx}(\cos x) = -\sin x$$

(iii) Let y = tan x

For a small increament δx in x, let the corresponding increament in y be δy.

$$\therefore \quad y + \delta y = \tan(x + \delta x)$$

and $$\delta y = \tan(x+\delta x) - \tan x = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cdot\cos x - \sin x \cdot \cos(x+\delta x)}{\cos(x+\delta x)\cos x} = \frac{\sin[(x+\delta x) - x]}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin \delta x}{\cos(x+\delta x)\cdot\cos x}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x+\delta x)\cos x}$$

or $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x+\delta x)\cos x}$$

$$= 1 \cdot \frac{1}{\cos^2 x} = \sec^2 x \qquad \left[\therefore \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1\right]$$

Thus, $$\frac{dy}{dx} = \sec^2 x$$

i.e. $$\frac{d}{dx}(\tan x) = \sec^2 x$$

(iv) Let y = sec x

For a small increament δ x in, let the corresponding increament in y be δ y .

$$\therefore \quad y + \delta y = \sec(x + \delta x)$$

and $$\delta y = \sec(x+\delta x) - \sec x = \frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x} = \frac{2\sin\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \frac{\sin x}{\cos^2 x}\cdot 1 = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x.\sec x$$

Thus, $$\frac{dy}{dx} = \sec x.\tan x$$

i.e. $$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

Similarly, we can show that

$$\frac{d}{dx}(\cot x) = -\cos ec^2 x$$

and $$\frac{d}{dx}(\cos ecx) = -\cos ec\, x \cdot \cot x$$

Example 27.1 Find the derivative of cot x^2 from first principle.

Solution: $y = \cot x^2$

For a small increament δx in x, let the corresponding increament in y be δy .

$$\therefore \quad y + \delta y = \cot(x+\delta x)^2$$

$$\delta y = \cot(x+\delta x)^2 - \cot x^2$$

$$= \frac{\cos(x+\delta x)^2}{\sin(x+\delta x)^2} - \frac{\cos x^2}{\sin x^2} = \frac{\cos(x+\delta x)^2 \sin x^2 - \cos x^2 \sin(x+\delta x)^2}{\sin(x+\delta x)^2 \sin x^2}$$

$$= \frac{\sin\left[x^2-(x+\delta x)^2\right]}{\sin(x+\delta x)^2 \sin x^2} = \frac{\sin\left[-2x\delta x-(\delta x)^2\right]}{\sin(x+\delta x)^2 \sin x^2} = \frac{-\sin\left[(2x+\delta x)\delta x\right]}{\sin(x+\delta x)^2 \sin x^2}$$

$\therefore$ $$\frac{\delta y}{\delta x} = \frac{-\sin\left[(2x+\delta x)\delta x\right]}{\delta x \sin(x+\delta x)^2 \sin x^2}$$

and $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\frac{\sin\left[(2x+\delta x)\delta x\right]}{\delta x(2x+\delta x)}\lim_{\delta x\to 0}\frac{2x+\delta x}{\sin(x+\delta x)^2 \sin x^2}$$

or $$\frac{dy}{dx} = -1\cdot\frac{2x}{\sin x^2 . \sin x^2} \qquad \left[\lim_{\delta x\to 0}\frac{\sin\left[(2x+\delta x)\delta x\right]}{\delta x(2x+\delta x)} = 1\right]$$

$$= \frac{-2x}{\left(\sin x^2\right)^2} = \frac{-2x}{\sin^2 x^2} = -2x.\cos ec^2 x^2$$

Hence $$\frac{d}{dx}\left(\cot x^2\right) = -2x\cdot\cos ec^2 x^2$$

Example 27.2 Find the derivative of $\sqrt{\cos ecx}$ from first principle.

Solution: Let $y = \sqrt{\cos ecx}$

and $$y+\delta y = \sqrt{\cos ec(x+\delta x)}$$

$\therefore$ $$\delta y = \frac{\left[\sqrt{\cos ec(x+\delta x)} - \sqrt{\cos ecx}\right]\left[\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}\right]}{\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}}$$

$$= \frac{\cos ec(x+\delta x) - \cos ecx}{\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}} = \frac{\dfrac{1}{\sin(x+\delta x)} - \dfrac{1}{\sin x}}{\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}}$$

$$= \frac{\sin x - \sin(x+\delta x)}{\left[\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}\right]\left[\sin(x+\delta x)\sin x\right]}$$

$$= -\frac{2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\left(\sqrt{\cos ec(x+\delta x)} + \sqrt{\cos ecx}\right)\left[\sin(x+\delta x)\sin x\right]}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\cos\left(x + \frac{\delta x}{2}\right)}{\sqrt{\cos ec\,(x+\delta x)} + \sqrt{\cos ec\, x}} \times \frac{\frac{\sin \delta x/2}{\delta x/2}}{\left[\sin(x+\delta x).\sin x\right]}$$

or
$$\frac{dy}{dx} = \frac{-\cos x}{(2\sqrt{(\cos ec\, x)}(\sin x)^2} = -\frac{1}{2}(\cos ec\, x)^{\frac{1}{2}}(\cos ec\, x \cot x)$$

Thus,
$$\frac{d}{dx}\left(\sqrt{\cos ec\, x}\right) = \frac{1}{2}(\cos ec\, x)^{\frac{1}{2}}(\cos ec\, x \cot x)$$

Example 27.3 Find the derivative of $\sec^2 x$ from first principle.

Solution: Let $y = \sec^2 x$

and
$$y + \delta y = \sec^2(x+\delta x)$$

then,
$$\delta y = \sec^2(x+\delta x) - \sec^2 x = \frac{\cos^2 x - \cos^2(x+\delta x)}{\cos^2(x+\delta x)\cos^2 x}$$

$$= \frac{\sin[(x+\delta x+x]\sin[(x+\delta x - x)]}{\cos^2(x+\delta x)\cos^2 x} = \frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x\delta x}$$

Now,
$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x\delta x}$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\cos^2 x\cos^2 x} = \frac{2\sin x\cos x}{\cos^2 x\cos^2 x} = 2\tan x.\sec^2 x$$

$$= 2\sec x(\sec x.\tan x) = 2\sec x(\sec x\tan x)$$

CHECK YOUR PROGRESS 27.1

1. Find derivative from principle of the following functions with respect to x:

   (a) cosec x (b) cot x (c) cos 2x

   (d) cot 2 x (e) $\cos ec^2 x$ (f) $\sqrt{\sin x}$

2. Find the derivative of each of the following functions:

   (a) $2\sin^2 x$ (b) $\cos ec^2 x$ (c) $\tan^2 x$

27.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You heve learnt how we can find the derivative of a trigonometric function from first principle and also how to deal with these functions as a function of a function as shown in the alternative method. Now we consider some more examples of these derivatives.

Example 27.4 Find the derivative of each of the following functions:

(i) $\sin 2x$ (ii) $\tan\sqrt{x}$ (iii) $\operatorname{cosec}\left(5x^3\right)$

Solution: (i) Let $y = \sin 2x,$

$= \sin t,$ where t = 2 x

$$\frac{dy}{dt} = \cos t \quad \text{and} \quad \frac{dt}{dx} = 2$$

By chain Rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$, we heve

$$\frac{dy}{dx} = \cos t(2) = 2.\cos t = 2\cos 2x$$

Hence, $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$

(ii) Let $y = \tan\sqrt{x}$

$= \tan t$ where t = $\sqrt{x}$

$\therefore \quad \dfrac{dy}{dt} = \sec^2 t \quad$ and $\quad \dfrac{dt}{dx} = \dfrac{1}{2\sqrt{x}}$

By chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$, we heve

$$\frac{dy}{dx} = \sec^2 t \cdot \frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Hence, $\dfrac{d}{dx}\left(\tan\sqrt{x}\right) = \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Alternatively: Let $y = \tan\sqrt{x}$

$$\frac{dy}{dx} = \sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $y = \operatorname{cosec}\left(5x^3\right)$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec}\left(5x^3\right)\cot\left(5x^3\right)\cdot\frac{d}{dx}\left[5x^3\right]$$

$$= -15x^2 \operatorname{cosec}\left(5x^3\right)\cot\left(5x^3\right)$$

or you may solve it by substituting $t = 5x^3$

Example 27.5 Find the derivative of each of the following functions:

(i) $y = x^4 \sin 2x$ (ii) $y = \dfrac{\sin x}{1+\cos x}$

Solution : $y = x^4 \sin 2x$

(i) $$\therefore \quad \frac{dy}{dx} = x^4 \frac{d}{dx}(\sin 2x) + \sin 2x \frac{d}{dx}\left(x^4\right) \quad \text{(Using product rule)}$$

$$= x^4\left(2\cos 2x\right) + \sin 2x\left(4x^3\right)$$

$$= 2x^4 \cos 2x + 4x^3 \sin 2x$$

$$= 2x^3\left[x\cos 2x + 2\sin 2x\right]$$

(ii) Let $y = \dfrac{\sin x}{1+\cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{(1+\cos x)} = \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2}\sec^2\frac{x}{2}$$

Example 27.6 Find the derivative of each of the following functions w.r.t. x:

(i) $\cos^2 x$ (ii) $\sqrt{\sin^3 x}$

Solution : (i) Let $y = \cos^2 x$

$= t^2$ where t = cos x

$$\therefore \qquad \frac{dy}{dt} = 2t \text{ and } \frac{dt}{dx} = -\sin x$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ we have}$$

$$\frac{dy}{dx} = 2\cos x.(-\sin x)$$

$$= -2\cos x \sin x = -\sin 2x$$

(ii) Let $y = \sqrt{\sin^3 x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{2}\left(\sin^3 x\right)^{-1/2} \cdot \frac{d}{dx}\left(\sin^3 x\right) = \frac{1}{2\sqrt{\sin^3 x}} \cdot 3\sin^2 x \cdot \cos x$$

$$= \frac{3}{2}\sqrt{\sin x}\cos x$$

Thus, $$\frac{d}{dx}\left(\sqrt{\sin^3 x}\right) = \frac{3}{2}\sqrt{\sin x}\cos x$$

Example 8.7 Find $\frac{dy}{dx}$, *when*

(i) $y = \sqrt{\dfrac{1-\sin x}{1+\sin x}}$

Solution : We have,

(i) $y = \sqrt{\dfrac{1-\sin x}{1+\sin x}}$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1-\sin x}{1+\sin x}\right]^{\frac{1}{2}} \cdot \frac{d}{dx}\left[\frac{1-\sin x}{1+\sin x}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \frac{(-\cos x)(1+\sin x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \left(\frac{-2\cos x}{(1+\sin x)^2}\right) = \sqrt{\frac{1+\sin x}{1-\sin x} \cdot \frac{\sqrt{1-\sin^2 x}}{(1+\sin x)^2}}$$

$$= -\frac{\sqrt{1+\sin x}\sqrt{1+\sin x}}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

Thus, $dy/dx = -\dfrac{1}{1+\sin x}$

Example 27.8 Find the derivative of each of the following functions at the indicated points :

(i) $y = \sin 2x + (2x-5)^2$ at $x = \dfrac{\pi}{2}$

(ii) $y = \cot x + \sec^2 x + 5$ at $x = \pi/6$

Solution :

(i) $y = \sin 2x + (2x-5)^2$

$$\therefore \quad \frac{dy}{dx} = \cos 2x \frac{d}{dx}(2x) + 2(2x-5)\frac{d}{dx}(2x-5)$$

$$= 2\cos 2x + 4(2x-5)$$

At $x = \dfrac{\pi}{2}$, $\quad \dfrac{dy}{dx} = 2\cos \pi + 4(\pi - 5) = -2 + 4\pi - 20 = 4\pi - 22$

(ii) $y = \cot x + \sec^2 x + 5$

$$\therefore \quad \frac{dy}{dx} = -\cos ec^2 x + 2\sec x(\sec x \tan x) = -\cos ec^2 x + 2\sec^2 x \tan x$$

At $x = \dfrac{\pi}{6}$, $\quad \dfrac{dy}{dx} = -\cos ec^2 \dfrac{\pi}{6} + 2\sec^2 \dfrac{\pi}{6}\tan\dfrac{\pi}{6} = -4 + 2\cdot\dfrac{4}{3}\dfrac{1}{\sqrt{3}} = -4 + \dfrac{8}{3\sqrt{3}}$

Example 27.9 If $\sin y = x \sin(a+y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Solution : It is given that

$\sin y = x \sin(a+y)$ or $x = \dfrac{\sin y}{\sin(a+y)}$

Differentiating w.r.t. x on both sides of (1) we get

$$1 = \left[\frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$1=\left[\frac{\sin(a+y-y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

Example 27.10 If y $=\sqrt{\sin x+\sqrt{\sin x+....to\inf inity}},$

prove that $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Solution : We are given that

$$y=\sqrt{\sin x+\sqrt{\sin x+...to\inf inity}}$$

or $$y=\sqrt{\sin x+y}$$ or $$y^2=\sin x+y$$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$ or $$(2y-1)\frac{dy}{dx}=\cos x$$

Thus, $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

CHECK YOUR PROGRESS 27.2

1. Find the derivative of each of the following functions w.r.tx:

(a) y = 3 sin 4 x (b) y = cos 5 x (c) $y=\tan\sqrt{x}$

(d) $y=\sin\sqrt{x}$ (e) $y=\sin x^2$ (f) $y=\sqrt{2}\tan 2x$

(g) $y=\pi\cot 3x$ (h) $y=\sec 10x$ (i) $y=\cos ec2x$

2. Find the derivative of each of the following functions:

(a) $f(x)=\frac{\sec x-1}{\sec x+1}$ (b) $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

(c) $f(x)=x\sin x$ (d) $f(x)=(1+x^2)\cos x$

(e) $f(x)=x\cos ec\,x$ (f) $f(x)=\sin 2x\cos 3x$

(g) $f(x)=\sqrt{\sin 3x}$

3. Find the derivative of each of the following functions:

(a) $y = \sin^3 x$ (b) $y = \cos^2 x$ (c) $y = \tan^4 x$

(d) $y = \cot^4 x$ (e) $y = \sec^5 x$ (f) $y = \cos^3 x$

(g) $y = \sec\sqrt{x}$ (h) $y = \sqrt{\dfrac{\sec x + \tan x}{\sec - + \tan x}}$

4. Find the derivative of the following functions at the indicated points:

(a) $y = \cos(2x + \pi/2), x = \dfrac{\pi}{3}$ (b) $y = \dfrac{1 + \sin x}{\cos x}, x = \dfrac{\pi}{4}$

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots,}}}$ to infinity

Show that $(2y - 1)\dfrac{dy}{dx} = \sec^2 x.$

6. If $\cos y = x\cos(a + y),$

Prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a + y)}{\sin a}$

27.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

We now find derivatives of standard inverse trignometric functions $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x,$ by first principle.

(i) We will show that by first principle the derivative $\sin^{-1} x$ w.r.t.x is given by

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{\left(1 - x^2\right)}}$$

Let $y = \sin^{-1} x$. Then x = sin y and so x + δx = sin (y+δy)

As $\delta x \to 0, \delta y \to 0.$

Now, $\delta x = \sin(y + \delta) - \sin y$

$$\therefore \quad 1 = \frac{\sin(y + \delta y) - \sin y}{\delta x}$$ [On dividing both sides by δx]

or $$1 = \frac{\sin(y+\delta y)-\sin y}{\delta x}\cdot\frac{\delta y}{\delta x}$$

$\therefore$ $$1 = \lim_{\delta x\to 0}\frac{\sin(y+\delta y)-\sin y}{\delta x}\cdot\lim_{\delta x\to 0}\frac{\delta y}{\delta x} \qquad [\because \delta y \to 0 \, when \, \delta x \to 0]$$

$$= \left[\lim_{\delta x\to 0}\frac{2\cos\left(y+\frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta x}\right]\cdot\frac{dy}{dx} = (\cos y).\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

$\therefore$ $$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{(1-x^2)}}$$

(ii) $$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{(1-x^2)}}.$$

For proof proceed exactly as in the case of $\sin^{-1} x$.

(iii) Now we show that,

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

Let $y = \tan^{-1} x$. Then x = tan y and so $x+\delta x = \tan(y+\delta y)$

As $\delta x \to 0, also\, \delta y \to 0$

Now, $\delta x = \tan(y+\delta y) - \tan y$

$\therefore$ $$1 = \frac{\tan(y+\delta y)-\tan y}{\delta y}.\frac{\delta y}{\delta x}.$$

$\therefore$ $$1 = \lim_{\delta x\to 0}\frac{\tan(y+\delta y)-\tan y}{\delta y}.\lim_{\delta x\to 0}\frac{\,}{\delta x}. \qquad [\because \delta y \to 0 \, when \, \delta x \to 0]$$

$$= \left[\lim_{\delta x \to 0} \left\{ \frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y} \right\} / \delta y \right] . \frac{dy}{dx}$$

$$= \frac{dy}{dx} . \lim_{\delta x \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y . \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} . \lim_{\delta x \to 0} \frac{\sin(y+\delta y - y)}{\delta y . \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} . \lim_{\delta x \to 0} \left[\frac{\sin \delta y}{\delta y} . \frac{1}{\cos(y+\delta y)\cos y} \right]$$

$$= \frac{dy}{dx} . \frac{1}{\cos^2 y} = \frac{dy}{dx} . \sec^2 y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

$$\therefore \quad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

(iv) $$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{1}{1+x^2}$$

For proof proceed exactly as in the case of $\tan^{-1} x$.

(v) We have by first principle $\frac{d}{dx}\left(\sec^{-1} x.\right) = \frac{1}{x\sqrt{\left(x^2+1\right)}}$

Let $y = \sec^{-1} x$. Then = sec y and so $x + \delta x = \sec(y+\delta y)$.

As $\delta x \to 0. also\, \delta y \to 0.$

Now $\delta x = \sec(y+\delta y) - \sec y.$

$$\therefore \quad 1 = \frac{\sec(y+\delta y) - \sec y}{\delta y} . \frac{\delta y}{\delta x}.$$

$$1 = \lim_{\delta x \to 0} \frac{\sec(y+\delta y) - \sec y}{\delta y} . \lim_{\delta x \to 0} \frac{\delta y}{\delta x}. \qquad [\because \delta y \to 0\ when\ \delta x \to 0]$$

$$= \frac{dy}{dx} \cdot \lim_{\delta x \to 0} \frac{2\sin\left(y + \frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y . \cos y \cos\left(y + \delta y\right)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta x \to 0} \left[\frac{\sin\left(y + \frac{1}{2}\delta y\right)}{\cos y \cos\left(y + \delta y\right)} \cdot \frac{\sin\left(\frac{1}{2}\delta y\right)}{\frac{1}{2}\delta y} \right]$$

$$= \frac{dy}{dx} \cdot \frac{\sin y}{\cos y \cos y} = \frac{dy}{dx} . \sec y \tan y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec\sqrt{\left(\sec^2 y - 1\right)}} = \frac{1}{x\sqrt{\left(x^2 - 1\right)}}$$

$$\therefore \quad \frac{d}{dx} = \left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}$$

(v) $$\frac{d}{dx} = \left(co\sec^{-1} x\right) = \frac{1}{x\sqrt{\left(x^2 - 1\right)}}.$$

For proof proceed as in the case of $\sec^{-1} x$.

Example 27.11 Find derivative of $\sin^{-1}\left(x^2\right)$ from first principle.

Solution: Let $y = \sin^{-1} x^2$

$$\therefore \quad x^2 = \sin y$$

Now, $$\left(x + \delta x\right)^2 = \sin\left(y + \delta y\right)$$

$$\frac{\left(x + \delta x\right)^2 - x^2}{\delta x} = \frac{\sin\left(y + \delta x\right) - \sin y}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\left(x + \delta x\right)^2 - x^2}{\left(x + \delta x\right) - x} = \lim_{\delta x \to 0} \frac{2\cos\left(y + \frac{\delta x}{2}\right)\sin\frac{\delta y}{2}}{2} \cdot \frac{\frac{\delta y}{2}}{\frac{\delta y}{2}} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$\Rightarrow \qquad 2x = \cos y . \frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2x}{\cos y} = \frac{2x}{\sqrt{1-\sin^2 y}} = \frac{2x}{\sqrt{1-x^4}}.$$

Example 27.12 Find derivative of $\sin^{-1}\sqrt{x}$ w.r.t.x by first principle

Solution: Let $\quad y = \sin^{-1}\sqrt{x}$

$$\Rightarrow \qquad \sin y = \sqrt{x}$$

Also $\qquad \sin(y+\delta y) = \sqrt{x+\delta x}$

From (1) and (2), we get

$$\sin(y+\delta y) - \sin y = \sqrt{x+\delta x} - \sqrt{x}$$

or $\qquad 2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right) = \dfrac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}}$

$$= \frac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\therefore \qquad \frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\delta x} = -\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

or $\qquad \dfrac{\delta y}{\delta x}.\cos\left(y+\dfrac{\delta y}{2}\right).\dfrac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}} = \dfrac{1}{\sqrt{x+\delta x}+\sqrt{x}}$

$$\therefore \qquad \lim_{\delta x \to 0}\frac{\delta y}{\delta x}.\lim_{\delta x \to 0}\cos\left(y+\frac{\delta y}{2}\right).\lim_{\delta x \to 0}\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}$$

$$= \lim_{\delta x \to 0}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}} \qquad (\because \delta y \to 0 \; as \; \delta x \to 0)$$

or $\qquad \dfrac{dy}{dx}\cos = \dfrac{1}{2\sqrt{x}}$ or $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}\cos y} = \dfrac{1}{2\sqrt{x}\sqrt{1-\sin^2 y}} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

CHECK YOUR PROGRESS 27.3

1. Find by first principle that derivative of each of the following:

(i) $\cos^{-1} x^2$ (ii) $\frac{\cos^{-1} x}{x}$ (iii) $\cos^{-1}\sqrt{x}$

(iv) $\tan^{-1} x^2$ (v) $\frac{\tan^{-1} x}{x}$ (vi) $\tan^{-1}\sqrt{x}$

27.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section, we have learnt to find derivatives of inverse trignometric functions by first principle. Now we learn to find derivatives of inverse trigonometric functions using these results

Example 27.13 Find the derivative of each of the following:

(i) $\sin^{-1}\sqrt{x}$ (ii) $\cos^{-1} x^2$ (iii) $\left(\cos^{-1} x\right)^2$

Solution:

(i) Let $y = \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{\sqrt{1-x}}.\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{\times}\sqrt{1-x}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{2\sqrt{\times}\sqrt{1-x}}$$

(iii) Let $y = \cos^{-1} x^2$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(x^2\right)^2}}.\frac{d}{dx}\left(x^2\right) = \frac{-1}{\sqrt{1-x^4}}.(2x)$$

$$\therefore \quad \frac{d}{dx}\left(\cos^{-1} x^2\right) = \frac{-2x}{\sqrt{1-x^4}}$$

(iii) Let $y = \left(\cos^{-1} x\right)^2$

$$\frac{dy}{dx}=2\left(\cos ec^{-1}x\right).\frac{d}{dx}\left(\cos ec^{-1}x\right)=2\left(\cos ec^{-1}x\right).\frac{1}{|x|\sqrt{x^2-1}}$$

$$=\frac{-2\cos ec^{-1}x}{|x|\sqrt{x^2-1}}$$

$$\therefore \quad \frac{d}{dx}\left(\cos ec^{-1}x\right)^2=\frac{-2\cos ec^{-1}x}{|x|\sqrt{x^2-1}}$$

Example 27.14 Find the derivative of each of the following:

(i) $\tan^{-1}\dfrac{\cos x}{1+\sin x}$ (ii) $\sin\left(2\sin^{-1}x\right)$

Solution:

(i) Let $y=\tan^{-1}\dfrac{\cos x}{1+\sin x}=\tan^{-1}\dfrac{\sin\left(\frac{\pi}{2}-2\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]=\tan\frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx}=-1/2$$

(ii) $y=\sin\left(2\sin^{-1}x\right)$

Let $y=\sin\left(2\sin^{-1}x\right)$

$$\therefore \quad \frac{dy}{dx}=\cos\left(2\sin^{-1}x\right).\frac{d}{dx}\left(2\sin^{-1}x\right)$$

$$\therefore \quad \frac{dy}{dx}=\cos\left(2\sin^{-1}x\right).\frac{2}{\sqrt{1-x^2}}$$

$$=\frac{2\cos\left(2\sin^{-1}x\right)}{\sqrt{1-x^2}}$$

Example 27.15 Show that the derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ $w.r.t \sin^{-1}\dfrac{2x}{1+x^2}$ is 1.

Solution: Let $y=\tan^{-1}\dfrac{2x}{1-x^2}$ and $z=\sin^{-1}\dfrac{2x}{1+x^2}$

Let $\quad x = \tan\theta$

$$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} \; and \;\; z = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \tan^{-1}(\tan 2\theta) \; and \;\; z = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta \qquad\qquad and \;\; z = 2\theta$$

$$\frac{dy}{d\theta} - 2 \qquad \text{and} \quad \frac{dz}{d\theta} = 2$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dz} = 2\cdot\frac{1}{2} = 1 \qquad \text{(By chain rule)}$$

CHECK YOUR PROGRESS 27.4

Find the derivative of each of the following functions w.r.t. x and express the result in the simplest form (1-3):

1. (a) $\sin^{-1}x^2$ (b) $\cos^{-1}\frac{x}{2}$ (c) $\cos^{-1}\frac{1}{x}$

2. (a) $\tan^{-1}(\operatorname{cosec} x - \cot x)$ (b) $\cot^{-1}(\sec x + \tan x)$ (c) $\cot^{-1}\frac{\cos x - \sin x}{\cos x + \sin x}$

3. (a) $\sin\left(\cos^{-1}x\right)$ (b) $\sec\left(\tan^{-1}x\right)$ (c) $\sin^{-1}\left(1-2x^2\right)$

   (d) $\cos^{-1}\left(4x^3-3x\right)$ (e) $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$

4. Find the derivative of:

   $$\frac{\tan^{-1}x}{1+\tan^{-1}x} \; w.r.t \tan^{-1}x.$$

27.5 SECOND ORDER DERIVATIVES

We know that the second order derivative of a functions is the derivative of the first derivative of that function. In this section, we shall find the second order derivatives of trigonometric and inverse trigonometric functions. In the process, we shall be using product rule, quotient rule and chain rule.

Let us take some examples.

Example 27.16 Find the second order derivative of

(i) sin x (ii) x cos x (iii) $\cos^{-1}x$

Solution: (i) Let $y = \sin x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) = -\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\sin x$$

(ii) Let $y = x \cos x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = x(-\sin x) + \cos .1$$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-x\sin x + \cos x) \quad = -(x.\cos x + \sin x) - \sin x$$

$$= -x.\cos x - 2\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -(x.\cos x + 2\sin x)$$

(iii) Let $y = \cos^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{1/2}} = -(1-x^2)^{\frac{1}{2}}$$

Differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}\cdot(1-x^2)^{-3/2}\cdot(-2x)\right] = -\frac{x}{(1-x^2)^{-3/2}}$$

$$\frac{d^2y}{dx^2} = \frac{-x}{(1-x^2)^{-3/2}}$$

Example 27.17 If $y=\sin^{-1}x$, show that $\left(1-x^2\right)y_2-xy_1=0$, where y_2 and y_1 respectively denote the second and first, order derivatives of y w.r.t. x.

Solution: We have, $y=\sin^{-1}x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

or $$\left(\frac{dy}{dx}\right)^2=\frac{1}{1-x^2} \qquad \text{(Squaring both sides)}$$

or $$\left(1-x^2\right)\left(y_1\right)^2=1$$

Differentiating w.r.t. x both sides, we get

$$\left(1-x^2\right)\cdot 2y_1\frac{d}{dx}\left(y_1\right)+\left(-2x\right)\cdot y_1^2=0$$

or $$\left(1-x^2\right)\cdot 2y_1y_2-2xy_1^2=0$$

or $$\left(1-x^2\right)y_2-xy_1=0$$

CHECK YOUR PROGRESS 27.5

1. Find the second order derivative of each of the following:

   (a) $\sin\left(\cos x\right)$ (b) $x^2\tan^{-1}x$

2. If $y=\frac{1}{2}\left(\sin^{-1}x\right)^2$, show that $\left(1-x^2\right)y_2-xy_1=1$.

3. If $y=\sin\left(\sin x\right)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$.

4. If y = x + tan x, show that $\cos^2 x\frac{d^2y}{dx^2}-2y+2x=0$

LET US SUM UP

- (i) $\frac{d}{dx}\left(\sin x\right)=\cos x$ (ii) $\frac{d}{dx}\left(\cos x\right)=-\sin x$

  (iii) $\frac{d}{dx}\left(\tan x\right)=\sec^2 x$ (iv) $\frac{d}{dx}\left(\cot x\right)=-co\sec^2 x$

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (vi) $\frac{d}{dx}(\operatorname{cosec} x) = -co\sec x \cot x$

- If u is a derivable function of x, then

(i) $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

(iii) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot u) = -\cos ec^2 u \frac{du}{dx}$

(v) $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$ (vi) $\frac{d}{dx}(\cos ecu) = -\cos ecu \cot u \frac{du}{dx}$

- (i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{-1}{1-x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

- If u is a derivable function of x, then

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

(iii) $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$

(v) $\frac{d}{dx}(\sec^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$ (vi) $\frac{d}{dx}(\cos ec^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

The second order derivative of a trignometric function is the derivative of their first order derivatives.

SUPPORTIVE WEB SITES

http://people.hofstra.edu/stefan_waner/trig/trig3.html

http://www.math.com/tables/derivatives/more/trig.htm

https://www.freemathhelp.com/trig-derivatives.html

TERMINAL EXERCISE

1. If $y = x^3 \tan^2 \frac{x}{2}$, *find* $\frac{dy}{dx}$.

2. Evaluate, $\frac{d}{dx}\sqrt{\sin^4 x + \cos^4 x}$ *at* $x = \frac{\pi}{2}$ and 0.

3. If $y = \frac{5x}{\sqrt[3]{(1-x)^2}} + \cos^2(2x+1)$, find $\frac{dy}{dx}$.

4. $y = \sec^{-1}\frac{\sqrt{x+1}}{\sqrt{x-1}} + \sin^{-1}\frac{\sqrt{x-1}}{\sqrt{x}}$, then show that $\frac{dy}{dx} = 0$

5. If $x = a\cos^3\theta, y = a\sin^3\theta$, then find $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

6. If $y = \sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}$, find $\frac{dy}{dx}$.

7. Find the derivative of $\sin^{-1x}$ *w.r.t* $\cos^{-1}\sqrt{1-x^2}$

8. If y = cos (cos x), prove that

   $$\frac{d^2y}{dx^2} - \cot x \cdot \frac{dy}{dx} + y.\sin^2 x = 0.$$

9. If $y = \tan^{-1} x$ show that

   $$(1+x)^2 y_2 + 2xy_1 = 0.$$

10. If $y = (\cos^{-1} x)^2$ show that

    $$(1+x)^2 y_2 - xy_1 - 2 = 0.$$

ANSWERS

CHECK YOUR PROGRESS 27.1

(1) (a) $-\cos ecx \cot x$ (b) $-\cos ec^2 x$ (c) -2 sin 2 x

(d) $-2\cos ec^2 2x$ (e) $-2x\cos ecx^2 \cot x^2$ (f) $\dfrac{\cos x}{2\sqrt{\sin x}}$

2. (a) 2 sin 2 x (b) $-2\cos ec^2 x \cot x$ (c) $2\tan x \sec^2 x$

CHECK YOUR PROGRESS 27.2

1. (a) 12 cos 4 x (b) - 5 sin 5 x (c) $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

(d) $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$ (e) $2x\cos x^2$ (f) $2\sqrt{2}\sec^2 2x$

(g) $-3\pi \cos ec^2 3x$ (h) 10 sec 10 x tan 10 x

(I) $-2\cos ec 2x \cot 2x$

2. (a) $\dfrac{2\sec x \tan x}{(\sec x+1)^2}$ (b) $\dfrac{-2}{(\sin x-\cos x)^2}$ (c) $x\cos x+\sin x$

(d) $2x\cos x-(1+x^2)\sin x$ (e) $\cos ecx(1-x\cot x)$

(f) $2\cos 2x\cos 3x-3\sin 2x \sin 3x$ (g) $\dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

3. (a) $3\sin^2 x\cos x$ (b) $-\sin 2x$ (c) $4\tan^3 x\sec^2 x$

(d) $-4\cot^3 x\cos ec^2 x$ (e) $5\sec^5 x\tan x$ (f) $-3\cos ec^3 x\cot x$

(g) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$ (h) $\sec x(\sec x+\tan x)$

4. (a) 1 (b) $\sqrt{2}+2$

CHECK YOUR PROGRESS 27.3

1. (i) $\dfrac{-2x}{\sqrt{1-x^4}}$ (ii) $\dfrac{-1}{x\sqrt{1-x^2}} - \dfrac{-\cos^{-1}x}{x^2}$

(iii) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$ (iv) $\dfrac{2x}{1+x^2}$

(v) $\dfrac{1}{x\left(1+x^2\right)} - \dfrac{\tan^{-1}x}{x^2}$ (vi) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$

CHECK YOUR PROGRESS 27.4

1. (a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{-1}{\sqrt{4-x^2}}$

(c) $\dfrac{1}{x\sqrt{x^2-1}}$

2. (a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$

(c) -1

3. (a) $\dfrac{\cos\left(\cos^{-1}x\right)}{\sqrt{1-x^2}}$ (b) $\dfrac{x}{1+x^2}\cdot\sec\left(\tan^{-1}x\right)$

(c) $\dfrac{-2}{\sqrt{1-x^2}}$ (d) $\dfrac{-3}{\sqrt{1-x^2}}$

(e) $\dfrac{-1}{2\left(1+x^2\right)}$

4. $\dfrac{1}{\left(1+\tan^{-1}x\right)^2}$

CHECK YOUR PROGRESS 27.5

1. (a) $-\cos x\cos\left(\cos x\right) - \sin^2 x\sin\left(\cos x\right)$

(b) $\dfrac{2x\left(2+x^2\right)}{\left(1+x^2\right)^2} + 2\tan^{-1}x$

TERMINAL EXERCISE

1. $x^3 \tan\frac{x}{2}\sec^2\frac{x}{2} + 3x^2\tan^2\frac{x}{2}$

2. 0,0

3. $\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}} - 2\sin(4x+2)$

5. $|\sec\theta|$

6. $\frac{1}{2y-1}$

7. $\frac{1}{2\sqrt{1-x^2}}$

28

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DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We are aware that population generally grows but in some cases decay also. There are many other areas where growth and decay are continuous in nature. Examples from the fields of Economics, Agriculture and Business can be cited, where growth and decay are continuous. Let us consider an example of bacteria growth. If there are 10,00,000 bacteria at present and say they are doubled in number after 10 hours, we are interested in knowing as to after how much time these bacteria will be 30,00,000 in number and so on.

Answers to the growth problem does not come from addition (repeated or otherwise), or multiplication by a fixed number. In fact Mathematics has a tool known as exponential function that helps us to find growth and decay in such cases. Exponential function is inverse of logarithmic function. We shall also study about Rolle's Theorem and Mean Value Theorems and their applications. In this lesson, we propose to work with this tool and find the rules governing their derivatives.

OBJECTIVES

After studying this lesson, you will be able to :

- define and find the derivatives of exponential and logarithmic functions;
- find the derivatives of functions expressed as a combination of algebraic, trigonometric, exponential and logarithmic functions; and
- find second order derivative of a function.
- state Rolle's Theorem and Lagrange's Mean Value Theorem; and
- test the validity of the above theorems and apply them to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Application of the following standard limits :

(i) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ (ii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$

(iii) $\lim\limits_{h \to 0} \dfrac{\left(e^h - 1\right)}{h} = 1$

MODULE - VIII
Calculus

Notes

- Definition of derivative and rules for finding derivatives of functions.

28.1 DERIVATIVE OF EXPONENTIAL FUNCTIONS

Let $y = e^x$ be an exponential function. .....(i)

$\therefore \quad y + \delta y = e^{(x+\delta x)}$ (Corresponding small increments) .....(ii)

From (i) and (ii), we have

$$\therefore \quad \delta y = e^{x+\delta x} - e^x$$

Dividing both sides by δx and taking the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \frac{\left[e^{\delta x} - 1\right]}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x.1 = e^x$$

Thus, we have $\mathbf{\frac{d}{dx}\left(e^x\right) = e^x}$.

Working rule : $\frac{d}{dx}\left(e^x\right) = e^x \cdot \frac{d}{dx}(x) = e^x$

Next, let $y = e^{ax+b}$.

Then $y + \delta y = e^{a(x+\delta x)+b}$

[δx and δy are corresponding small increments]

$$\therefore \quad \delta y = e^{a(x+\delta x)+b} - e^{ax+b}$$

$$= e^{ax+b}\left[e^{a\delta x} - 1\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = e^{ax+b} \frac{\left[e^{a\delta x} - 1\right]}{\delta x}$$

$$= a \cdot e^{ax+b} \frac{e^{a\delta x} - 1}{a\delta x} \qquad \text{[Multiply and divide by a]}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a \cdot e^{ax+b} \cdot \lim_{\delta x \to 0} \frac{e^{a\delta x} - 1}{a\ \delta x}$$

or $$\frac{dy}{dx} = a \cdot e^{ax+b} \cdot 1 \qquad \left[\lim_{x \to 0} \frac{e^x - 1}{x} = 1\right] \qquad = ae^{ax+b}$$

Working rule :

$$\frac{d}{dx}\left(e^{ax+b}\right) = e^{ax+b} \cdot \frac{d}{dx}(ax+b) = e^{ax+b} \cdot a$$

$$\therefore \quad \frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

Example 28.1 Find the derivative of each of the follwoing functions :

(i) e^{5x} (ii) e^{ax} (iii) $e^{-\frac{3x}{2}}$

Soution : (i) Let $y = e^{5x}$.

Then $\quad y = e^t$ where $5x = t$

$$\therefore \quad \frac{dy}{dt} = e^t \quad \text{and } 5 = \frac{dt}{dx}$$

We know that, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = e^t \cdot 5 = 5e^{5x}$

Alternatively $\dfrac{d}{dx}\left(e^{5x}\right) = e^{5x} \cdot \dfrac{d}{dx}(5x) = e^{5x} \cdot 5 = 5e^{5x}$

(ii) Let $\quad y = e^{ax}$.

Then $\quad y = e^t$ when $t = ax$

$$\therefore \quad \frac{dy}{dt} = e^t \text{ and } \frac{dt}{dx} = a$$

We know that, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = e^t \cdot a$

Thus, $\dfrac{dy}{dx} = a \cdot e^{ax}$

(iii) Let $\quad y = e^{\frac{-3x}{2}}$

$$\therefore \quad \frac{dy}{dt} = e^{\frac{-3}{2}x} \cdot \frac{d}{dx}\left(\frac{-3}{2}x\right)$$

Thus, $\dfrac{dy}{dt} = \dfrac{-3}{2} e^{\frac{-3x}{2}}$

Example 28.2 Find the derivative of each of the following :

(i) $y = e^x + 2\cos x$ (ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$

Solution : (i) $y = e^x + 2\cos x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + 2\frac{d}{dx}(\cos x) = e^x - 2\sin x$$

(ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$

$$\therefore \quad \frac{dy}{dx} = e^{x^2}\frac{d}{dx}\left(x^2\right) + 2\cos x - \frac{5}{3}e^x + 0 \quad \text{[Since e is constant]}$$

$$= 2xe^{x^2} + 2\cos x - \frac{5}{3}e^x$$

Example 28.3 Find $\frac{dy}{dx}$, when

(i) $y = e^{x\cos x}$ (ii) $y = \frac{1}{x}e^x$ (iii) $y = e^{\frac{1-x}{1+x}}$

Solution : (i) $y = e^{x\cos x}$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\frac{d}{dx}(x\cos x)$$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(x)\right]$$

$$= e^{x\cos x}\left[-x\sin x + \cos x\right]$$

(ii) $y = \frac{1}{x}e^x$

$$\therefore \quad \frac{dy}{dx} = e^x\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}\left(e^x\right) \quad \text{[Using product rule]}$$

$$= \frac{-1}{x^2}e^x + \frac{1}{x}e^x$$

$$= \frac{e^x}{x^2}[-1+x] = \frac{e^x}{x^2}[x-1]$$

(iii) $y = e^{\frac{1-x}{1+x}}$

$$\frac{dy}{dx} = e^{\frac{1-x}{1+x}}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-1.(1+x)-(1-x).1}{(1+x)^2}\right]$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-2}{(1+x)^2}\right] = \frac{-2}{(1+x)^2}e^{\frac{1-x}{1+x}}$$

Example 28.4 Find the derivative of each of the following functions :

(i) $e^{\sin x} \cdot \sin e^x$ (ii) $e^{ax} \cdot \cos(bx+c)$

Solution : $y = e^{\sin x} \cdot \sin e^x$

$$\therefore \quad \frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}\left(\sin e^x\right) + \sin e^x \frac{d}{dx} e^{\sin x}$$

$$= e^{\sin x} \cdot \cos e^x \cdot \frac{d}{dx}\left(e^x\right) + \sin e^x \cdot e^{\sin x} \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cdot \cos e^x \cdot e^x + \sin e^x \cdot e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}[e^x \cdot \cos e^x + \sin e^x \cdot \cos x]$$

(ii) $y = e^{ax} \cos(bx+c)$

$$\therefore \quad \frac{dy}{dx} = e^{ax} \cdot \frac{d}{dx}\cos(bx+c) + \cos(bx+c)\frac{d}{dx}e^{ax}$$

$$= e^{ax} \cdot [-\sin(bx+c)]\frac{d}{dx}(bx+c) + \cos(bx+c)e^{ax}\frac{d}{dx}(ax)$$

$$= -e^{ax}\sin(bx+c) \cdot b + \cos(bx+c)e^{ax}.a$$

$$= e^{ax}[-b\sin(bx+c) + a\cos(bx+c)]$$

Example 28.5 Find $\frac{dy}{dx}$, if $y = \frac{e^{ax}}{\sin(bx+c)}$

Solution :

$$\frac{dy}{dx} = \frac{\sin(bx+c)\frac{d}{dx}e^{ax} - e^{ax}\frac{d}{dx}[\sin(bx+c)]}{\sin^2(bx+c)}$$

$$= \frac{\sin(bx+c).e^{ax}.a - e^{ax}\cos(bx+c).b}{\sin^2(bx+c)}$$

$$= \frac{e^{ax}[a\sin(bx+c) - b\cos(bx+c)]}{\sin^2(bx+c)}$$

CHECK YOUR PROGRESS 28.1

1. Find the derivative of each of the following functions :

(a) e^{5x} (b) e^{7x+4} (c) $e^{\sqrt{2}x}$ (d) $e^{\frac{-7}{2}x}$ (e) e^{x^2+2x}

2. Find $\frac{dy}{dx}$, if

(a) $y=\frac{1}{3}e^x-5e$ (b) $y=\tan x+2\sin x+3\cos x-\frac{1}{2}e^x$

(c) $y=5\sin x-2e^x$ (d) $y=e^x+e^{-x}$

3. Find the derivative of each of the following functions :

(a) $f(x)=e^{\sqrt{x+1}}$ (b) $f(x)=e^{\sqrt{\cot x}}$

(c) $f(x)=e^{x\sin^2 x}$ (d) $f(x)=e^{x\sec^2 x}$

4. Find the derivative of each of the following functions :

(a) $f(x)=(x-1)e^x$ (b) $f(x)=e^{2x}\sin^2 x$

5. Find $\frac{dy}{dx}$, if

(a) $y=\frac{e^{2x}}{\sqrt{x^2+1}}$ (b) $y=\frac{e^{2x}\cdot\cos x}{x\sin x}$

28.2 DERIVATIVE OF LOGARITHMIC FUNCTIONS

We first consider logarithmic function

Let $y=\log x$ .....(i)

$\therefore$ $y+\delta y=\log(x+\delta x)$ .....(ii)

(δx and δy are corresponding small increments in x and y)

From (i) and (ii), we get

$$\delta y=\log(x+\delta x)-\log x$$

$$=\log\frac{x+\delta x}{x}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$

$$=\frac{1}{x}\cdot\frac{x}{\delta x}\log\left[1+\frac{\delta x}{x}\right] \qquad \text{[Multiply and divide by x]}$$

$$=\frac{1}{x}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Taking limits of both sides, as $\delta x\to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{1}{x}\lim_{\delta x\to 0}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log\left\{\lim_{\delta x \to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right\}$$

$$= \frac{1}{x}\log e \qquad \left[\because \lim_{\delta x \to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e\right]$$

$$= \frac{1}{x}$$

Thus, $$\mathbf{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

Next, we consider logarithmic function

$$y = \log(ax+b) \qquad ...(i)$$

$\therefore$ $$y + \delta y = \log[a(x+\delta x)+b] \qquad ...(ii)$$

[δx and δy are corresponding small increments]

From (i) and (ii), we get

$$\delta y = \log[a(x+\delta x)+b] - \log(ax+b)$$

$$= \log\frac{a(x+\delta x)+b}{ax+b}$$

$$= \log\frac{(ax+b)+a\delta x}{ax+b}$$

$$= \log\left[1+\frac{a\delta x}{ax+b}\right]$$

$\therefore$ $$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$= \frac{a}{ax+b}\cdot\frac{ax+b}{a\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right] \qquad \left[\text{Multiply and divide by } \frac{a}{ax+b}\right]$$

$$= \frac{a}{ax+b}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

Taking limits on both sides as $\delta x \to 0$

$\therefore$ $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{a}{ax+b}\lim_{\delta x \to 0}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

or $$\frac{dy}{dx} = \frac{a}{ax+b}\log e \qquad \left[\because \lim_{x \to 0}(1+x)^{\frac{1}{x}} = e\right]$$

or, $$\frac{dy}{dx} = \frac{a}{ax+b}$$

Working rule :

$$\frac{d}{dx}\log(ax+b) = \frac{1}{ax+b}\frac{d}{dx}(ax+b)$$

$$= \frac{1}{ax+b} \times a = \frac{a}{ax+b}$$

Example 28.6 Find the derivative of each of the functions given below :

(i) $y = \log x^5$ (ii) $y = \log\sqrt{x}$ (iii) $y = (\log x)^3$

Solution : (i) $y = \log x^5 = 5\log x$

$\therefore$ $$\frac{dy}{dx} = 5 \cdot \frac{1}{x} = \frac{5}{x}$$

(ii) $y = \log\sqrt{x} = \log x^{\frac{1}{2}}$ or $y = \frac{1}{2}\log x$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x} = \frac{1}{2x}$$

(iii) $y = (\log x)^3$

$\therefore$ $y = t^3,$ when $t = \log x$

$\Rightarrow$ $$\frac{dy}{dt} = 3t^2 \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

We know that, $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 3t^2 \cdot \frac{1}{x}$$

$\therefore$ $$\frac{dy}{dx} = 3(\log x)^2 \cdot \frac{1}{x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{3}{x}(\log x)^2$$

Example 28.7 Find, $\frac{dy}{dx}$ if

(i) $y = x^3 \log x$ (ii) $y = e^x \log x$

Solution :

(i) $y = x^3 \log x$

$\therefore$ $$\frac{dy}{dx} = \log x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(\log x)$$ [Using Product rule]

$$= 3x^2 \log x + x^3 \cdot \frac{1}{x}$$

$$= x^2(3\log x + 1)$$

(ii) $y = e^x \log x$

$\therefore$ $$\frac{dy}{dx} = e^x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot \frac{1}{x} + e^x \cdot \log x$$

$$= e^x \left[\frac{1}{x} + \log x\right]$$

Example 28.8 Find the derivative of each of the following functions :

(i) $\log \tan x$ (ii) $\log [\cos (\log x)]$

Solution : (i) Let $y = \log \tan x$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{1}{\tan x} \cdot \sec^2 x$$

$$= \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}$$

$$= \text{cosec } x . \sec x$$

(ii) Let $y = \log [\cos (\log x)]$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\cos(\log x)} \cdot \frac{d}{dx}[\cos(\log x)]$$

$$= \frac{1}{\cos(\log x)} \cdot \left[-\sin \log x \frac{d}{dx}(\log x)\right]$$

$$= \frac{-\sin(\log x)}{\cos(\log x)} \cdot \frac{1}{x}$$

$$= -\frac{1}{x} \tan(\log x)$$

Example 28.9 Find $\frac{dy}{dx}$, if $y = \log(\sec x + \tan x)$

Solution : $y = \log (\sec x + \text{tax } x)$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{\sec x + \tan x} \cdot \left[\sec x \tan x + \sec^2 x\right]$$

$$= \frac{1}{\sec x + \tan x} \cdot \sec x \left[\sec x + \tan x\right]$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 28.10 Find $\frac{dy}{dx}$, if

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x-7)^{\frac{3}{4}}}$$

Solution : Although, you can find the derivative directly using quotient rule (and product rule) but if you take logarithm on both sides, the product changes to addition and division changes to subtraction. This simplifies the process:

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x-7)^{\frac{3}{4}}}$$

Taking logarithm on both sides, we get

$$\therefore \quad \log y = \log\left[\frac{\left(4x^2 - 1\right)\left(1 + x^2\right)^{\frac{1}{2}}}{x^3 (x-7)^{\frac{3}{4}}}\right]$$

or $$\log y = \log\left(4x^2 - 1\right) + \frac{1}{2}\log\left(1 + x^2\right) - 3\log x - \frac{3}{4}\log(x-7)$$

[Using log properties]

Now, taking derivative on both sides, we get

$$\frac{d}{dx}(\log y) = \frac{1}{4x^2 - 1} \cdot 8x + \frac{1}{2(1+x^2)} \cdot 2x - \frac{3}{x} - \frac{3}{4} \cdot \left(\frac{1}{x-7}\right)$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

$$= \frac{(4x^2 - 1)\sqrt{1+x^2}}{x^3 (x-7)^{\frac{3}{4}}}\left[\frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

CHECK YOUR PROGRESS 28.2

1. Find the derivative of each the functions given below:

   (a) $f(x) = 5 \sin x \;\; 2 \log x$ (b) $f(x) = \log \cos x$

2. Find $\frac{dy}{dx}$, if

   (a) $y = e^{x^2} \log x$ (b) $y = \frac{e^{x^2}}{\log x}$

3. Find the derivative of each of the following functions :

   (a) $y = \log(\sin \log x)$ (b) $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

   (c) $y = \log\left[\frac{a + b\tan x}{a - b\tan x}\right]$ (d) $y = \log(\log x)$

4. Find $\frac{dy}{dx}$, if

   (a) $y = (1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}$ (b) $y = \frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}$

28.3 DERIVATIVE OF LOGARITHMIC FUNCTION (CONTINUED)

We know that derivative of the function x^n w.r.t. x is $n x^{n-1}$, where n is a constant. This rule is not applicable, when exponent is a variable. In such cases we take logarithm of the function and then find its derivative.

Therefore, this process is useful, when the given function is of the type $[f(x)]^{g(x)}$. For example,

a^x, x^x etc.

Note : Here f(x) may be constant.

Derivative of a^x w.r.t. x

Let $y = a^x$, $a > 0$

Taking log on both sides, we get

$$\log y = \log a^x = x \log a \qquad [\log m^n = n \log m]$$

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log a) \quad \text{or} \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$$

or $$\frac{dy}{dx} = y \log a$$

$$= a^x \log a$$

Thus, $\quad \dfrac{d}{dx} a^x = a^x \log a, \qquad a > 0$

Example 28.11 Find the derivative of each of the following functions :

(i) $y = x^x$ $\qquad$ (ii) $y = x^{\sin x}$

Solution : (i) $\quad y = x^x$

Taking logrithms on both sides, we get

$$\log y = x \log x$$

Taking derivative on both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x) \qquad \text{[Using product rule]}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \log x + x \cdot \frac{1}{x}$$

$$= \log x + 1$$

$\therefore \qquad \dfrac{dy}{dx} = y[\log x + 1]$

Thus, $\qquad \dfrac{dy}{dx} = x^x(\log x + 1)$

(ii) $\qquad y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \sin x \log x$$

$\therefore \qquad \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{d}{dx}(\sin x \log x)$

or $\qquad \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \cos x . \log x + \sin x \cdot \dfrac{1}{x}$

or $\qquad \dfrac{dy}{dx} = y\left[\cos x \log x + \dfrac{\sin x}{x}\right]$

Thus, $\qquad \dfrac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \dfrac{\sin x}{x}\right]$

Example 28.12 Find the derivative, if

$$y = (\log x)^x + \left(\sin^{-1} x\right)^{\sin x}$$

Solution : Here taking logarithm on both sides will not help us as we cannot put

$(\log x)^x + (\sin^{-1} x)^{\sin x}$ in simpler form. So we put

$$u = (\log x)^x \quad \text{and} \quad v = (\sin^{-1} x)^{\sin x}$$

Then, $\quad y = u + v$

$$\therefore \qquad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{......(i)}$$

Now $\quad u = (\log x)^x$

Taking log on both sides, we have

$$\log u = \log(\log x)^x$$

$$\therefore \qquad \log u = x \log(\log x) \qquad \left[\because \log m^n = n \log m\right]$$

Now, finding the derivative on both sides, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = 1 \cdot \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$$

Thus, $$\frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

Thus, $$\frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right] \qquad \text{.....(ii)}$$

Also, $\quad v = (\sin^{-1} x)^{\sin x}$

$$\therefore \qquad \log v = \sin x \log(\sin^{-1} x)$$

Taking derivative on both sides, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[\sin x \log(\sin^{-1} x)]$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \cos x \cdot \log\left(\sin^{-1} x\right)$$

or, $$\frac{dv}{dx} = v\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \cdot \log \sin^{-1} x\right]$$

$$= \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log\left(\sin^{-1} x\right)\right] \qquad \text{....(iii)}$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log \sin^{-1} x\right]$$

Example 28.13 If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Solution : It is given that $\quad x^y = e^{x-y} \quad$ ...(i)

Taking logarithm on both sides, we get

$$y \log x = (x-y)\log e$$
$$= (x-y)$$

or $\quad y(1+\log x) = x \quad [\because \log e = 1]$

or $\quad y = \frac{x}{1+\log x} \quad$ (ii)

Taking derivative with respect to x on both sides of (ii), we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 28.14 Find , $\frac{dy}{dx}$ if

$$e^x \log y = \sin^{-1} x + \sin^{-1} y$$

Solution : We are given that

$$e^x \log y = \sin^{-1} x + \sin^{-1} y$$

Taking derivative with respect to x of both sides, we get

$$e^x\left(\frac{1}{y}\frac{dy}{dx}\right) + e^x \log y = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

or $\quad \left[\frac{e^x}{y} - \frac{1}{\sqrt{1-y^2}}\right]\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - e^x \log y$

or $\quad \frac{dy}{dx} = \frac{y\sqrt{1-y^2}\left[1-e^x\sqrt{1-x^2}\log y\right]}{\left[e^x\sqrt{1-y^2} - y\right]\sqrt{1-x^2}}$

Example 28.15 Find $\frac{dy}{dx}$, if $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$

Solution : We are given that

$$y=(\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}=(\cos x)^{y}$$

Taking logarithm on both sides, we get

$$\log y = y\,\log\cos x$$

Differentiating (i) w.r.t.x, we get

$$\frac{1}{y}\frac{dy}{dx}=y\cdot\frac{1}{\cos x}(-\sin x)+\log(\cos x)\cdot\frac{dy}{dx}$$

or $$\left[\frac{1}{y}-\log(\cos x)\right]\frac{dy}{dx}=-y\tan x$$

or $$\left[1-y\log(\cos x)\right]\frac{dy}{dx}=-y^{2}\tan x$$

or $$\frac{dy}{dx}=\frac{-y^{2}\tan x}{1-y\log(\cos x)}$$

CHECK YOUR PROGRESS 28.3

1. Find the derivative with respect to x of each the following functions :

   (a) $y=5^{x}$ (b) $y=3^{x}+4^{x}$ (c) $y=\sin(5^{x})$

2. Find $\frac{dy}{dx}$, if

   (a) $y=x^{2x}$ (b) $y=(\cos x)^{\log x}$ (c) $y=(\log x)^{\sin x}$

   (d) $y=(\tan x)^{x}$ (e) $y=(1+x^{2})^{x^{2}}$ (f) $y=x^{(x^{2}+\sin x)}$

3. Find the derivative of each of the functions given below :

   (a) $y=(\tan x)^{\cot x}+(\cot x)^{x}$ (b) $y=x^{\log x}+(\sin x)^{\sin^{-1}x}$

   (c) $y=x^{\tan x}+(\sin x)^{\cos x}$ (d) $y=(x)^{x^{2}}+(\log x)^{\log x}$

4. If $y=(\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}}$, show that

   $$\frac{dy}{dx}=\frac{y^{2}\cot x}{1-y\log(\sin x)}$$

5. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\ldots\infty}}}$, show that

   $$\frac{dy}{dx}=\frac{1}{x(2x-1)}$$

28.4 SECOND ORDER DERIVATIVES

In the previous lesson we found the derivatives of second order of trigonometric and inverse trigonometric functions by using the formulae for the derivatives of trigonometric and inverse trigonometric functions, various laws of derivatives, including chain rule, and power rule discussed earlier in lesson 21. In a similar manner, we will discuss second order derivative of exponential and logarithmic functions :

Example 28.16 Find the second order derivative of each of the following :

(i) e^x (ii) $\cos(\log x)$ (iii) x^x

Solution : (i) Let $y = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{dy}{dx} = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(e^x) = e^x$

$$\therefore \qquad \frac{d^2y}{dx^2} = e^x$$

(ii) Let $y = \cos(\log x)$

Taking derivative w.r.t x on both sides, we get

$$\frac{dy}{dx} = -\sin(\log x)\cdot\frac{1}{x} = \frac{-\sin(\log x)}{x}$$

Taking derivative w.r.t. x on both sides, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{\sin(\log x)}{x}\right]$$

or
$$= -\frac{x\cdot\cos(\log x)\cdot\frac{1}{x} - \sin(\log x)}{x^2}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{\sin(\log x) - \cos(\log x)}{x^2}$$

(iii) Let $y = x^x$

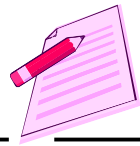
Taking logarithm on both sides, we get

$$\log y = x \log x \qquad \text{....(i)}$$

Taking derivative w.r.t. x of both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x = 1 + \log x$$

or
$$\frac{dy}{dx} = y(1 + \log x) \qquad \text{....(ii)}$$



Taking derivative w.r.t. x on both sides we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[y(1+\log x)]$$

$$= y \cdot \frac{1}{x} + (1+\log x)\frac{dy}{dx} \qquad \text{...(iii)}$$

$$= \frac{y}{x} + (1+\log x)y(1+\log x)$$

$$= \frac{y}{x} + (1+\log x)^2 y \qquad \text{(Using (ii))}$$

$$= y\left[\frac{1}{x} + (1+\log x)^2\right]$$

$$\therefore \qquad \frac{d^2y}{dx^2} = x^x\left[\frac{1}{x} + (1+\log x)^2\right]$$

Example 28.17 If $y = e^{a\cos^{-1}x}$, show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \cdot$$

Solution : We have, $y = e^{a\cos^{-1}x}$ .....(i)

$$\therefore \qquad \frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{-a}{\sqrt{1-x^2}}$$

$$= -\frac{ay}{\sqrt{1-x^2}} \qquad \text{Using (i)}$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)^2\left(1-x^2\right) - a^2y^2 = 0 \qquad \text{.....(ii)}$$

Taking derivative of both sides of (ii), we get

$$\left(\frac{dy}{dx}\right)^2(-2x) + 2\left(1-x^2\right) \times \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - a^2 \cdot 2y \cdot \frac{dy}{dx} = 0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \qquad \left[\text{Dividing through out by } 2 \cdot \frac{dy}{dx}\right]$$

CHECK YOUR PROGRESS 28.4

1. Find the second order derivative of each of the following :

(a) $x^4 e^{5x}$ (b) $\tan\left(e^{5x}\right)$ (c) $\dfrac{\log x}{x}$

2. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

3. If $y = e^{\tan^{-1} x}$, prove that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

28.5 DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$, then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$

Example 28.18 Find $\dfrac{dy}{dx}$, when $x = a\sin t$, $y = a\cos t$

Differentiating w.r. to ‘t’, we get

$$\frac{dx}{dt} = a\cos t \text{ and } \frac{dy}{dt} = -a\sin t$$

Hence, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a\cos t} = -\tan t$$

Example 28.19 Find $\dfrac{dy}{dx}$, if $x = 2at^2$ and $y = 2at$.

Solution : Given $x = 2at^2$ and $y = 2at$.

Differentiating w.r. to ‘t’, we get

$$\frac{dx}{dt} = 4at \text{ and } \frac{dy}{dt} = 2a$$

Hence $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2a}{4at} = \dfrac{1}{2t}$

Example 28.20 Find $\dfrac{dy}{dx}$, If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Solution : Given

$$x = a(\theta - \sin\theta) \text{ and}$$

$$y = a(1 + \cos\theta)$$

Differentiating both w.r. to 'θ', we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a(-\sin\theta)$$

Hence $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\cot\theta/2$$

Example 28.21 Find $\dfrac{dy}{dx}$, if $x = a\cos^3 t$ and $y = a\sin^3 t$

Solution : Given $x = a\cos^3 t$ and $y = a\sin^3 t$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = 3a\cos^2 t\frac{d}{dt}(\cos t) = -3a\cos^2 t\sin t$$

and $$\frac{dy}{dt} = 3a\sin^2 t\frac{d}{dt}(\sin t) = 3a\sin^2 t\cos t$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

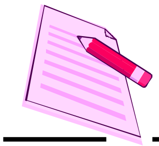
Example 28.22 Find $\dfrac{dy}{dx}$, If $x = a\dfrac{1-t^2}{1+t^2}$ and $y = \dfrac{2bt}{1+t^2}$.

Solution : Given $x = a\dfrac{1-t^2}{1+t^2}$ and $y = \dfrac{2bt}{1+t^2}$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = a\left\{\frac{(1+t^2).(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}\right\} = \frac{-4at}{(1+t^2)^2}$$

and $$\frac{dy}{dt} = 2b\left\{\frac{(1+t^2).(1)-t.(0+2t)}{(1+t^2)^2}\right\} = \frac{2b(1-t^2)}{(1+t^2)^2}$$



Notes

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} = \frac{-b(1-t^2)}{2at}$$

CHECK YOUR PROGRESS 28.5

Find $\frac{dy}{dx}$, when :

1. $x = 2at^3$ and $y = at^4$
2. $x = a \cos \theta$ and $y = a \sin \theta$
3. $x = 4t$ and $y = \frac{4}{t}$
4. $x = b\sin^2\theta$ and $y = a\cos^2\theta$
5. $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$
6. $x = a\sec\theta$ and $y = b\tan\theta$
7. $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$
8. $x = \sin 2t$ and $y = \cos 2t$

28.6 SECOND ORDER DERIVATIVE OF PARAMETRIC FUNCTIONS

If two parametric functions $x = f(t)$ and $y = g(t)$ are given then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t) \quad \left(\text{let here } \frac{dx}{dt} \neq 0\right)$$

Hence $$\frac{d^2y}{dx^2} = \frac{d}{dt}\big((h(t)\big) \times \frac{dt}{dx}$$

Example 28.23 Find $\frac{d^2y}{dx^2}$, if $x = at^2$ and $y = 2at$

Solution : Differentiating both w.r.t. to 't', we get

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating both sides w.r.t. to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$$

Example 28.24 Find $\frac{d^2y}{dx^2}$, if $x = a \sin^3 \theta$ and $y = b \cos^3 \theta$

Solution : Given $x = a \sin^3 \theta$ and $y = b \cos^3 \theta$

Differentiating both w.r. to ‘θ’, we get

$$\frac{dx}{d\theta} = 3a \sin^2\theta \cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3b\cos^2\theta(-\sin\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3b\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\frac{b}{a}\cot\theta$$

Differentiating both sides w.r. to ‘x’, we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dx}(\cot\theta) = \frac{-b}{a}\frac{d}{d\theta}(\cot\theta) \times \frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a}(-\text{cosec}^2\theta) \times \frac{1}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{b}{3a^2}\text{cosec}^4\theta\sec\theta$$

Example 28.25 If $x = a\sin t$ and $y = b\cos t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Solution : Given $x = a\sin t$ and $y = b\cos t$

Differentiating both w.r. to ‘t’, we get

$$\frac{dx}{dt} = a\cos t \quad \text{and} \quad \frac{dy}{dt} = -b\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b\sin t}{a\cos t} = \frac{-b}{a}\tan t$$

Differentiating both sides w.r. to ‘x’, we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dt}(\tan t)\times\frac{dt}{dx} = \frac{-b}{a}\sec^2 t\times\frac{1}{a\cos t}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{-b}{a^2}\sec^3 t$$

$$\left(\frac{d^2y}{dx^2}\right)at\ t=\frac{\pi}{4} = \frac{-b}{a^2}\sec^3\frac{\pi}{4} = \frac{-b}{a^2}(\sqrt{2})^3 = \frac{-2\sqrt{2}b}{a^2}$$

CHECK YOUR PROGRESS 28.6

Find $\dfrac{d^2y}{dx^2}$, when

1. $x = 2at$ and $y = at^2$
2. $x = a(t+\sin t)$ and $y = a(1-\cos t)$
3. $x = 10(\theta-\sin\theta)$ and $y = 12(1-\cos\theta)$
4. $x = a\sin t$ and $y = b\cos 2t$
5. $x = a-\cos 2t$ and $y = b-\sin 2t$

LET US SUM UP

- (i) $\dfrac{d}{dx}(e^x) = e^x$ (ii) $\dfrac{d}{dx}(a^x) = a^x\log a$; $a>0$
- If μ is a derivable function of x, then

  (i) $\log a\cdot\dfrac{dx}{dx}$; $a>0$

  (iii) $\dfrac{d}{dx}(e^{ax+b}) = e^{ax+b}\cdot a = ae^{ax+b}$
- (i) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$ (ii) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}\cdot\dfrac{d\mu}{dx}$, if μ is a derivable function of x.
- (iii) $\dfrac{d}{dx}\log(ax+b) = \dfrac{1}{ax+b}\cdot a = \dfrac{a}{ax+b}$
- If $x = f(t)$ and $y = g(t)$,

  then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$,. provided $\dfrac{dx}{dt}\neq 0$

- If $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t)$,

  then $\frac{d^2y}{dx^2} = \frac{d}{dt}[h(t)] \times \frac{dt}{dx}$

SUPPORTIVE WEB SITES

http://www.themathpage.com/acalc/exponential.htm

http://www.math.brown.edu/utra/explog.html

http://www.freemathhelp.com/derivative-log-exponent.html

TERMINAL EXERCISE

1. Find the derivative of each of the following functions :

   (a) $(x^x)^x$ (b) $x^{(x^x)}$

2. Find $\frac{dy}{dx}$, if

   (a) $y = a^{x \log \sin x}$ (b) $y = (\sin x)^{\cos^{-1} x}$

   (c) $y = \left(1 + \frac{1}{x}\right)^{x^2}$ (d) $y = \log\left[e^x \left(\frac{x-4}{x+4}\right)^{\frac{3}{4}}\right]$

3. Find the derivative of each of the functions given below :

   (a) $f(x) = \cos x \log(x) e^{x^2} x^x$ (b) $f(x) = (\sin^{-1} x)^2 \cdot x^{\sin x} \cdot e^{2x}$

4. Find the derivative of each of the following functions :

   (a) $y = (\tan x)^{\log x} + (\cos x)^{\sin x}$ (b) $y = x^{\tan x} + (\sin x)^{\cos x}$

5. Find $\frac{dy}{dx}$, if (a) If $y = \frac{x^4\sqrt{x+6}}{(3x+5)^2}$ (b) If $y = \frac{e^x + e^{-x}}{(e^x - e^{-x})}$

6. Find $\frac{dy}{dx}$, if (a) If $y = a^x \cdot x^a$ (b) $y = 7^{x^2+2x}$

7. Find the derivative of each of the following functions :

(a) $y = x^2 e^{2x} \cos 3x$ (b) $y = \dfrac{2^x \cot x}{\sqrt{x}}$

8. If $y = x^{x^{x^{x \ldots\ldots \infty}}}$, prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y \log x}$

Find derivative of each of the following function

9. $(\sin x)^{\cos x}$

10. $(\log x)^{\log x}$

11. $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

12. $\left(x + \dfrac{1}{x}\right)^x + x^{x + \frac{1}{x}}$

13. $x = a\left(\cos t + \log \dfrac{t}{2}\right)$ and $y = a \sin t$

14. $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

15. $x = e^t(\sin t + \cos t)$ and $y = e^t(\sin t - \cos t)$

16. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

17. $x = a\left(t + \dfrac{1}{t}\right)$ and $y = a\left(t - \dfrac{1}{t}\right)$

18. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$

19. If $x = \dfrac{2bt}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$, find $\dfrac{dy}{dx}$ at $t = 2$.

20. If $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$, prove that $\dfrac{dy}{dx} = -\cot 3t$

21. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\dfrac{dy}{dx} = \tan\left(\dfrac{3\theta}{2}\right)$

22. If $x = \cos t$ and $y = \sin t$, prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3}}$ at $t = \dfrac{2\pi}{3}$

23. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

24. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$

25. If $x = a\sin pt$ and $y = b\cos pt$, find the value of $\dfrac{d^2y}{dx^2}$ at $t = 0$

26. If $x = \log t$ and $y = \dfrac{1}{t}$, find $\dfrac{d^2y}{dx^2}$

27. If $x = a(1 + \cos t)$ and $y = a(t + \sin t)$, find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{2}$

28. If $x = at^2$ and $y = 2at$, find $\dfrac{d^2y}{dx^2}$.

ANSWERS

CHECK YOUR PROGRESS 28.1

1. (a) $5e^{5x}$ (b) $7e^{7x+4}$ (c) $\sqrt{2}e^{\sqrt{2}x}$ (d) $-\frac{7}{2}e^{-\frac{7}{2}x}$ (e) $2(x+1)e^{x^2+2x}$

2. (a) $\frac{1}{3}e^x$ (b) $\sec^2 x+2\cos x-3\sin x-\frac{1}{2}e^x$ (c) $5\cos x-2e^x$ (d) e^x-e^{-x}

3. (a) $\dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$ (b) $e^{\sqrt{\cot x}}\left[\dfrac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}}\right]$

   (c) $e^{x\sin^2 x}[\sin x+2x\cos x]\sin x$ (d) $e^{x\sec^2 x}\left[\sec^2 x+2x\sec^2 x\tan x\right]$

4. (a) xe^x (b) $2e^{2x}\sin x(\sin x+\cos x)$

5. (a) $\dfrac{2x^2-x+2}{(x^2+1)^{3/2}}e^{2x}$ (b) $\dfrac{e^{2x}[(2x-1)\cot x-x\operatorname{cosec}^2 x]}{x^2}$

CHECK YOUR PROGRESS 28.2

1. (a) $5\cos x-\frac{2}{x}$ (b) $-\tan x$

2. (a) $e^{x^2}\left[2x\log x+\frac{1}{x}\right]$ (b) $\dfrac{2x^2\log x-1}{x(\log x)^2}.e^{x^2}$

3. (a) $\dfrac{\cot(\log x)}{x}$ (b) $\sec x$ (c) $\dfrac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$ (d) $\dfrac{1}{x\log x}$

4. (a) $(1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}\times\left[\dfrac{1}{2(1+x)}-\dfrac{2}{3(2-x)}+\dfrac{2x}{7(x^2-5)}-\dfrac{3}{2(x+9)}\right]$

   (b) $\dfrac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}\left[\dfrac{1}{2x}-\dfrac{3}{1-2x}-\dfrac{5}{3+4x}+\dfrac{7x}{2(3-7x^2)}\right]$

CHECK YOUR PROGRESS 28.3

1. (a) $5^x\log 5$ (b) $3^x\log 3+4^x\log 4$ (c) $\cos 5^x 5^x\log 5$

2. (a) $2x^{2x}(1+\log x)$ (b) $(\cos x)^{\log x}\left[\dfrac{\log\cos x}{x}-\tan x\log x\right]$

   (c) $(\log x)^{\sin x}\left[\cos x\log(\log x)+\dfrac{\sin x}{x\log x}\right]$

(d) $(\tan x)^x \left[\log \tan x + \dfrac{x}{\sin x \cos x}\right]$ (e) $(1+x)^{x^2}\left[2x\log(1+x^2) + 2\dfrac{x^3}{1+x^2}\right]$

(f) $x^{(x^2+\sin x)}\left[\dfrac{x^2+\sin x}{x} + (2x+\cos x)\log x\right]$

3. (a) $\operatorname{cosec}^2 x(1-\log\tan x)(\tan x)^{\cot x} + \left(\log\cot x - x\operatorname{cosec}^2 x\tan x\right)(\cot x)^x$

(b) $2x^{(\log x - 1)}\log x + (\sin x)^{\sin^{-1}x}\left[\cot x\sin^{-1}x + \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $x^{\tan x}\left(\dfrac{\tan x}{x} + \sec^2 x\log x\right) + (\sin x)^{\cos x}\left[\cos x\cot x - \sin x\log\sin x\right]$

(d) $(x)^{x^2}\cdot x(1+2\log x) + (\log x)^{\log x}\left[\dfrac{1+\log(\log x)}{x}\right]$

CHECK YOUR PROGRESS 28.4

1. (a) $e^{5x}\left(25x^4+40x^3+12x^2\right)$ (b) $25e^{5x}\sec^2\left(e^{5x}\right)\left\{1+2e^{5x}\tan e^{5x}\right\}$

(c) $\dfrac{2\log x - 3}{x^3}$

CHECK YOUR PROGRESS 28.5

1. $\dfrac{2t}{3}$ 2. $-\cot\theta$ 3. $-\dfrac{1}{t^2}$ 4. $-\dfrac{a}{b}$ 5. $\dfrac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$

6. $\dfrac{b}{a}\operatorname{cosec}\theta$ 7. $\dfrac{2t}{1-t^2}$ 8. $-\tan 2t$

CHECK YOUR PROGRESS 28.6

1. $\dfrac{1}{2a}$ 2. $\dfrac{\sec^4 t/2}{4a}$ 3. $\dfrac{-3}{100}\operatorname{cosec}^4 \theta/2$ 4. $\dfrac{-4b}{a^2}$ 5. $\operatorname{cosec}^3 2t$

TERMINAL EXERCISE

1. (a) $(x^x)^x[x+2x\log x]$ (b) $x^{(x)^x}[x^{x-1} + \log x(\log x + 1)x^x]$

2. (a) $a^{x\log\sin x}[\log\sin x + x\cot x]\log a$

(b) $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x\cot x - \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $\left(1+\frac{1}{x}\right)^{x^2}\left[2x\log\left(x+\frac{1}{x}\right)-\frac{1}{1+\frac{1}{x}}\right]$ (d) $1+\frac{3}{4(x-4)}-\frac{3}{4(x+4)}$

3. (a) $\cos x\log(x)e^{x^2}\cdot x^x\left[-\tan x+\frac{1}{x\log x}+2x+1+\log x\right]$

(b) $(\sin^{-1}x)^2.x^{\sin x}e^{2x}\left[\frac{2}{\sqrt{1-x^2}\sin^{-1}x}+\cos x\log x+\frac{\sin x}{x}+2\right]$

4. (a) $(\tan x)^{\log x}\left[2\operatorname{cosec} 2x\log x+\frac{1}{x}\log\tan x\right]$

$+(\cos x)^{\sin x}[-\sin x\tan x+\cos x\log(\cos x)]$

(b) $x^{\tan x}\left[\frac{\tan x}{x}+\sec^2 x\log x\right]+(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x)]$

5. (a) $\frac{x^4\sqrt{x+6}}{(3x+5)^2}\left[\frac{4}{x}+\frac{1}{2(x+6)}-\frac{6}{(3x+5)}\right]$ (b) $\frac{-4e^{2x}}{(e^{2x}-1)^2}$

6. (a) $a^x.x^{a-1}[a+x\log_e a]$ (b) $7^{x^2+2x}(2x+2)\log_e 7$

7. (a) $x^2e^{2x}\cos 3x\left\{\frac{2}{x}+2-3\tan 3x\right\}$ (b) $\frac{2^x\cot x}{\sqrt{x}}\left[\log 2-2\operatorname{cosec}2x-\frac{1}{2x}\right]$

9. $(\sin x)^{\cos x}[-\sin x\log\sin x+\cos x.\cot x]$

10. $(\log x)^{\log x}\left[\frac{\log(\log x)+1}{x}\right]$

11. $\frac{(x-1)(x-2)}{(x-3)(x-4)}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}\right]$

12. $\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]+x^{x+\frac{1}{x}}\left[\frac{x^2-1}{x^2}\log x+\frac{x^2+1}{x^2}\right]$

13. $\tan t$ 14. $\tan\theta$ 15. $\tan t$ 16. $\frac{-y\log x}{x\log y}$ 17. $\frac{x}{y}$ 18. $-\sqrt{3}$

19. $\frac{4a}{3b}$ 20. $\frac{\sec^3\theta}{a\theta}$ 21. $\frac{1}{a}$ 22. $\frac{-b}{a^2}$ 23. $\frac{1}{t}$ 24. $\frac{-1}{a}$

25. $\frac{-1}{2at^3}$ 26. $\frac{-1}{t}$ 27. -2 28. $\frac{1}{t}$

29

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APPLICATIONS OF DERIVATIVES

In the previous lesson, we have learnt that the slope of a line is the tangent of the angle which the line makes with the positive direction of x-axis. It is denoted by the letter 'm'. Thus, if θ is the angle which a line makes with the positive direction of x-axis, then m is given by tan θ.

We have also learnt that the slope m of a line, passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

In this lesson, we shall find the equations of tangents and normals to different curves, using derinatives.

OBJECTIVES

After studying this lesson, you will be able to :

- find rate of change of quantities
- find approximate value of functions
- define tangent and normal to a curve (graph of a function) at a point;
- find equations of tangents and normals to a curve under given conditions;
- define monotonic (increasing and decreasing) functions;
- establish that $\frac{dy}{dx} > 0$ in an interval for an increasing function and $\frac{dy}{dx} < 0$ for a decreasing function;
- define the points of maximum and minimum values as well as local maxima and local minima of a function from the graph;
- establish the working rule for finding the maxima and minima of a function using the first and the second derivatives of the function; and
- work out simple problems on maxima and minima.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of coordinate geometry and
- Concept of tangent and normal to a curve
- Concept of diferential coefficient of various functions
- Geometrical meaning of derivative of a function at a point
- Solution of equetions and the inequations.

29.1 RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function of x and let there be a small change Δx in x, and the corresponding change in y be Δy.

$\therefore$ Average change in y per unit change in $x = \dfrac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the limiting value of the average rate of change of y with respect to x.

So the rate of change of y per unit change in x

$$= \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Hence, $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x.

Thus,

The value of $\dfrac{dy}{dx}$ at $x = x_0$ i.e. $\left(\dfrac{dy}{dx}\right)_{x=x_0} = f'(x_0)$

$f'(x_0)$ represent the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying one with respect to another variable t i.e. if $y = f(t)$ and $x = g(t)$, then by chain rule.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

Hence, the rate of change y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Example 29.1 Find the rate of change of area of a circle with respect to its variable radius r, when $r = 3$ cm.

Solution : Let A be the area of a circle of radius r,

then $\quad A = \pi r^2$

$\therefore$ The rate of change of area A with respect to its radius r

$$\Rightarrow \quad \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

when $r = 3$ cm, $\dfrac{dA}{dr} = 2\pi \times 3 = 6\pi$

Hence, the area of the circle is changing at the rate of 6π cm^2/cm

Example 29.2 A balloon which always remains spherical, has a variable diameter $\dfrac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.

Solution : Radius (say r) of the spherical balloon $= \dfrac{1}{2}$ (diameter)

$$= \frac{1}{2} \times \frac{3}{2}(2x+3) = \frac{3}{4}(2x+3)$$

Let V be the volume of the balloon, then

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{4}(2x+3)\right)^3$$

$$\Rightarrow \qquad V = \frac{9}{16}\pi(2x+3)^3$$

$\therefore$ The rate of change of volume w.r. to 'x'

$$\frac{dV}{dx} = \frac{9}{16}\pi \times 3(2x+3)^2 \times 2 = \frac{27}{8}\pi(2x+3)^2$$

Hence, the volume is changing at the rate of $\frac{27}{8}\pi(2x+3)^2 \text{unit}^3/\text{unit}$

Example 29.3 A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

Solution : Let r be the radius of the spherical balloon and V be its volume at any time t, then

$$V = \frac{4}{3}\pi r^3$$

Diff. w.r. to 't' we get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= \frac{4}{3}\pi.3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

But $\qquad \frac{dV}{dt} = 900 \text{ cm}^3/\text{sec. (given)}$

So, $\qquad 4\pi r^2\frac{dr}{dt} = 900$

$$\Rightarrow \qquad \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

when $r = 15$ cm,

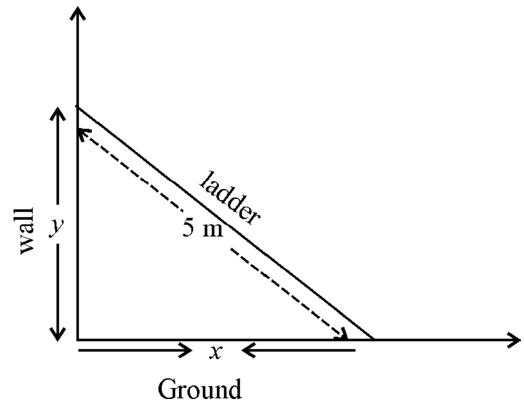
$$\frac{dr}{dt} = \frac{225}{\pi \times 15^2} = \frac{1}{\pi}$$

Hence, the radius of balloon is increasing at the rate of $\frac{1}{\pi}$ cm/sec, when its radius is 15 cm.

Example 29.4 A ladder 5 m long is leaning against a wall. The foot of the ladder is pulled along the ground, away from the wall, at the rate of 2m/sec. How fast is its height on the wall decreasing when the foot of ladder is 4m away from the wall?

Solution : Let the foot of the ladder be at a distance x metres from the wall and y metres be the height of the ladder at any time t, then

$$x^2 + y^2 = 25 \quad \text{...(i)}$$



Diff. w.r. to 't'. We get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

But $$\frac{dx}{dt} = 2 \text{ m/sec. (given)}$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y} \times 2 = -\frac{2x}{y}$$

...(ii)

When $x = 4$m, from (i) $y^2 = 25 - 16 \Rightarrow y = 3$m

Putting $x = 4$m and $y = 3$m in (ii), we get

$$\frac{dy}{dx} = -\frac{2 \times 4}{3} = \frac{-8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ m/sec.

Example 29.5 The total revenue received from the sale of x units of a product is given by

$$R(x) = 10x^2 + 13x + 24$$

Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue w.r. to the number of items sold at an instant.

Solution : Given $R(x) = 10x^2 + 13x + 24$

Since marginal revenue is the rate of change of the revenue with respect to the number of units sold, we have

$$\text{marginal revenue (MR)} = \frac{dR}{dx} = 20x + 13$$

when $x = 5$, $MR = 20 \times 5 + 13 = 113$

Hence, the marginal revenue = ` 113

Example 29.6 The total cost associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced, where by marginal cost we mean the instantaneous rate of change of the total cost at any level of output.

Solution : Given $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

Since marginal cost is the rate of change of total cost w.r. to the output, we have

$$\text{Marginal Cost (MC)} = \frac{dC}{dx}$$

$$= 0.007 \times 3x^2 - 0.003 \times 2x + 15$$

$$= 0.021x^2 - 0.006x + 15$$

when x = 17,

$$MC = 0.021 \times 17^2 - 0.006 \times 17 + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, marginal cost = ` 20.967

CHECK YOUR PROGRESS 29.1

1. The side of a square sheet is increasing at rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?
2. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long.
3. Find the rate of change of the area of a circle with respect to its radius when the radius is 6 cm.
4. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.
5. Find the rate of change of the volume of a cube with respect to its edge when the edge is 5 cm.

29.2 APPROXIMATIONS

In this section, we shall give a meaning to the symbols dx and dy in such a way that the original meaning of the symbol $\frac{dy}{dx}$ coincides with the quotient when dy is divided by dx.

Let $y = f(x)$ be a function of x and Δx be a small change in x and let Δy be the corresponding change in y. Then,

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$$\Rightarrow \qquad \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \varepsilon, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$$\Rightarrow \qquad \Delta y = \frac{dy}{dx}\Delta x + \varepsilon \Delta x$$

$\because$ $\varepsilon \Delta x$ is a very-very small quantity that can be neglected, therefore

we have $\Delta y = \dfrac{dy}{dx}\Delta x$, approximately

This formula is very useful in the calculation of small change (or errors) in dependent variable corresponding to small change (or errors) in the independent variable.

SOME IMPORTANT TERMS

ABSOLUTE ERROR : The error Δx in x is called the absolute error in x.

RELATIVE ERROR : If Δx is an error in x, then $\dfrac{\Delta x}{x}$ is called relative error in x.

PERCENTAGE ERROR : If Δx is an error in x, then $\dfrac{\Delta x}{x} \times 100$ is called percentage error in x.

Note : We have $\Delta y = \dfrac{dy}{dx}\Delta x + \varepsilon.\Delta x$

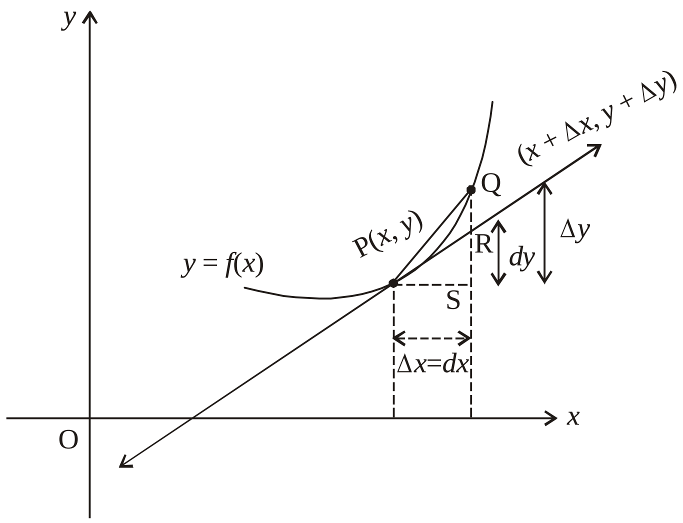
$\because$ $\varepsilon.\Delta x$ is very smal, therefore principal value of $\Delta y = \dfrac{dy}{dx}\Delta x$ which is called differential of y.

i.e. $\Delta y = \dfrac{dy}{dx}.\Delta x$

So, the differential of x is given by

$$dx = \frac{dx}{dx}.\Delta x = \Delta x$$

Hence, $dy = \dfrac{dy}{dx} dx$



To understand the geometrical meaning of dx, Δx, dy and Δy. Let us focus our attention to the portion of the graph of $y = f(x)$ in the neighbourhood of the point P(x, y) where a tangent can be drawn the curve. If Q$(x + \Delta x, y + \Delta y)$ be another point $(\Delta x \neq 0)$ on the curve, then the slope of line PQ will be $\frac{\Delta y}{\Delta x}$ which approaches the limiting value $\frac{dy}{dx}$ (slope of tangent at P).

Therefore, when $\Delta x \to 0$, Δy is approximately equal to dy.

Example 29.7 Using differentials, find the approximate value of $\sqrt{25.3}$

Solution : Let $y = \sqrt{x}$

Differentiating w.r. to 'x' we get

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Take $x = 25$ and $x + \Delta x = 25.3$, then $dx = \Delta x = 0.3$ when x = 25, $y = \sqrt{25} = 5$

$$\Delta y = \frac{dy}{dx}\Delta x = \frac{1}{2\sqrt{x}}\Delta x = \frac{1}{2\sqrt{25}} \times 0.3 = \frac{1}{10} \times 0.3 = 0.03$$

$\Rightarrow \Delta y = 0.03$ ($\because$ dy is approximately equal to Δy)

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{25.3}$$

$\Rightarrow$ $\sqrt{25.3} = 5 + 0.03 = 5.03$ approximately

Example 29.8 Using differentials find the approximate value of $(127)^{\frac{1}{3}}$

Solution : Take $y = x^{\frac{1}{3}}$

Let $x = 125$ and $x + \Delta x = 127$, then $dx = \Delta x = 2$

When $x = 125$, $y = (125)^{\frac{1}{3}} = 5$

Now $$y = x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$$

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{2/3}}dx = \frac{1}{3(125)^{2/3}} \times 2 = \frac{2}{75}$$

$\Rightarrow$ $$\Delta y = \frac{2}{75}$$

($\because$ $\Delta y = dy$)

Hence, $$(127)^{\frac{1}{3}} = y + \Delta y = 5 + \frac{2}{75} = 5.026 \text{ (Approximate)}$$

Example 29.9 Find the approximate value of $f(3.02)$, where

$$f(x) = 3x^2 + 5x + 3$$

Solution : Let x = 3 and $x + \Delta x = 3.02$, then $dx = \Delta x = 0.02$

We have $$f(x) = 3x^2 + 5x + 3$$

when $x = 3$

$\Rightarrow$ $$f(3) = 3(3)^2 + 5(3) + 3 = 45$$

Now $y = f(x)$

$\Rightarrow$ $$\Delta y = \frac{dy}{dx}\Delta x = (6x + 5)\Delta x$$

$\Rightarrow$ $$\Delta y = (6 \times 3 + 5) \times 0.02 = 0.46$$

$\therefore$ $f(3.02) = f(x + \Delta x) = y + \Delta y = 45 + 0.46 = 45.46$

Hence, the approximate value of $f(3.02)$ is 45.46.

Example 29.10 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Solution : Let r be the radius of the sphere and Δr be the error in measuring the radius. Then

$r = 9$ cm and $\Delta r = 0.03$ cm

Let S be the surface area of the sphere. Then

$S = 4\pi r^2$

$\Rightarrow$ $$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r$$

$$\left(\frac{dS}{dr}\right)_{\text{at } r=9} = 8\pi \times (9) = 72\pi$$

Let ΔS be the error in S, then

$$\Delta S = \frac{dS}{dr}\Delta r = 72\pi \times 0.03 = 2.16\pi \text{ cm}^2$$

Hence, approximate error in calculating the surface area is 2.16π cm^2.

Example 29.11 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution : Let Δx be the change in x and ΔV be the corresponding change in V.

Given that $\frac{\Delta x}{x} \times 100 = 2 \Rightarrow \Delta x = \frac{2x}{100}$

we have $\qquad V = x^3$

$$\Rightarrow \qquad \frac{dV}{dx} = 3x^2$$

Now $\qquad \Delta V = \frac{dV}{dx}\Delta x$

$$\Rightarrow \qquad \Delta V = 3x^2 \times \frac{2x}{100}$$

$$\Rightarrow \qquad \Delta V = \frac{6}{100}.V$$

Hence, the approximate change in volume is 6%.

CHECK YOUR PROGRESS 29.2

1. Using differentials, find the approximate value of $\sqrt{36.6}$.
2. Using differentials, find the appoximate value of $(25)^{\frac{1}{3}}$.
3. Using differentials, find the approximate value of $(15)^{\frac{1}{4}}$.
4. Using differentials, find the approximate value of $\sqrt{26}$.
5. If the radius of a sphere is measured as 7 m with an error of 0.02 m, find the approximate error in calculating its volume.
6. Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the box.

29.3 SLOPE OF TANGENT AND NORMAL

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by

$$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) \qquad ....(i)$$

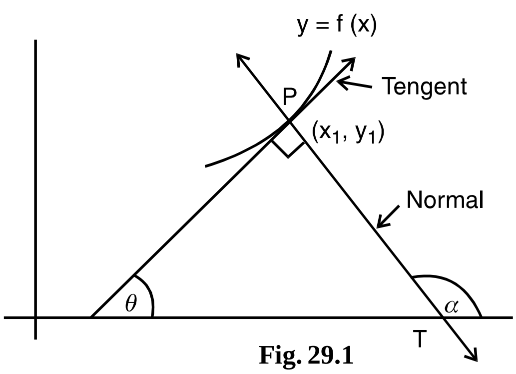


Fig. 29.1

and (i) is equal to $\tan\theta$

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\quad \alpha = \frac{\pi}{2} + \theta$ (From Fig. 10.1)

$$\Rightarrow \qquad \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= -\frac{1}{\tan\theta}$$

$\therefore$ Slope of normal $\quad = -\frac{1}{m} = \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x_1, y_1) or $-\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

Note

1. The tangent to a curve at any point will be parallel to x-axis if $\theta = 0$, i.e, the derivative at the point will be zero.

   i.e. $\quad \left(\frac{dx}{dy}\right)$ at $(x_1, y_1) = 0$

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis if $\frac{dy}{dx} = 0$ at that point.

Let us consider some examples :

Example 29.12 Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1, 1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5 \qquad .....(i)$$

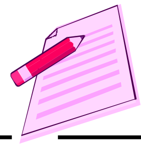
Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0 \qquad .....(ii)$$

Substituting $x = 1, \quad y = 1$, in (ii), we get

$$2\times1 + 3\times1 + 3\left[\frac{dy}{dx} + 1\right] + 2\frac{dy}{dx} = 0$$

or $\quad 5\frac{dy}{dx} = -8 \Rightarrow \quad \frac{dy}{dx} = -\frac{8}{5}$



$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 29.13 Show that the tangents to the curve $y=\frac{1}{6}\left[3x^5+2x^3-3x\right]$ at the points $x=\pm 3$ are parallel.

Solution : The equation of the curve is $y=\frac{3x^5+2x^3-3x}{6}$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\left(15x^4+6x^2-3\right)}{6}$$

$$\left(\frac{dy}{dx}\right)\underset{x=3}{\text{at}}=\frac{\left[15(3)^4+6(3)^2-3\right]}{6}$$

$$=\frac{1}{6}[15\times 9\times 9+54-3]$$

$$=\frac{3}{6}[405+17]=211$$

$$\left(\frac{dy}{dx}\right)\text{at}\, x=-3=\frac{1}{6}\left[15(-3)^4+6(-3)^2-3\right]=211$$

$\therefore$ The tangents to the curve at $x=\pm 3$ are parallel as the slopes at $x=\pm 3$ are equal.

Example 29.14 The slope of the curve $6y^3=px^2+q$ at $(2,-2)$ is $\frac{1}{6}$.

Find the values of p and q.

Solution : The equation of the curve is

$$6y^3=px^2+q \qquad .....(i)$$

Differentiating (i) w.r.t. x, we get

$$18y^2\frac{dy}{dx}=2px \qquad .....(ii)$$

Putting $x=2,\ y=-2,$ we get

$$18(-2)^2\frac{dy}{dx}=2p\cdot 2=4p$$

$$\therefore \qquad \frac{dy}{dx} = \frac{p}{18}$$

It is given equal to $\frac{1}{6}$

$$\therefore \qquad \frac{1}{6} = \frac{p}{18} \Rightarrow p = 3$$

$\therefore$ The equation of curve becomes

$$6y^3 = 3x^2 + q$$

Also, the point (2, –2) lies on the curve

$$\therefore \qquad 6(-2)^3 = 3(2)^2 + q$$

$$\Rightarrow \qquad -48 - 12 = q \quad \text{or} \quad q = -60$$

$\therefore$ The value of $p = 3,\ q = -60$

CHECK YOUR PROGRESS 29.3

1. Find the slopes of tangents and normals to each of the curves at the given points :

   (i) $y = x^3 - 2x$ at $x = 2$ (ii) $x^2 + 3y + y^2 = 5$ at $(1, 1)$

   (iii) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$

2. Find the values of p and q if the slope of the tangent to the curve $xy + px + qy = 2$ at (1, 1) is 2.

3. Find the points on the curve $x^2 + y^2 = 18$ at which the tangents are parallel to the line x+y = 3.

4. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendiculat to the line $2y + x - 7 = 0$.

29.4 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in the section before, the slope of tangent to the curve y = f (x) at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1)

$\therefore$ Equation of tangent to the curve y = f (x) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} [x - x_1]$$

And, the equation of normal to the curve y = f(x) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} [x - x_1]$$

Note

(i) The equation of tangent to a curve is parallel to x-axis if $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$. In that case the equation of tangent is $y = y_1$.

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \to \infty$, the tangent at (x_1, y_1) is parallel to $y-$axis and its equation is $x = x_1$

Let us take some examples and illustrate

Example 29.15 Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3)

Solution : The equation of circle is

$$x^2 + y^2 = 25 \qquad \text{....(i)}$$

Differentialing (1), w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{\text{at }(4,3)} = -\frac{4}{3}$$

$\therefore$ Equation of tangent to the circle at (4, 3) is

$$y - 3 = -\frac{4}{3}(x - 4)$$

or $\qquad 4(x-4) + 3(y-3) = 0 \qquad$ or, $\qquad 4x + 3y = 25$

Also, slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(4,3)}} = \dfrac{3}{4}$

$\therefore$ Equation of the normal to the circle at (4,3) is

$$y-3=\frac{3}{4}(x-4)$$

or $$4y-12=3x-12$$

$\Rightarrow$ $$3x=4y$$

$\therefore$ Equation of the tangent to the circle at (4,3) is 4x+3y = 25

Equation of the normal to the circle at (4,3) is 3x = 4y

Example 29.16 Find the equation of the tangent and normal to the curve $16x^2+9y^2=144$ at the point (x_1, y_1) where $y_1>0$ and $x_1=2$

Solution : The equation of curve is

$$16x^2+9y^2=144 \qquad .....(i)$$

Differentiating (i), w.r.t. x we get

$$32x+18y\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{16x}{9y}$$

As $x_1=2$ and (x_1, y_1) lies on the curve

$\therefore$ $$16(2)^2+9\left(y^2\right)=144$$

$\Rightarrow$ $$y^2=\frac{80}{9} \Rightarrow y=\pm\frac{4}{3}\sqrt{5}$$

As $$y_1>0 \Rightarrow y=\frac{4}{3}\sqrt{5}$$

$\therefore$ Equation of the tangent to the curve at $\left(2, \frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(-\frac{16x}{9y}\right)_{\text{at}\left(2,\frac{4\sqrt{5}}{3}\right)}[x-2]$$

or $$y-\frac{4}{3}\sqrt{5}=-\frac{16}{9}\cdot\frac{2\times 3}{4\sqrt{5}}(x-2)$$ or $$y-\frac{4}{3}\sqrt{5}+\frac{8}{3\sqrt{5}}(x-2)=0$$

or $$3\sqrt{5}y-\frac{4}{3}\sqrt{5}\cdot 3\sqrt{5}+8(x-2)=0$$

$$3\sqrt{5}y-20+8x-16=0$$ or $$3\sqrt{5}y+8x=36$$

Also, equation of the normal to the curve at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(\frac{9y}{16x}\right)_{\text{at}\left(2,\frac{4}{3}\sqrt{5}\right)}[x-2]$$

$$y-\frac{4}{3}\sqrt{5}=\frac{9}{16}\times\frac{2\sqrt{5}}{3}(x-2)$$

$$y-\frac{4}{3}\sqrt{5}=\frac{3\sqrt{5}}{8}(x-2)$$

$$3\times 8(y)-32\sqrt{5}=9\sqrt{5}(x-2)$$

$$24y-32\sqrt{5}=9\sqrt{5}\,x-18\sqrt{5}$$ or $$9\sqrt{5}x-24y+14\sqrt{5}=0$$

Example 29.17 Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9}-\frac{y^2}{16}=1 \quad ....(i)$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9}-\frac{2y}{16}\cdot\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=\frac{16x}{9y}$$

For tangent to be parallel to x-axis, $\frac{dy}{dx}=0$

$\Rightarrow$ $$\frac{16x}{9y}=0 \quad \Rightarrow \quad x=0$$

Putting x = 0 in (i), we get $y^2=-16$ $y=\pm 4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$ is parallel to x-axis.

Example 29.18 Find the equation of all lines having slope – 4 that are tangents to the curve $y=\frac{1}{x-1}$

Solution : $$y=\frac{1}{x-1} \qquad .....(i)$$

$\therefore$ $$\frac{dy}{dx}=-\frac{1}{(x-1)^2}$$

It is given equal to – 4

$\therefore$ $$\frac{-1}{(x-1)^2}=-4$$

$$\Rightarrow \quad (x-1)^2=\frac{1}{4}, \quad \Rightarrow \quad x=1\pm\frac{1}{2}\Rightarrow x=\frac{3}{2},\frac{1}{2}$$

Substituting $x=\frac{1}{2}$ in (i), we get

$$y=\frac{1}{\frac{1}{2}-1}=\frac{1}{-\frac{1}{2}}=-2$$

When $$x=\frac{3}{2}, \quad y=2$$

$\therefore$ The points are $\left(\frac{3}{2},2\right),\left(\frac{1}{2},-2\right)$

$\therefore$ The equations of tangents are

(a) $$y-2=-4\left(x-\frac{3}{2}\right), \quad \Rightarrow \quad y-2=-4x+6 \quad \text{or} \quad 4x+y=8$$

(b) $$y+2=-4\left(x-\frac{1}{2}\right)$$

$$\Rightarrow \qquad y+2=-4x+2 \quad \text{or} \quad 4x+y=0$$

Example 29.19 Find the equation of the normal to the curve $y=x^3$ at (2 , 8)

Solution : $$y=x^3 \qquad \Rightarrow \frac{dy}{dx}=3x^2$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{at\,x=2}=12$$

$\therefore$ Slope of the normal $= -\dfrac{1}{12}$

$\therefore$ Equation of the normal is

$$y - 8 = -\frac{1}{12}(x - 2)$$

or $\quad 12(y-8)+(x-2)=0 \quad$ or $\quad x+12y=98$

CHECK YOUR PROGRESS 29.4

1. Find the equation of the tangent and normal at the indicated points :

   (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

   (ii) $y = x^2$ at $(1,\ 1)$

   (iii) $y = x^3 - 3x + 2$ at the point whose x–coordinate is 3

2. Find the equation of the targent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1)

3. Find the equation of the tangent to the hyperbola

   $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad$ at (x_0, y_0)

4. Find the equation of normals to the curve

   $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

5. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$

29.5 Mathematical formulation of Rolle's Theorem

Let f be a real function defined in the closed interval [a, b] such that

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open inteval (a, b)

(iii) f (a) = f (b)

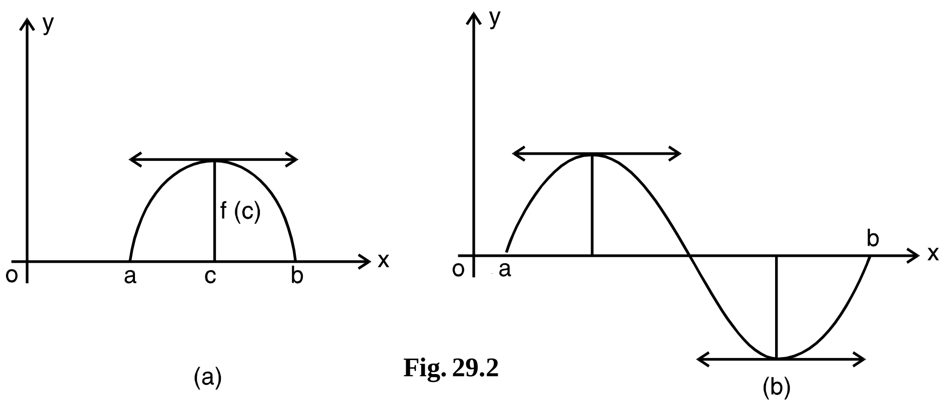


Fig. 29.2

Then there is at least one point c in the open inteval (a, b) such that $f'(c) = 0$

Remarks

(i) The remarks "at least one point" says that there can be more than one point $c \in (a, b)$ such that $f'(c) = 0$.

(ii) The condition of continuity of f on [a, b] is essential and can not be relaxed

(iii) The condition of differentiability of f on (a,b) is also essential and can not be relaxed.

For example $f(x) = |x|, x \in [-1, 1]$ is continuous on $[-1, 1]$ and differentiable on (–1, 1) and Rolle's Theorem is valid for this

Let us take some examples

Example 29.20 Verify Rolle's for the function

$$f(x) = x(x-1)(x-2),\ x \in [0, 2]$$

Solution :

$$f(x) = x(x-1)(x-2)$$

$$= x^3 - 3x^2 + 2x$$

(i) f (x) is a polynomial function and hence continuous in [0, 2]

(ii) f (x) is differentiable on (0,2)

(iii) Also f (0) = 0 and f (2) = 0

$$\therefore \qquad f(0) = f(2)$$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Also,
$$f'(x) = 3x^2 - 6x + 2$$

$\therefore f'(c) = 0$ gives $3c^2 - 6c + 2 = 0 \quad \Rightarrow \quad c = \dfrac{6 \pm \sqrt{36-24}}{6}$

$$\Rightarrow \qquad c = 1 \pm \frac{1}{\sqrt{3}}$$

We see that both the values of c lie in (0, 2)

Example 29.21 Discuss the applicability of Rolle's Theorem for

$$f(x) = \sin x - \sin 2x, \quad x \in [0, \pi] \qquad ....(i)$$

(i) is a sine function. It is continuous and differentiable on $(0, \pi)$

Again, we have, $f(0) = 0$ and $f(\pi) = 0$

$$\Rightarrow \qquad f(\pi) = f(0) = 0$$

$\therefore$ All the conditions of Rolle's theorem are satisfied

Now $\qquad f'(c) = 2\left[2\cos^2 c - 1\right] - \cos c = 0$

or $\qquad 4\cos^2 c - \cos c - 2 = 0$

$$\therefore \qquad \cos c = \frac{1 \pm \sqrt{1+32}}{8}$$

$$= \frac{1 \pm \sqrt{33}}{8}.$$

As $\sqrt{33} < 6$

$$\therefore \qquad \cos c < \frac{7}{8} = 0.875$$

which shows that c lies between 0 and π

CHECK YOUR PROGRESS 29.5

Verify Rolle's Theorem for each of the following functions :

(i) $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{3} + 2x, \; x \in [0, 3]$ (ii) $f(x) = x^2 - 1$ on $[-1, 1]$

(iii) $f(x) = \sin x + \cos x - 1$ on $\left(0, \dfrac{\pi}{2}\right)$ (iv) $f(x) = \left(x^2 - 1\right)\left(x - 2\right)$ on $[-1, 2]$

29.6 LANGRANGE'S MEAN VALUE THEOREM

This theorem improves the result of Rolle's Theorem saying that it is not necessary that tangent may be parallel to x-axis. This theorem says that the tangent is parallel to the line joining the end points of the curve. In other words, this theorem says that there always exists a point on the graph, where the tangent is parallel to the line joining the end-points of the graph.

29.6.1 Mathematical Formulation of the Theorem

Let f be a real valued function defined on the closed interval [a, b] such that

(a) f is continuous on [a, b], and

(b) f is differentiable in (a, b)

(c) $f(b) \neq f(a)$

then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks

When f (b) = f (a), f '(c) = 0 and the theorem reduces to Rolle's Theorem

Let us consider some examples

Example 29.22 Verify Langrange's Mean value theorem for

$$f(x) = (x-3)(x-6)(x-9) \text{ on } [3, 5]$$

Solution : $$f(x) = (x-3)(x-6)(x-9)$$

$$= (x-3)(x^2 - 15x + 54)$$

or $$f(x) = x^3 - 18x^2 + 99x - 162 \qquad \text{...(i)}$$

(i) is a polynomial function and hence continuous and differentiable in the given interval

Here, $f(3) = 0,\ f(5) = (2)(-1)(-4) = 8$

$\therefore$ $$f(3) \neq f(5)$$

$\therefore$ All the conditions of Mean value Theorem are satisfied

$\therefore$ $$f'(c) = \frac{f(5) - f(3)}{5 - 3} = \frac{8 - 0}{2} = 4$$

Now $$f'(x) = 3x^2 - 36x + 99$$

$\therefore$ $$3c^2 - 36c + 99 = 4 \text{ or } 3c^2 - 36c + 95 = 0$$

$\therefore$ $$c = \frac{36 \pm \sqrt{1296 - 1140}}{6} = \frac{36 \pm 12.5}{6}$$

$$= 8.08 \text{ or } 3.9$$

$$c = 3.9 \in (3, 5)$$

$\therefore$ Langranges mean value theorem is verified

Example 29.23 Find a point on the parabola $y = (x-4)^2$ where the tangent is parallel to the chord joining $(4, 0)$ and $(5, 1)$

Solution : Slope of the tangent to the given curve at any point is given by $(f'(x))$ at that point.

$$f'(x) = 2(x-4)$$

Slope of the chord joining $(4, 0)$ and $(5, 1)$ is

$$\frac{1-0}{5-4}=1 \qquad \left[\because m=\frac{y_2-y_1}{x_2-x_1}\right]$$

$\therefore$ According to mean value theorem

$$2(x-4)=1 \quad \text{or} \quad (x-4)=\frac{1}{2}$$

$$\Rightarrow \qquad x=\frac{9}{2}$$

which lies between 4 and 5

Now $\qquad y=(x-4)^2$

When $\qquad x=\frac{9}{2},\ y=\left(\frac{9}{2}-4\right)^2=\frac{1}{4}$

$\therefore$ The required point is $\left(\frac{9}{2},\frac{1}{4}\right)$

CHECK YOUR PROGRESS 29.6

1. Check the applicability of Mean Value Theorem for each of the following functions :

   (i) $f(x)=3x^2-4$ on [2, 3]

   (ii) $f(x)=\log x$ on [1, 2]

   (iii) $f(x)=x+\frac{1}{x}$ on [1, 3]

   (iv) $f(x)=x^3-2x^2-x+3$ on [0, 1]

2. Find a point on the parabola $y=(x+3)^2$, where the tangent is parallel to the chord joining $(3,0)$ and $(-4,1)$

29.7 INCREASING AND DECREASING FUNCTIONS

You have already seen the common trends of an increasing or a decreasing function. Here we will try to establish the condition for a function to be an increasing or a decreasing.

Let a function $f(x)$ be defined over the closed interval [a,b].

Let $x_1, x_2 \in [a, b]$, then the function f(x) is said to be an increasing function in the given interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. It is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1,\ x_1, x_2 \in [a, b]$.

In Fig. 29.3, sin x increases from -1 to +1 as x increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

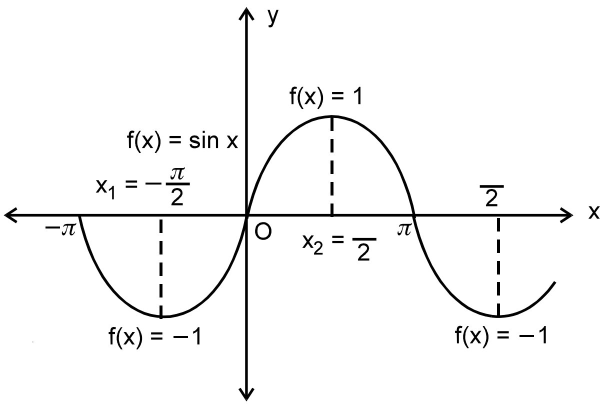


Fig. 29. 3

Note : A function is said to be an increasing function in an interval if $f(x+h) > f(x)$ for all x belonging to the interval when h is positive.

A function f(x) defined over the closed interval [a, b] is said to be a decreasing function in the given interval, if $f(x_2) \leq f(x_1)$, whenever $x_2 > x_1,\ x_1, x_2 \in [a, b]$. It is said to be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1,\ x_1, x_2 \in [a, b]$.

In Fig. 29.4, cos x decreases from 1 to -1 as x increases from 0 to π.

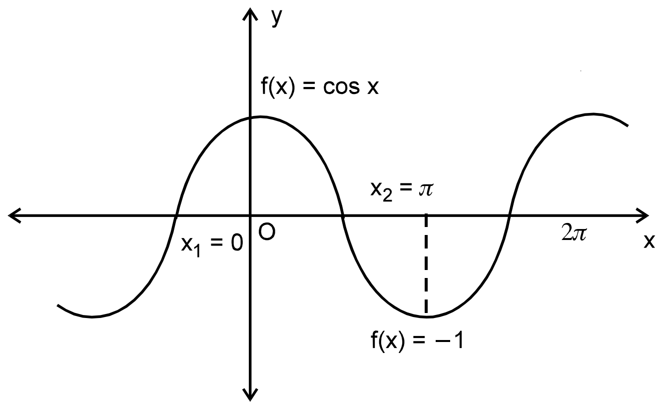


Fig. 29.4

Note : A function is said to be a decreasing in an internal if $f(x+h) < f(x)$ for all x belonging to the interval when h is positive.

29.7.1 MONOTONIC FUNCTIONS

Let x_1, x_2 be any two points such that $x_1 < x_2$ in the interval of definition of a function f(x). Then a function f(x) is said to be monotonic if it is either increasing or decreasing. It is said to be monotonically increasing if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$ belonging to the interval and monotonically decreasing if $f(x_1) \geq f(x_2)$.

Example 29.24 Prove that the function $f(x) = 4x + 7$ is monotonic for all values of $x \in R$.

Solution : Consider two values of x say $x_1, x_2 \in R$

such that $$x_2 > x_1 \qquad .....(1)$$

Multiplying both sides of (1) by 4, we have $4x_2 > 4x_1$ .....(2)

Adding 7 to both sides of (2), to get

$$4x_2 + 7 > 4x_1 + 7$$

We have $$f(x_2) > f(x_1)$$

Thus, we find $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence the given function $f(x) = 4x + 7$ is monotonic function. (monotonically increasing).

Example 29.25 Show that

$$f(x) = x^2$$

is a strictly decreasing function for all $x < 0$.

Solution : Consider any two values of x say x_1, x_2 such that

$$x_2 > x_1, \qquad x_1, x_2 < 0 \qquad ....(i)$$

Order of the inequality reverses when it is multiplied by a negative number. Now multiplying (i) by x_2, we have

$$x_2 \cdot x_2 < x_1 \cdot x_2$$

or, $$x_2^2 < x_1 x_2 \qquad ......(ii)$$

Now multiplying (i) by x_1, we have

$$x_1 \cdot x_2 < x_1 \cdot x_1$$

or, $$x_1 x_2 < x_1^2 \qquad .....(iii)$$

From (ii) and (iii), we have

$$x_2^2 < x_1 x_2 < x_1^2$$

or, $$x_2^2 < x_1^2$$

or, $\quad f(x_2) < f(x_1)$ .....(iv)

Thus, from (i) and (iv), we have for

$$x_2 > x_1,$$

$$f(x_2) < f(x_1)$$

Hence, the given function is strictly decreasing for all $x < 0$.

CHECK YOUR PROGRESS 29.7

1. (a) Prove that the function

$$f(x) = 3x + 4$$

is monotonic increasing function for all values of $x \in R$.

(b) Show that the function

$$f(x) = 7 - 2x$$

is monotonically decreasing function for all values of $x \in R$.

(c) Prove that $f(x) = ax + b$ where a, b are constants and $a > 0$ is a strictly increasing function for all real values of x.

2. (a) Show that $f(x) = x^2$ is a strictly increasing function for all real $x > 0$.

(b) Prove that the function $f(x) = x^2 - 4$ is monotonically increasing for $x > 2$ and monotonically decreasing for $-2 < x < 2$ where $x \in R$.

Theorem 1 : If f(x) is an increasing function on an open interval]a, b[, then its derivative f '(x) is positive at this point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f (x)] be a point on the curve y = f (x)

For a positive δx, we have

$$x + \delta x > x$$

Now, function f(x) is an increasing function

$\therefore \quad f(x + \delta x) > f(x)$

or, $\quad f(x + \delta x) - f(x) > 0$

or, $\quad \dfrac{f(x+\delta x) - f(x)}{\delta x} > 0 \quad [\because \delta x > 0]$

Taking δx as a small positive number and proceeding to limit, when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} > 0$$

or, $\quad f'(x) > 0$

Thus, if $y = f(x)$ is an increasing function at a point, then f '(x) is positive at that point.

Theorem 2 : If f (x) is a decreasing function on an open interval]a, b[then its derivative f '(x) is negative at that point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f(x)] be a point on the curve y = f(x)

For a positive δx, we have $\quad x + \delta x > x$

Since the function is a decreasing function

$\therefore \quad f(x + \delta x) < f(x) \quad \delta x > 0$

or, $\quad f(x + \delta x) - f(x) < 0$

Dividing by δx, we have $\quad \dfrac{f(x + \delta x) - f(x)}{\delta x} < 0 \quad \delta x > 0$

or, $\quad \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x} < 0$

or, $\quad f'(x) < 0$

Thus, if $y = f(x)$ is a decreasing function at a point, then, $f'(x)$ is negative at that point.

Note : If $f(x)$ is derivable in the closed interval [a,b], then f (x) is

(i) increasing over [a,b], if f '(x)>0 in the open interval]a,b[

(ii) decreasing over [a,b], if f '(x)<0 in the open interval]a,b[.

29.8 RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

Consider a function whose curve is shown in the Fig. 29.5

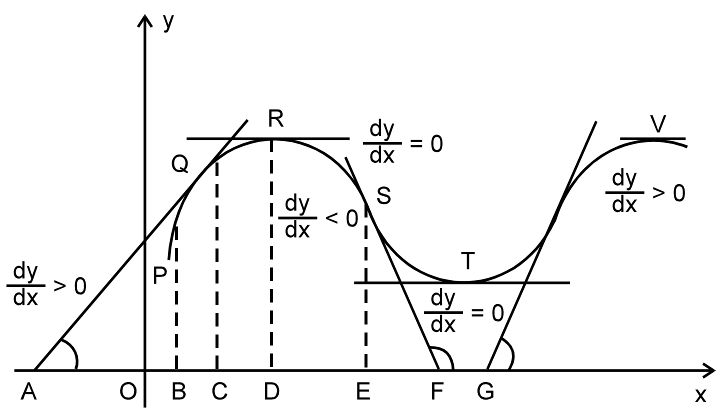


Fig. 29.5

We divide, our study of relation between sign of derivative of a function and its increasing or decreasing nature (monotonicity) into various parts as per Fig. 29.5

(i) P to R (ii) R to T (iii) T to V

(i) We observe that the ordinate (y-coordinate) for every succeeding point of the curve from P to R increases as also its x-coordinate. If (x_2, y_2) are the coordinates of a point that succeeds (x_1, y_1) then $x_2 > x_1$ yields $y_2 > y_1$ or $f(x_2) > f(x_1)$.

Also the tangent at every point of the curve between P and R makes acute angle with the positive direction of x-axis and thus the slope of the tangent at such points of the curve (except at R) is positive. At R where the ordinate is maximum the tangent is parallel to x-axis, as a result the slope of the tangent at R is zero.

We conclude for this part of the curve that

(a) The function is monotonically increasing from P to R.

(b) The tangent at every point (except at R) makes an acute angle with positive direction of x-axis.

(c) The slope of tangent is positive i.e. $\frac{dy}{dx} > 0$ for all points of the curve for which y is increasing.

(d) The slope of tangent at R is zero i.e. $\frac{dy}{dx} = 0$ where y is maximum.

(ii) The ordinate for every point between R and T of the curve decreases though its x-coordinate increases. Thus, for any point $x_2 > x_1$ yelds $y_2 < y_1$, or $f(x_2) < f(x_1)$.

Also the tangent at every point succeeding R along the curve makes obtuse angle with positive direction of x-axis. Consequently, the slope of the tangent is negative for all such points whose ordinate is decreasing. At T the ordinate attains minimum value and the tangent is parallel to x-axis and as a result the slope of the tangent at T is zero.

We now conclude :

(a) The function is monotonically decreasing from Rto T.

(b) The tangent at every point, except at T, makes obtuse angle with positive direction of x-axis.

(c) The slope of the tangent is negative i.e., $\frac{dy}{dx} < 0$ for all points of the curve for which y is decreasing.

(d) The slope of the tangent at T is zero i.e. $\frac{dy}{dx} = 0$ where the ordinate is minimum.

(iii) Again, for every point from T to V

The ordinate is constantly increasing, the tangent at every point of the curve between T and V makes acute angle with positive direction of x-axis. As a result of which the slope of the tangent at each of such points of the curve is positive.

Conclusively,

$$\frac{dy}{dx} > 0$$

at all such points of the curve except at T and V, where $\frac{dy}{dx} = 0$. The derivative $\frac{dy}{dx} < 0$ on one side, $\frac{dy}{dx} > 0$ on the other side of points R, T and V of the curve where $\frac{dy}{dx} = 0$.

Example 29.26 Find for what values of x, the function

$$f(x) = x^2 - 6x + 8$$

is increasing and for what values of x it is decreasing.

Solution : $f(x) = x^2 - 6x + 8$

$$f'(x) = 2x - 6$$

For $f(x)$ to be increasing, $f'(x) > 0$

i.e., $2x - 6 > 0$ or, $2(x-3) > 0$

or, $x - 3 > 0$ or, $x > 3$

The function increases for x>3.

For $f(x)$ to be decreasing,

$$f'(x) < 0$$

i.e., $2x - 6 < 0$ or, $x - 3 < 0$

or, $x < 3$

Thus, the function decreases for $x < 3$.

Example 29.27 Find the interval in which $f(x) = 2x^3 - 3x^2 - 12x + 6$ is increasing or decreasing.

Solution : $f(x) = 2x^3 - 3x^2 - 12x + 6$

$$f'(x) = 6x^2 - 6x - 12$$

$$= 6(x^2 - x - 2)$$

$$= 6(x-2)(x+1)$$

For $f(x)$ to be increasing function of x,

$$f'(x) > 0$$

i.e. $6(x-2)(x+1) > 0$ or, $(x-2)(x+1) > 0$

Since the product of two factors is positive, this implies either both are positive or both are negative.

Either $x-2>0$ and $x+1>0$ or $x-2<0$ and $x+1<0$

i.e. $x>2$ and $x>-1$ | i.e. $x<2$ and $x<-1$

$x>2$ implies $x>-1$ | $x<-1$ implies $x<2$

$\therefore$ $x>2$ | $\therefore$ $x<-1$

Hence f (x) is increasing for $x>2$ or $x<-1$.

Now, for f (x) to be decreasing,

$$f'(x)<0$$

or, $6(x-2)(x+1)<0$ or, $(x-2)(x+1)<0$

Since the product of two factors is negative, only one of them can be negative, the other positive.

Therefore,

Either | or

$x-2>0$ and $x+1<0$
i.e. $x>2$ and $x<-1$
There is no such possibility that $x>2$ and at the same time $x<-1$

$x-2<0$ and $x+1>0$
i.e. $x<2$ and $x>-1$
This can be put in this form
$-1<x<2$

$\therefore$ The function is decreasing in $-1<x<2$.

Example 29.28 Determine the intervals for which the function

$$f(x)=\frac{x}{x^2+1} \text{ is increasing or decreasing.}$$

Solution :

$$f'(x)=\frac{\left(x^2+1\right)\frac{dx}{dx}-x\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)-x\cdot(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{1-x^2}{\left(x^2+1\right)^2}$$

$$\therefore \quad f'x=\frac{(1-x)(1+x)}{\left(x^2+1\right)^2}$$

As $\left(x^2+1\right)^2$ is positive for all real x.

Therefore, if $-1 < x < 0, (1-x)$ is positive and $(1+x)$ is positive, so $f'(x) > 0$;

$\therefore$ If $\quad 0 < x < 1, (1-x)$ is positive and $(1+x)$ is positive, so f'$f'(x) > 0$;

If $\quad x < -1, (1-x)$ is positive and $(1+x)$ is negative, so $f'(x) < 0$;

$x > 1, (1-x)$ is negative and $(1+x)$ is positive, so $f'(x) < 0$;

Thus we conclude that

the function is increasing $\quad$ for $-1 < x < 0$ and $0 < x < 1$

or, $\quad$ for $-1 < x < 1$

and the function is decreasing for $x < -1$ or $x > 1$

Note : Points where $f'(x) = 0$ are critical points. Here, critical points are $x = -1$, $x = 1$.

Example 29.29 Show that

(a) $\quad f(x) = \cos x$ is decreasing in the interval $0 \le x \le \pi$.

(b) $\quad f(x) = x - \cos x$ is increasing for all x.

Solution :(a) $\quad f(x) = \cos x$

$$f'(x) = -\sin x$$

$f(x)$ is decreasing

If $\quad f'(x) < 0$

i.e., $\quad -\sin x < 0$

i.e., $\quad \sin x > 0$

sin x is positive in the first quadrant and in the second quadrant, therefore, sin x is positive in $0 \le x \le \pi$

$\therefore \quad f(x)$ is decreasing in $0 \le x \le \pi$

(b) $\quad f(x) = x - \cos x$

$$f'(x) = 1 - (-\sin x)$$

$$= 1 + \sin x$$

Now, we know that the minimum value of sinx is –1 and its maximum; value is 1 i.e.,sin x lies between –1 and 1 for all x,

i.e., $\quad -1 \le \sin x \le 1 \quad$ or $\quad 1-1 \le 1+\sin x \le 1+1$

or $\quad 0 \le 1+\sin x \le 2$

or $\quad 0 \le f'(x) \le 2$

or $\quad 0 \le f'(x)$

or $\qquad f'(x) \geq 0$

$\Rightarrow f(x) = x - \cos x$ is increasing for all x.

CHECK YOUR PROGRESS 29.8

Find the intervals for which the followiong functions are increasing or decreasing.

1. (a) $f(x) = x^2 - 7x + 10$ (b) $f(x) = 3x^2 - 15x + 10$

2. (a) $f(x) = x^3 - 6x^2 - 36x + 7$ (b) $f(x) = x^3 - 9x^2 + 24x + 12$

3. (a) $y = -3x^2 - 12x + 8$ (b) $f(x) = 1 - 12x - 9x^2 - 2x^3$

4. (a) $y = \dfrac{x-2}{x+1}, x \neq -1$ (b) $y = \dfrac{x^2}{x-1}, x \neq 1$ (c) $y = \dfrac{x}{2} + \dfrac{2}{x}, x \neq 0$

5. (a) Prove that the function log sin x is decreasing in $\left[\dfrac{\pi}{2}, \pi\right]$

   (b) Prove that the function cos x is increasing in the interval $[\pi, 2\pi]$

   (c) Find the intervals in which the function $\cos\left(2x + \dfrac{\pi}{4}\right), 0 \leq x \leq \pi$ is decreasing or increasing.

Find also the points on the graph of the function at which the tangents are parallel to x-axis.

29.9 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We have seen the graph of a continuous function. It increases and decreases alternatively. If the value of a continious function increases upto a certain point then begins to decrease, then this point is called point of maximum and corresponding value at that point is called maximum value of the function. A stage comes when it again changes from decreasing to increasing . If the value of a continuous function decreases to a certain point and then begins to increase, then value at that point is called minimum value of the function and the point is called point of minimum.

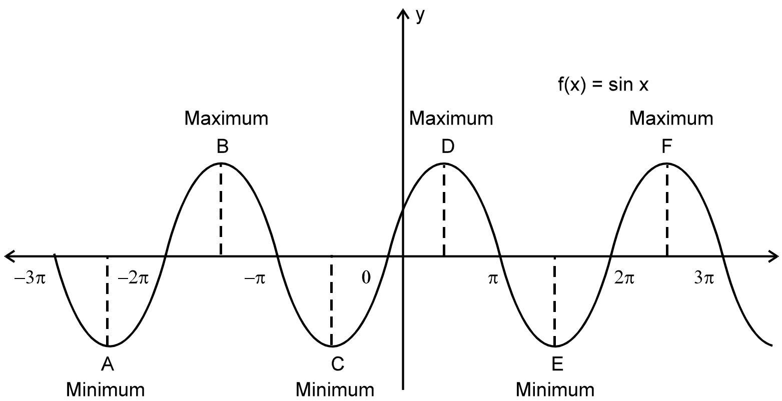


Fig. 29.6

Fig. 29.6 shows that a function may have more than one maximum or minimum values. So, for continuous function we have maximum (minimum) value in an interval and these values are not absolute maximum (minimum) of the function. For this reason, we sometimes call them as local maxima or local minima.

A function f (x) is said to have a maximum or a local maximum at the point x = a where a – b < a < a + b (See Fig. 29.7), if $f(a) \geq f(a \pm b)$ for all sufficiently small positive b.

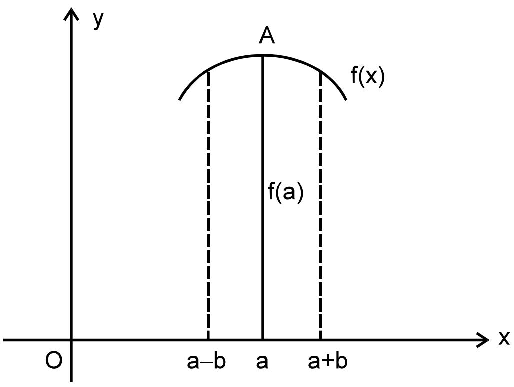


Fig. 29.7

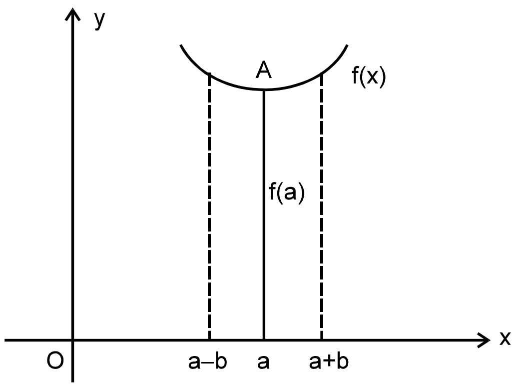


Fig. 29.8

A maximum (or local maximum) value of a function is the one which is greater than all other values on either side of the point in the immediate neighbourhood of the point.

A function f (x) is said to have a minimum (or local minimum) at the point x = a if $f(a) \leq f(a \pm b)$ where a – b< a < a + b

for all sufficiently small positive b.

In Fig. 25.8, the function f(x) has local minimum at the point x = a.

A minimum (or local miunimum) value of a function is the one which is less than all other values, on either side of the point in the immediate neighbourhood of the point.

Note : A neighbourhood of a point x $\in$ R is defined by open internal]x–$\in$ [, when $\in$ >0.

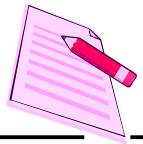
29.9.1 CONDITIONS FOR MAXIMUM OR MINIMUM

We know that derivative of a function is positive when the function is increasing and the derivative is negative when the function is decreasing. We shall apply this result to find the condition for maximum or a function to have a minimum. Refer to Fig. 25.6, points B,D, F are points of maxima and points A,C,E are points of minima.

Now, on the left of B, the function is increasing and so $f'(x) > 0$, but on the right of B, the function is decreasing and, therefore, $f'(x) < 0$. This can be achieved only when f '(x) becomes zero somewhere in betwen. We can rewrite this as follows :

A function f (x) has a maximum value at a point if (i) f '(x) = 0 and (ii) f '(x) changes sign from positive to negative in the neighbourhood of the point at which f '(x)=0 (points taken from left to right).

MODULE - VIII
Calculus



Notes

Now, on the left of C (See Fig. 29.6), function is decreasing and f '(x) therefore, is negative and on the right of C, f (x) is increasing and so f '(x) is positive. Once again f '(x) will be zero before having positive values. We rewrite this as follows :

A function f (x) has a minimum value at a point if (i) f '(x)=0, and (ii) f '(x) changes sign from negative to positive in the neighbourhood of the point at which f '(x) = 0.

We should note here that f '(x) = 0 is necessary condition and is not a sufficient condition for maxima or minima to exist. We can have a function which is increasing, then constant and then again increasing function. In this case, f '(x) does not change sign. The value for which f '(x) = 0 is not a point of maxima or minima. Such point is called point of inflexion.

For example, for the function $f(x) = x^3, x = 0$ is the point of inflexion as $f'(x) = 3x^2$ does not change sign as x passes through 0. f'(x) is positive on both sides of the value '0' (tangents make acute angles with x-axis) (See Fig. 29.9).

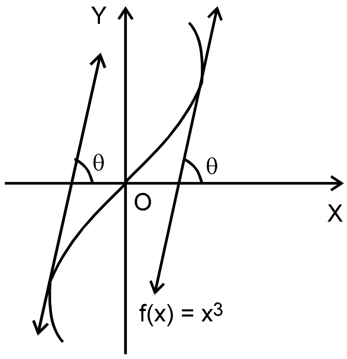


Fig. 29.9

Hence $f(x) = x^3$ has a point of inflexion at x = 0.

The points where f '(x) = 0 are called stationary points as the rate ofchange of the function is zero there. Thus points of maxima and minima are stationary points.

Remarks

The stationary points at which the function attains either local maximum or local minimum values are also called extreme points and both local maximum and local minimum values are called extreme values of f (x). Thus a function attains an extreme value at x=a if f (a) is either a local maximum or a local minimum.

29.9.2 METHOD OF FINDING MAXIMA OR MINIMA

We have arrived at the method of finding the maxima or minima of a function. It is as follows :

(i) Find f'(x)

(ii) Put f '(x)=0 and find stationary points

(iii) Consider the sign of f '(x) in the neighbourhood of stationary points. If it changes sign from +ve to –ve, then f (x) has maximum value at that point and if f '(x) changes sign from –ve to +ve, then f (x) has minimum value at that point.

(iv) If f '(x) does not change sign in the neighbourhood of a point then it is a point of inflexion.

Example 29.30 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$.

Solution : Here

$$f(x) = x^3 - 3x^2 - 9x$$

$$f'(x) = 3x^2 - 6x - 9$$

Step I. Now $f'(x)=0$ gives us $3x^2-6x-9=0$

or $x^2-2x-3=0$

or $(x-3)(x+1)=0$

or $x=3,-1$

$\therefore$ Stationary points are $x=3, x=-1$

Step II. At $x=3$

For $x<3$ $f'(x)<0$

and for $x>3$ $f'(x)>0$

$\therefore$ $f'(x)$ changes sign from –ve to +ve in the neighbourhood of 3.

$\therefore$ $f(x)$ has minimum value at $x = 3$.

Step III. At $x=-1$,

For $x<-1$, $f'(x)>0$

and for $x>-1$, $f'(x)<0$

$\therefore$ $f'(x)$ changes sign from +ve to –ve in the neighbourhood of –1.

$\therefore$ $f(x)$ has maximum value at $x=-1$.

$\therefore$ $x = -1$ and $x = 3$ give us points of maxima and minima respectively. If we want to find maximum value (minimum value), then we have

$$\text{maximum value} = f(-1)=(-1)^3-3(-1)^2-9(-1)$$

$$=-1-3+9=5$$

and $$\text{minimum value} = f(3)=3^3-3(3)^2-9(3)=-27$$

$\therefore$ $(-1,5)$ and $(3,-27)$ are points of local maxima and local minima respectively.

Example 29.31 Find the local maximum and the local minimum of the function

$$f(x)=x^2-4x$$

Solution : $f(x)=x^2-4x$

$\therefore$ $f'(x)=2x-4=2(x-2)$

Putting $f'(x)=0$ yields $2x-4=0$, i.e., $x=2$.

We have to examine whether $x = 2$ is the point of local maximum or local minimum or neither maximum nor minimum.

Let us take $x = 1.9$ which is to the left of 2 and $x = 2.1$ which is to the right of 2 and find $f'(x)$ at these points.

$$f'(1.9) = 2(1.9-2) < 0$$

$$f'(2.1) = 2(2.1-2) > 0$$

Since $f'(x) < 0$ as we approach 2 from the left and $f'(x) > 0$ as we approach 2 from the right, therefore, there is a local minimum at $x = 2$.

We can put our findings for sign of derivatives of f(x) in any tabular form including the one given below :

sign of $f'(x)$

point $x = 2$

left of 2	right of 2
$f'(x) < 0$	$f'(x) < 0$

Local minimum

Example 29.32 Find all local maxima and local minima of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 8$$

Solution : $\qquad f(x) = 2x^3 - 3x^2 - 12x + 8$

$\therefore \qquad f'(x) = 6x^2 - 6x - 12$

$$= 6\left(x^2 - x - 2\right)$$

$\therefore \qquad f'(x) = 6(x+1)(x-2)$

Now solving f '(x)=0 for x, we get

$$6(x+1)(x-2) = 0$$

$\Rightarrow \qquad x = -1, 2$

Thus $\qquad f'(x) = 0$ at $x = -1, 2$.

We examine whether these points are points of local maximum or local minimum or neither of them.

Consider the point $x = -1$

Let us take $x = -1.1$ which is to the left of -1 and $x = -0.9$ which is to the right of -1 and find $f'(x)$ at these points.

$$f'(-1.1) = 6(-1.1+1)(-1.1-2)$$, which is positive i.e. $f'(x)>0$

$$f'(-0.9) = 6(-0.9+1)(-0.9-2)$$, which is negative i.e. $f'(x)<0$

Thus, at $x = -1$, there is a local maximum.

Consider the point $x = 2$.

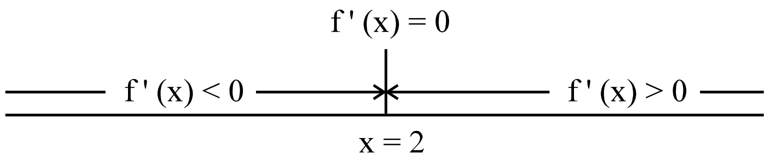
Now, let us take $x = 1.9$ which is to the left of $x = 2$ and $x = 2.1$ which is to the right of $x = 2$ and find $f'(x)$ at these points.

$$f'(1.9) = 6(1.9+1)(1.9-2)$$
$$= 6\times(\text{Positive number})\times(\text{negative number})$$
$$= \text{a negative number}$$

i.e. $\quad f'(1.9) < 0$

and $\quad f'(2.1)=6(2.1+1)(2.1-2)$, which is positive

i.e., $\quad f'(2.1)>0$



$\because \quad f'(x) < 0$ as we approach 2 from the left

and $\quad f'(x) > 0$ as we approach 2 from the right.

$\therefore \quad x = 2$ is the point of local minimum

Thus $f(x)$ has local maximum at $x = -1$, maximum value of $f(x) = -2-3+12+8 = 15$

$f(x)$ has local minimum at $x = 2$, minimum value of $f(x) = 2(8)-3(4)-12(2)+8 = -12$

Sign of $f'(x)$

Point $x = -1$			Point $x = 2$	
Left of -1	Right of -1		Left of 2	Right of 2
positive	negative		negative	positive
local maximum			local minimum	

Example 29.33 Find local maximum and local minimum, if any, of the following function

$$f(x) = \frac{x}{1+x^2}$$

Solution :
$$f(x) = \frac{x}{1+x^2}$$

Then $$f'(x) = \frac{(1+x^2)1-(2x)x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

For finding points of local maximum or local minimum, equate f '(x) to 0.

i.e. $$\frac{1-x^2}{(1+x^2)^2} = 0$$

$\Rightarrow$ $$1-x^2 = 0$$

or $(1+x)(1-x) = 0$ or $x = 1, -1.$

Consider the value x = 1.

The sign of f '(x) for values of x slightly less than 1 and slightly greater than 1 changes from positive to negative. Therefore there is a local maximum at x =1, and the local maximum

value $= \dfrac{1}{1+(1)^2} = \dfrac{1}{1+1} = \dfrac{1}{2}$

Now consider x = – 1.

f '(x) changes sign from negative to positive as x passes through –1, therefore, f (x) has a local minimum at x = – 1

Thus, the local minimum value $= \dfrac{-1}{2}$

Example 29.34 Find the local maximum and local minimum, if any, for the function

$$f(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$$

Solution : We have $f(x) = \sin x + \cos x$

$$f'(x) = \cos x - \sin x$$

For local maxima/minima, $f'(x) = 0$

$\therefore$ $$\cos x - \sin x = 0$$

or, $\tan x = 1$ or, $x = \dfrac{\pi}{4}$ in $0 < x < \dfrac{\pi}{2}$

At $x = \dfrac{\pi}{4}$,

For $x < \dfrac{\pi}{4}$, $\cos x > \sin x$

$\therefore$ $$f'(x) = \cos x - \sin x > 0$$

For $x > \frac{\pi}{4}$, $\cos x - \sin x < 0$

$\therefore$ $f'(x) = \cos x - \sin x < 0$

$\therefore$ f '(x) changes sign from positive to negative in the neighbourhood of $\frac{\pi}{4}$.

$\therefore$ $x = \frac{\pi}{4}$ is a point of local maxima.

$$\text{Maximum value } = f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$\therefore$ Point of local maxima is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

CHECK YOUR PROGRESS 29.9

Find all points of local maxima and local minima of the following functions. Also , find the maxima and minima at such points.

1. $x^2 - 8x + 12$
2. $x^3 - 6x^2 + 9x + 15$
3. $2x^3 - 21x^2 + 36x - 20$
4. $x^4 - 62x^2 + 120x + 9$
5. $(x-1)(x-2)^2$
6. $\dfrac{x-1}{x^2 + x + 2}$

29.10 USE OF SECOND DERIVATIVE FOR DETERMINATION OF MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We now give below another method of finding local maximum or minimum of a function whose second derivative exists. Various steps used are :

(i) Let the given function be denoted by f (x).

(ii) Find f ' (x) and equate it to zero.

(iii) Solve f ' (x)=0, let one of its real roots be x = a.

(iv) Find its second derivative, f "(x). For every real value 'a' of x obtained in step (iii), evaluate f" (a). Then if

f "(a) <0 then x =a is a point of local maximum.

f "(a)>0 then x = a is a point of local minimum.

f "(a)=0 then we use the sign of f '(x) on the left of 'a' and on the right of 'a' to arrive at the result.

Example 29.35 Find the local minimum of the following function :

$$2x^3 - 21x^2 + 36x - 20$$

Solution : Let $\quad f(x) = 2x^3 - 21x^2 + 36x - 20$

Then $\quad f'(x) = 6x^2 - 42x + 36$

$$= 6\left(x^2 - 7x + 6\right)$$

$$= 6(x-1)(x-6)$$

For local maximum or minimum

$$f'(x) = 0$$

or $\quad 6(x-1)(x-6) = 0 \quad \Rightarrow x = 1, 6$

Now $\quad f''(x) = \dfrac{d}{dx} f'(x)$

$$= \frac{d}{dx}\left[6\left(x^2 - 7x + 6\right)\right]$$

$$= 12x - 42$$

$$= 6(2x - 7)$$

For $\quad x = 1, f''(1) = 6(2.1 - 7) = -30 < 0$

$x = 1$ is a point of local maximum.

and $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20 = -3$ is a local maximum.

For $x = 6$,

$f''(6) = 6(2.6 - 7) = 30 > 0$

$\therefore \quad x = 6$ is a point of local minimum

and $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20 = -128$ is a local minimum.

Example 29.36 Find local maxima and minima (if any) for the function

$$f(x) = \cos 4x; \quad 0 < x < \frac{\pi}{2}$$

Solution : $\quad f(x) = \cos 4x$

$\therefore \quad f'(x) = -4\sin 4x$

Now, $\quad f'(x) = 0 \quad \Rightarrow \quad -4\sin 4x = 0$

or, $\quad \sin 4x = 0 \quad$ or, $\quad 4x = 0, \pi, 2\pi$

or, $$x = 0, \frac{\pi}{4}, \frac{\pi}{2}$$

$\therefore$ $$x = \frac{\pi}{4} \qquad \left[\because \; 0 < x < \frac{\pi}{2}\right]$$

Now, $$f''(x) = -16\cos 4x$$

at $$x = \frac{\pi}{4}, f''(x) = -16\cos\pi$$
$$= -16(-1) = 16 > 0$$

$\therefore$ f (x) is minimum at $x = \frac{\pi}{4}$

Minimum value $$f\left(\frac{\pi}{4}\right) = \cos\pi = -1$$

Example 29.37 (a) Find the maximum value of $2x^3 - 24x + 107$ in the interval [–3,–1].

(b) Find the minimum value of the above function in the interval [1,3].

Solution : Let $f(x) = 2x^3 - 24x + 107$

$$f'(x) = 6x^2 - 24$$

For local maximum or minimum,

$$f'(x) = 0$$

i.e. $$6x^2 - 24 = 0 \qquad \Rightarrow \qquad x = -2, 2$$

Out of two points obtained on solving f '(x)=0, only–2 belong to the interval [–3,–1]. We shall, therefore, find maximum if any at x=–2 only.

Now $$f''(x) = 12x$$

$\therefore$ $$f''(-2) = 12(-2) = -24$$

or $$f''(-2) < 0$$

which implies the function f (x) has a maximum at x =–2.

$\therefore$ Required maximum value $$= 2(-2)^3 - 24(-2) + 107$$
$$= 139$$

Thus the point of maximum belonging to the given interval [–3,–1] is –2 and, the maximum value of the function is 139.

(b) Now $$f''(x) = 12x$$

$\therefore \quad f''(2) = 24 > 0, \quad [\because 2 \text{ lies in } [1, 3]]$

which implies, the function f (x) shall have a minimum at x =2.

$\therefore$ Required minimum $= 2(2)^3 - 24(2) + 107$

$=75$

Example 29.38 Find the maximum and minimum value of the function

$$f(x) = \sin x(1+\cos x) \text{ in } (0, \pi).$$

Solution : We have, $f(x) = \sin x(1+\cos x)$

$$f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos^2 x - (1-\cos^2 x) = 2\cos^2 x + \cos x - 1$$

For stationary points, $f'(x) = 0$

$$\therefore \quad 2\cos^2 x + \cos x - 1 = 0$$

$$\therefore \quad \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$$

$$\therefore \quad x = \pi, \frac{\pi}{3}$$

Now, $f(0) = 0$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $f(\pi) = 0$

$\therefore$ f (x) has maximum value $\frac{3\sqrt{3}}{4}$ at $x = \frac{\pi}{3}$

and miminum value 0 at $x = 0$ and $x = \pi$.

CHECK YOUR PROGRESS 29.10

Find local maximum and local minimum for each of the following functions using second order derivatives.

1. $2x^3 + 3x^2 - 36x + 10$
2. $-x^3 + 12x^2 - 5$

3. $(x-1)(x+2)^2$

4. $x^5-5x^4+5x^3-1$

5. $\sin x(1+\cos x), 0<x<\dfrac{\pi}{2}$

6. $\sin x+\cos x, 0<x<\dfrac{\pi}{2}$

7. $\sin 2x-x, \dfrac{-\pi}{2}\le x\le\dfrac{\pi}{2}$

29.11 APPLICATIONS OF MAXIMA AND MINIMA TO PRACTICAL PROBLEMS

The application of derivative is a powerful tool for solving problems that call for minimising or maximising a function. In order to solve such problems, we follow the steps in the following order :

(i) Frame the function in terms of variables discussed in the data.

(ii) With the help of the given conditions, express the function in terms of a single variable.

(iii) Lastly, apply conditions of maxima or minima as discussed earlier.

Example 29.39 Find two positive real numbers whose sum is 70 and their product is maximum.

Solution : Let one number be x. As their sum is 70, the other number is 70–x. As the two numbers are positive, we have, $x>0, 70-x>0$

$$70-x>0 \quad \Rightarrow \quad x<70$$

$$\therefore \quad 0<x<70$$

Let their product be f (x)

Then $\quad f(x)=x(70-x)=70x-x^2$

We have to maximize the prouct f (x).

We, therefore, find f '(x) and put that equal to zero.

$$f'(x) = 70 - 2x$$

For maximum product, $f'(x)=0$

or $\quad 70-2x = 0$

or $\quad x = 35$

Now $f''(x) = -2$ which is negative. Hence f (x) is maximum at x = 35

The other number is $70-x=35$

Hence the required numbers are 35, 35.

Example 29.40 Show that among rectangles of given area, the square has the least perimeter.

Solution : Let x, y be the length and breadth of the rectangle respectively.

$\therefore$ Its area = xy

Since its area is given, represent it by A, so that we have

$$A = xy$$

or $$y = \frac{A}{x} \qquad \text{... (i)}$$

Now, perimeter say P of the rectangle = 2 (x + y)

or $$P = 2\left(x + \frac{A}{x}\right)$$

$$\therefore \quad \frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \qquad \text{...(ii)}$$

For a minimum $P, \frac{dP}{dx} = 0.$

i.e. $$2\left(1 - \frac{A}{x^2}\right) = 0$$

or $$A = x^2 \qquad \text{or} \qquad \sqrt{A} = x$$

Now, $$\frac{d^2P}{dx^2} = \frac{4A}{x^3}, \text{ which is positive.}$$

Hence perimeter is minimum when $x = \sqrt{A}$

$$\therefore \quad y = \frac{A}{x}$$

$$= \frac{x^2}{x} = x \qquad (\because A = x^2)$$

Thus, the perimeter is minimum when rectangle is a square.

Example 29.41 An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution : Let x be the side of the square base of the box and y its height.

Total surface area of othe box $= x^2 + 4xy$

$$\therefore \quad x^2 + 4xy = a^2 \qquad \text{or} \qquad y = \frac{a^2 - x^2}{4x}$$

Volume of the box, $V = \text{base area} \times \text{height}$

$$= x^2 y = x^2\left(\frac{a^2 - x^2}{4x}\right)$$

or $$V = \frac{1}{4}\left(a^2 x - x^3\right) \qquad \text{...(i)}$$

$\therefore$ $$\frac{dV}{dx} = \frac{1}{4}\left(a^2 - 3x^2\right)$$

For maxima/minima $\frac{dV}{dx} = 0$

$\therefore$ $$\frac{1}{4}\left(a^2 - 3x^2\right) = 0$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\text{Volume} = \frac{1}{4}\left(\frac{\left(a^3\right)}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}} \qquad \text{..(iii)}$$

Again $$\frac{d^2V}{dx^2} = \frac{d}{dx}\frac{1}{4}\left(a^2 - 3x^2\right) = -\frac{3}{2}x$$

x being the length of the side, is positive.

$\therefore$ $$\frac{d^2V}{dx^2} < 0$$

$\therefore$ The volume is maximum.

Hence maximum volume of the box $= \frac{a^3}{6\sqrt{3}}$.

Example 29.42 Show that of all rectangles inscribed in a given circle, the square has the maximum area.

Solution : Let ABCD be a rectangle inscribed in a circle of radius r. Then diameter AC= 2r

Let $\quad AB = x$ and $BC = y$

Then $\quad AB^2 + BC^2 = AC^2 \quad$ or $\quad x^2 + y^2 = (2r)^2 = 4r^2$

Now area A of the rectangle = xy

$\therefore$ $$A = x\sqrt{4r^2 - x^2}$$

$\therefore$ $$\frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \cdot 1$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

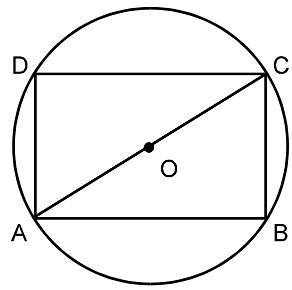


Fig. 29.10

For maxima/minima, $\frac{dA}{dx} = 0$

$$\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \Rightarrow x = \sqrt{2}r$$

Now $$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}(-4x) - (4r^2 - 2x^2)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{-4x(4r^2 - x^2) + x(4r^2 - 2x^2)}{(4r^2 - x^2)^{\frac{3}{2}}}$$

$$= \frac{-4\sqrt{2}(2r^2) + 0}{(2r^2)^{\frac{3}{2}}}$$ ... (Putting $x = \sqrt{2}r$)

$$= \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Thus, A is maximum when $x = \sqrt{2}r$

Now, from (i), $y = \sqrt{4r^2 - 2r^2} = \sqrt{2}\ r$

x = y. Hence, rectangle ABCD is a square.

Example 29.43 Show that the height of a closed right circular cylinder of a given volume andleast surface is equal to its diameter.

Solution : Let V be the volume, r the radius and h the height of the cylinder.

Then, $$V = \pi r^2 h$$

or $$h = \frac{V}{\pi r^2}$$ ... (i)

Now surface area $$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r.\frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

Now $$\frac{dS}{dr}=\frac{-2V}{r^2}+4\pi r$$

For minimum surface area, $\frac{dS}{dr}=0$

$\therefore$ $$\frac{-2V}{r^2}+4\pi r=0$$

or $$V=2\pi r^3$$

From (i) and (ii), we get $$h=\frac{2\pi r^3}{\pi r^2}=2r \quad ..(ii)$$

Again, $$\frac{d^2S}{dr^2}=\frac{4V}{r^3}+4\pi=8\pi+4\pi \quad \text{... [Using (ii)]}$$

$$=12\pi>0$$

$\therefore$ S is least when h = 2r

Thus, height of the cylidner = diameter of the cylinder.

Example 29.44 Show that a closed right circular cylinder of given surface has maximum volume if its height equals the diameter of its base.

Solution : Let S and V denote the surface area and the volume of the closed right circular cylinder of height h and base radius r.

Then $$S=2\pi rh+2\pi r^2 \quad .....(i)$$
(Here surface is a constant quantity, being given)

$$V=\pi r^2 h$$

$\therefore$ $$V=\pi r^2\left[\frac{S-2\pi r^2}{2\pi r}\right]$$

$$=\frac{r}{2}\left[S-2\pi r^2\right]$$

$$V=\frac{Sr}{2}-\pi r^3$$

$$\frac{dV}{dr}=\frac{S}{2}-\pi\left(3r^2\right)$$

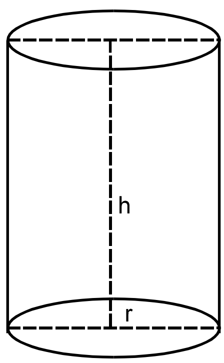


Fig. 29.11

For maximum or minimum, $\frac{dV}{dr}=0$

i.e., $$\frac{S}{2}-\pi\left(3r^2\right)=0$$

or $$S = 6\pi r^2$$

From (i), we have $$6\pi r^2 = 2\pi rh + 2\pi r^2$$

$\Rightarrow$ $$4\pi r^2 = 2\pi rh$$

$\Rightarrow$ $$2r = h \qquad \text{...(ii)}$$

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left[\frac{S}{2} - 3\pi r^2\right]$$

$$= -6\pi r, \qquad \because \frac{d}{dr}\left(\frac{S}{2}\right) = 0$$

= a negative quantity

Hence the volume of the right circular cylinder is maximum when its height is equal to twice its radius i.e. when h = 2r.

Example 29.45 A square metal sheet of side 48 cm. has four equal squares removed from the corners and the sides are then turned up so as to form an open box. Determine the size of the square cut so that volume of the box is maximum.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is (48–2x) cm. and height x cm.

Now x > 0, 48–2x > 0, i.e. x < 24

$\therefore$ x lies between 0 and 24 or $0 < x < 24$

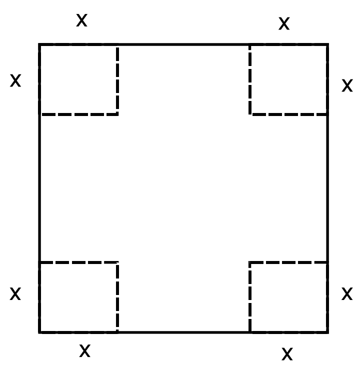


Fig. 29.12

Now, Volume V of the box

$$= (48-2x)(48-2x)x$$

i.e. $$V = (48-2x)^2 \cdot x$$

$\therefore$ $$\frac{dV}{dx} = (48-2x)^2 + 2(48-2x)(-2)x$$

$$= (48-2x)(48-6x)$$

Condition for maximum or minimum is $\frac{dV}{dx} = 0$

i.e., $(48-2x)(48-6x)=0$

We have either $x = 24$, or $x = 8$

$\because$ $0 < x < 24$

$\therefore$ Rejecting x = 24, we have, x = 8 cm.

Now, $$\frac{d^2V}{dx^2} = 24x - 384$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=8} = 192 - 384 = -192 < 0,$$

Hence for x = 8, the volume is maximum.

Hence the square of side 8 cm. should be cut from each corner.

Example 29.46 The profit function P (x) of a firm, selling x items per day is given by

$$P(x) = (150 - x)x - 1625.$$

Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

Solution : It is given that 'x' is the number of items produced and sold out by the firm every day. In order to maximize profit,

$$P'(x) = 0 \text{ i.e. } \frac{dP}{dx} = 0$$

or $$\frac{d}{dx}\left[(150-x)x - 1625\right] = 0$$

or $$150-2x = 0$$

or $$x = 75$$

Now, $\frac{d}{dx}P'(x) = P''(x) = -2$ = a negative quantity. Hence P (x) is maximum for x = 75.

Thus, the firm should manufacture only 75 items a day to make maximum profit.

Now, Maximum Profit $= P(75) = (150-75)75 - 1625$

= Rs. (75×75–1625)

= Rs. (5625–1625)

= Rs. 4000

Example 29.47 Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

Solution : Let h be the height and R the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder.

Then $\qquad V = \pi R^2 h \qquad$ ...(i)

From Δ OCB, we have

$$r^2 = \left(\frac{h}{2}\right)^2 + R^2 \qquad \ldots\left(\because OB^2 = OC^2 + BC^2\right)$$

$$\therefore \qquad R^2 = r^2 - \frac{h^2}{4} \qquad \ldots(ii)$$

Now $\qquad V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi r^2 h - \pi\frac{h^3}{4}$

$$\therefore \qquad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

For maxima/minima, $\frac{dV}{dh} = 0$

$$\therefore \qquad \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\Rightarrow \qquad h^2 = \frac{4r^2}{3} \qquad \Rightarrow \qquad h = \frac{2r}{\sqrt{3}}$$

Now $\qquad \frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$

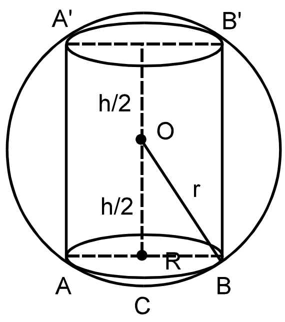


Fig. 29.13

$$\therefore \qquad \frac{d^2V}{dh^2}\left(\text{at } h = \frac{2r}{\sqrt{3}}\right) = -\frac{3\pi \times 2r}{2\times\sqrt{3}}$$

$$= -\sqrt{3}\pi r < 0$$

$\therefore \qquad$ V is maximum at $h = \frac{2r}{\sqrt{3}}$

Putting $h = \frac{2r}{\sqrt{3}}$ in (ii), we get

$$R^2 = r^2 - \frac{4r^2}{4\times 3} = \frac{2r^2}{3}, \ \therefore \ R = \sqrt{\frac{2}{3}}r$$

Maximum volume of the cylinder $= \pi R^2 h$

$$= \pi\cdot\left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}\,cm^3.$$

CHECK YOUR PROGRESS 29.11

1. Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.

2. Divide 15 into two parts such that the sum of their squares is minimum.

3. Show that among the rectangles of given perimeter, the square has the greatest area.

4. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

5. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

6. Find the radius of a closed right circular cylinder of volume 100 c.c. which has the minimum total surface area.

7. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder.

8. Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third that of the cone.

9. A conical tent of the given capacity (volume) has to be constructed. Find the ratio of the height to the radius of the base so as to minimise the canvas requried for the tent.

10. A manufacturer needs a container that is right circular cylinder with a volume 16π cubic meters. Determine the dimensions of the container that uses the least amount of surface (sheet) material.

11. A movie theatre's management is considering reducing the price of tickets from Rs.55 in order to get more customers. After checking out various facts they decide that the average number of customers per day 'q' is given by the function where x is the amount of ticket price reduced. Find the ticket price othat result in maximum revenue.

$$q = 500+100\,x$$

where x is the amount of ticket price reduced. Find the ticket price that result is maximum revenue.

30

INTEGRATION

311en30

In the previous lesson, you have learnt the concept of derivative of a function. You have also learnt the application of derivative in various situations.

Consider the reverse problem of finding the original function, when its derivative (in the form of a function) is given. This reverse process is given the name of integration. In this lesson, we shall study this concept and various methods and techniques of integration.

OBJECTIVES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like x^n, $\sin x$, $\cos x$,

  $\sec^2 x$, $\text{cosec}^2 x$, $\sec x \tan x$, $\text{cosec}\, x \cot x$, $\frac{1}{x}$, e^x etc.;
- state the following results :

  (i) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

  (ii) $\int [\pm k f(x)]\, dx = \pm k \int f(x)\, dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c},$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px + q)\, dx}{ax^2 + bx + c}, \int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$$

- derive and use the result

$$\int \frac{f'(x)}{f(x)} = \ell n\left|f(x)\right| + C$$

- state and use the method of integration by parts;

- evaluate integrals of the type :

$$\int \sqrt{x^2 \pm a^2}\,dx,\ \int \sqrt{a^2 - x^2}\,dx,\ \int e^{ax} \sin bx\ dx,\ \int e^{ax} \cos bx\ dx,$$

$$\int (px + q)\sqrt{ax^2 + bx + c}\ dx,\ \int \sin^{-1} x\ dx,\ \int \cos^{-1} x\ dx,$$

$$\int \sin^n x \cos^m x\ dx,\ \int \frac{dx}{a + b\sin x},\ \int \frac{dx}{a + b\cos x}$$

- derive and use the result

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c\text{; and}$$

- integrate rational expressions using partial fractions.

EXPECTED BACKGROUND KNOWLEDGE

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

30.1 INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ALMB as shown in Fig. 30.1.

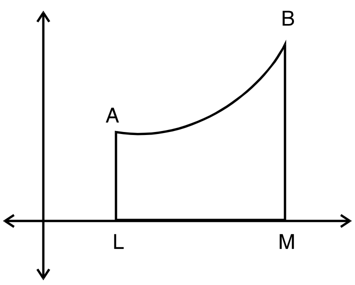


Fig. 30.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.30.2).

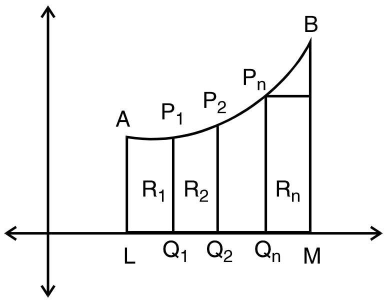


Fig. 30.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

This is the technique which Archimedes used two thousand years ago for finding areas, volumes, etc. The names of Newton (1642-1727) and Leibnitz (1646-1716) are often mentioned as the creators of present day of Calculus.

The integral calculus is the study of integration of functions. This finds extensive applications in Geometry, Mechanics, Natural sciences and other disciplines.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

30.2. INTEGRATION AS INVERSE OF DIFFERENTIATION

Consider the following examples :

(i) $\frac{d}{dx}\left(x^2\right) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}\left(e^x\right) = e^x$

Let us consider the above examples in a different perspective

(i) 2x is a function obtained by differentiation of x^2.

$\Rightarrow$ x^2 is called the antiderivative of $2x$

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow$ $\sin x$ is called the antiderivative of cos x

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

1. Integration of 2 x is x^2
2. Integration of cos x is sin x
3. Integration of e^x is e^x

The operation of integration is denoted by the symbol $\int$.

Thus

1. $\int 2x\,dx = x^2$ 2. $\int \cos x\,dx = \sin x$ 3. $\int e^x\,dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx}[f(x)] = f'(x)$, then f (x) is said to be an integral of $f'(x)$ and is written as $\int f'(x)dx = f(x)$

The function f '(x) which is integrated is called the integrand.

Constant of integration

$$\text{If } y = x^2, \text{ then } \frac{dy}{dx} = 2x$$

$$\therefore \quad \int 2x\,dx = x^2$$

Now consider $\frac{d}{dx}\left(x^2 + 2\right)$ or $\frac{d}{dx}\left(x^2 + c\right)$ where c is any real constant . Thus, we see that integral of 2x is not unique. The different values of $\int 2x\,dx$ differ by some constant. Therefore, $\int 2x\,dx = x^2 + C$, where c is called the constant of integration.

Thus $\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + c$

In general $\int f'(x)\,dx = f(x) + C$. The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note : $\int f(x)\,dx$, $\int f(y)\,dy$, $\int f(z)\,dz$ but not like $\int f(z)\,dx$

	Integral	**Verification**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where n is a constant and $n \neq -1$.	$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \sin x\,dx = -\cos x + C$	$\because \frac{d}{dx}(-\cos x + C) = \sin x$
3.	$\int \cos x\,dx = \sin x + C$	$\because \frac{d}{dx}(\sin x + C) = \cos x$
4.	$\int \sec^2 x\,dx = \tan x + C$	$\because \frac{d}{dx}(\tan x + C) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$	$\because \frac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
6.	$\int \sec x \tan x\,dx = \sec x + C$	$\because \frac{d}{dx}(\sec x + C) = \sec x \tan x$
7.	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$	$\because \frac{d}{dx}(-\operatorname{cosec} x + C) = \operatorname{cosec} x \cot x$
8.	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$	$\because \frac{d}{dx}\left(\sin^{-1} x + C\right) = \frac{1}{\sqrt{1-x^2}}$

9. $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C \qquad \because \frac{d}{dx}\left(\tan^{-1}x + C\right) = \frac{1}{1+x^2}$

10. $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C \qquad \because \frac{d}{dx}\left(\sec^{-1}x + C\right) = \frac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C \qquad \because \frac{d}{dx}\left(e^x + C\right) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C \qquad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x} \text{ if } x > 0$

13. $\int \frac{1}{x}dx = \log|x| + C \qquad \because \frac{d}{dx}\left(\log|x| + C\right)$

WORKING RULE

1. To find the integral of x^n, increase the index of x by 1, divide the result by new index and add constant C to it.

2. $\int \frac{1}{f(x)}dx$ will be very often written as $\int \frac{dx}{f(x)}$.

CHECK YOUR PROGRESS 30.1

1. Write any five different values of $\int x^{\frac{5}{2}}dx$

2. Write indefinite integral of the following :

(a) x^5 (b) $\cos x$ (c) 0

3. Evaluate :

(a) $\int x^6 dx$ (b) $\int x^{-7}dx$ (c) $\int \frac{1}{x}dx$ (d) $\int 3^x 5^{-x}dx$

(e) $\int \sqrt[3]{x}\,dx$ (f) $\int x^{-9}\,dx$ (g) $\int \frac{1}{\sqrt{x}}\,dx$ (h) $\int \sqrt[9]{x^{-8}}\,dx$

4. Evaluate :

(a) $\int \frac{\cos\theta}{\sin^2\theta}d\theta$ (b) $\int \frac{\sin\theta}{\cos^2\theta}d\theta$

(c) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} d\theta$ (d) $\int \frac{1}{\sin^2\theta} d\theta$

30.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions, e.g. if $f(x) = x^7 + x^3$, then

$$\int f(x)\,dx = \int [x^7 + x^3]\,dx = \int x^7 dx + \int x^3 dx = \frac{x^8}{8} + \frac{x^4}{4} + C$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Similarly, if the given function

$$f(x) = x^7 - x^2$$

we can write it as $\int f(x)\,dx = \int (x^7 - x^2)\,dx = \int x^7 dx - \int x^2 dx$

$$= \frac{x^8}{8} - \frac{x^3}{3} + C$$

The integral of the difference of two functions is equal to the difference of their integrals.

i.e. $$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

If we have a function f(x) as a product of a constant (k) and another function $[g(x)]$

i.e. $f(x) = kg(x)$, then we can integrate f(x) as

$$\int f(x)\,dx = \int kg(x)\,dx = k\int g(x)\,dx$$

Integral of product of a constant and a function is product of that constant and integral of the function.

i.e. $$\int kf(x)\,dx = k\int f(x)\,dx$$

Example 30.1 Evaluate :

(ii) $\int 4^x dx$ (ii) $\int (2^x)(3^{-x})\,dx$

Solution :(i) $\int 4^x dx = \frac{4^x}{\log 4} + C$

(ii) $$\int (2^x)(3^{-x})\,dx = \int \frac{2^x}{3^x}\,dx = \int \left(\frac{2}{3}\right)^x dx = \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Remember in (ii) it would not be correct to say that

$$\int 2^x 3^{-x}\,dx = \int 2^x\,dx \int 3^{-x}\,dx$$

Because $$\int 2^x\,dx \int 3^{-x}\,dx = \frac{2^x}{\log 2}\left(\frac{3^{-x}}{\log 3}\right) + C \neq \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Therefore, **integral of a product of two functions is not always equal to the product of the integrals.** We shall deal with the integral of a product in a subsequent lesson.

Example 30.2 Evaluate :

(i) $\displaystyle\int \frac{dx}{\cos^n x}$, when n = 0 and n = 2 (ii) $\displaystyle\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta$

Solution :

(i) When n = 0, $$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^o x}$$

$$= \int \frac{dx}{1} = \int dx$$

Now $\int dx$ can be written as $\int x^o dx$.

$$\therefore \qquad \int dx = \int x^o dx = \frac{x^{o+1}}{0+1} + C = x + C$$

When n = 2,

$$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x\,dx$$

$$= \tan x = C$$

(ii) $$\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{-1}{\sin^2\theta}\,d\theta = -\int \operatorname{cosec}^2\theta\,d\theta$$

$$= \cot\theta + C$$

Example 30.3 Evaluate :

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \frac{x^2 + 1}{x^3}\, dx$

(iii) $\int \frac{1 - x}{\sqrt{x}}\, dx$ (iv) $\int \left(\frac{1}{1 + x^2} - \frac{1}{\sqrt{1 - x^2}} \right) dx$

Solution : (i) $\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + \sin x + C$

(ii) $$\int \frac{x^2 + 1}{x^3}\, dx = \int \left(\frac{x^2}{x^3} + \frac{1}{x^3} \right) dx = \int \frac{1}{x}\, dx + \int \frac{1}{x^3}\, dx$$

$$= \log|x| + \frac{x^{-3+1}}{-3+1} + C = \log|x| - \frac{1}{2x^2} + C$$

(iii) $$\int \frac{1 - x}{\sqrt{x}}\, dx = \int \left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx = \int \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right) dx$$

$$= 2\sqrt{x} - \frac{2}{3} x^{3/2} + C$$

(iv) $$\int \left(\frac{1}{1 + x^2} - \frac{1}{\sqrt{1 - x^2}} \right) dx = \int \frac{dx}{1 + x^2} - \int \frac{dx}{\sqrt{1 - x^2}}$$

$$= \tan^{-1} x - \sin^{-1} x + C$$

Example 30.4 Evaluate :

(i) $\int \sqrt{1 - \sin 2\theta}\, d\theta$ (ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2 - 1}} \right) dx$

(iii) $\int (\tan x + \cot x)^2\, dx$ (iv) $\int \left(\frac{x^6 - 1}{x^2 - 1} \right) dx$

Solution : (i) $\sqrt{1 - \sin 2\theta} = \sqrt{\cos^2 \theta + \sin^2 \theta - 2 \sin\theta \cos\theta}$

$$\left[\because \sin^2 \theta + \cos^2 \theta = 1 \right]$$

$$= \sqrt{(\cos\theta - \sin\theta)^2} = \pm(\cos\theta - \sin\theta)$$

(sign is selected depending upon the value of θ)

(a) If $\sqrt{1 - \sin 2\theta} = \cos\theta - \sin\theta$

then $\int \sqrt{1 - \sin 2\theta}\, d\theta = \int (\cos\theta - \sin\theta)\, d\theta$

$$= \int \cos\theta \, d\theta - \int \sin\theta \, d\theta = \sin\theta + \cos\theta + C$$

(b) If $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (-\cos\theta + \sin\theta)\, d\theta = -\int \cos\theta\, d\theta + \int \sin\theta\, d\theta$

$$= -\sin\theta - \cos\theta + C$$

(ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}}\right) dx = \int 4e^x dx - \int \frac{3}{x\sqrt{x^2-1}}\, dx$

$$= 4\int e^x dx - 3\int \frac{dx}{x\sqrt{x^2-1}} = 4e^x - 3\sec^{-1} x + C$$

(iii) $\int (\tan x + \cot x)^2\, dx = \int \left(\tan^2 x + \cot^2 x + 2\tan x \cot x\right) dx$

$$= \int \left(\tan^2 x + \cot^2 x + 2\right) dx$$

$$= \int \left(\tan^2 x + 1 + \cot^2 x + 1\right) dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

(iv) $\int \left(\frac{x^6-1}{x^2+1}\right) dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2+1}\right) dx$ (dividing x^6-1 by x^2+1)

$$= \int x^4 dx - \int x^2 dx + \int dx - 2\int \frac{dx}{x^2+1}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$$

Example 30.5 Evaluate :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx$ (ii) $\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}\right) dx$

Solution :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx = \int \left(x^{3/2} + 3x\frac{1}{\sqrt{x}} + 3\sqrt{x}\frac{1}{x} + \frac{1}{x^{3/2}}\right) dx$

$$= \int x^{3/2} dx + 3\int \sqrt{x}\, dx + 3\int \frac{1}{\sqrt{x}}\, dx + \int \frac{dx}{x^{3/2}}$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + 3\frac{x^{1/2}}{\frac{1}{2}} - \frac{2}{\sqrt{x}} + C$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + C$$

(ii) $$\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}} \right) dx = \int \frac{4e^{5x}}{e^{3x}} dx - \int \frac{9e^{4x}}{e^{3x}} dx - \int \frac{3dx}{e^{3x}}$$

$$= 4\int e^{2x}\, dx - 9\int e^{x}\, dx - 3\int e^{-3x}\, dx$$

$$= 2e^{2x} - 9e^{x} + e^{-3x} + C$$

CHECK YOUR PROGRESS 30.2

1. Evaluate :

(a) $\int \left(x + \frac{1}{2} \right) dx$ (b) $\int \frac{-x^2}{1+x^2} dx$

(c) $\int \left(10x^9 - \sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$ (d) $\int \left(\frac{5 + 3x - 6x^2 - 7x^4 - 8x^6}{x^6} \right) dx$

(e) $\int \frac{x^4}{1+x^2} dx$ (f) $\int \left(\sqrt{x} + \frac{2}{\sqrt{x}} \right)^2 dx$

2. Evaluate :

(a) $\int \frac{dx}{1 + \cos 2x}$ (b) $\int \tan^2 x dx$ (c) $\int \frac{2 \cos x}{\sin^2 x} dx$

(d) $\int \frac{dx}{1 - \cos 2x}$ (e) $\int \frac{\sin x}{\cos^2 x} dx$ (f) $\int (\operatorname{cosec} x - \cot x) \operatorname{cosec} x \, dx$

3. Evaluate :

(a) $\int \sqrt{1 + \cos 2x}\, dx$ (b) $\int \sqrt{1 - \cos 2x}\, dx$ (c) $\int \frac{1}{1 - \cos 2x} dx$

4. Evaluate :

(a) $\int \sqrt{x + 2}\, dx$

30.5 TECHNIQUES OF INTEGRATION

11.5.1 Integration By Substitution

This method consists of expressing $\int f(x)\, dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson. First, we will consider the functions of the type $f(ax + b)$, $a \neq 0$ where f (x) is a standard function.

Example 30.6 Evaluate :

(i) $\int \sin(ax+b)\,dx$

Solution : (i) $\int \sin(ax+b)\,dx$

Put $\quad ax+b=t.$

Then $\quad a=\dfrac{dt}{dx} \quad$ or $\quad dx=\dfrac{dt}{a}$

$$\therefore \quad \int \sin(ax+b)\,dx = \int \sin t\,\frac{dt}{a} \text{ (Here the integration factor will be replaced by dt.)}$$

$$=\frac{1}{a}\int \sin t\,dt = \frac{1}{a}(-\cos t)+C = -\frac{\cos(ax+b)}{a}+C$$

Example 30.7 Evaluate :

(i) $\int (ax+b)^n\,dx$, where $n \neq -1$ $\qquad$ (ii) $\int \dfrac{1}{(ax+b)}\,dx$

Solution : (i) $\int (ax+b)^n\,dx$, where $n \neq -1$

Put $\quad ax+b=t \quad \Rightarrow \quad a=\dfrac{dt}{dx}$ or $dx=\dfrac{dt}{a}$

$$\therefore \quad \int (ax+b)^n\,dx = \frac{1}{a}\int t^n dt = \frac{1}{a}\cdot\frac{t^{n+1}}{(n+1)}+C$$

$$=\frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+C \qquad \text{where } n \neq -1$$

(ii) $\int \dfrac{1}{(ax+b)}\,dx$

Put $\quad ax+b=t \quad \Rightarrow \quad dx=\dfrac{1}{a}dt$

$$\therefore \quad \int \frac{1}{(ax+b)}\,dx = \int \frac{1}{a}\cdot\frac{dt}{t} = \frac{1}{a}\log|t|+C$$

$$=\frac{1}{a}\log|ax+b|+C$$

Example 30.8 Evaluate :

(i) $\int e^{5x+7} dx$

Solution : (i) $\int e^{5x+7} dx$

Put $5x + 7 = t \quad \Rightarrow \quad dx = \frac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx = \frac{1}{5}\int e^{t} dt = \frac{1}{5} e^{t} + C$$

$$= \frac{1}{5} e^{5x+7} + C$$

Likewise $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

Similarly, using the substitution $ax + b = t$, the integrals of the following functions will be :

$$\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a}\log|ax+b| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$$

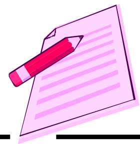
$$\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

Example 30.9 Evaluate :

(i) $\int \sin^2 x \, dx$ (ii) $\int \sin^3 x \, dx$ (iii) $\int \cos^3 x \, dx$ (iv) $\int \sin 3x \sin 2x \, dx$

Solution : We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x



(i) $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$= \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos 2x \, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(ii) $\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} dx \qquad \left[\because \sin 3x = 3\sin x - 4\sin^3 x\right]$

$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$

(iii) $\int \cos^3 x \, dx = \int \frac{\cos 3x + 3\cos x}{4} dx \qquad \left[\because \cos 3x = 4\cos^3 x - 3\cos x\right]$

$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + C$

(iv) $\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int 2\sin 3x \sin 2x \, dx$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx = \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$

CHECK YOUR PROGRESS 30.3

1. Evaluate :

(a) $\int \sin(4-5x)\, dx$ (b) $\int \sec^2(2+3x)\, dx$

(c) $\int \sec\left(x+\frac{\pi}{4}\right) dx$ (d) $\int \cos(4x+5)\, dx$

(e) $\int \sec(3x+5)\tan(3x+5)\, dx$

(f) $\int \operatorname{cosec}(2+5x)\cot(2+5x)\, dx$

2. Evaluate :

(a) $\int \frac{dx}{(3-4x)^4}$ (b) $\int (x+1)^4 \, dx$ (c) $\int (4-7x)^{10} \, dx$

(d) $\int (4x-5)^3 \, dx$ (e) $\int \frac{1}{3x-5} dx$ (f) $\int \frac{1}{\sqrt{5-9x}} dx$

(g) $\int (2x+1)^2 \, dx$ (h) $\int \frac{1}{x+1} dx$

3. Evaluate :

(a) $\int e^{2x+1} dx$ (b) $\int e^{3-8x} dx$ (c) $\int \frac{1}{e^{(7+4x)}} dx$

4. Evaluate :

(a) $\int \cos^2 x \, dx$ (b) $\int \sin^3 x \cos^3 x \, dx$

(c) $\int \sin 4x \cos 3x \, dx$ (d) $\int \cos 4x \cos 2x \, dx$

30.5.2 Integration of Function of The Type $\frac{f'(x)}{f(x)}$

To evaluate $\int \frac{f'(x)}{f(x)} dx$, we put f (x) = t. Then f ' (x) dx = dt.

$$\therefore \quad \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log|t| + C = \log|f(x)| + C$$

Integral of a function, whose numerator is derivative of the denominator, is equal to the logarithm of the denominator.

Example 30.10 Evaluate :

(i) $\int \frac{2x}{x^2+1} dx$ (ii) $\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)}$

Solution :

(i) Now 2x is the derivative of x^2+1.

$\therefore$ By applying the above result, we have

$$\int \frac{2x}{x^2+1} dx = \log\left|x^2+1\right| + C$$

(ii) $\frac{1}{2\sqrt{x}}$ is the derivative of $3+\sqrt{x}$

$$\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)} = \log\left|3+\sqrt{x}\right| + C$$

Example 30.11 Evaluate :

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ (ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Solution :

(i) $e^x + e^{-x}$ is the derivative of $e^x - e^{-x}$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \log\left|e^x - e^{-x}\right| + C$$

Alternatively,

For $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$,

Put $e^x - e^{-x} = t.$

Then $\left(e^x + e^{-x}\right) dx = dt$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{dt}{t} = \log|t| + C = \log\left|e^x - e^{-x}\right| + C$$

(ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Here $e^{2x} - 1$ is not the derivative of $e^{2x} + 1$. But if we multiply the numerator and denominator by e^{-x}, the given function will reduce to

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log\left|e^x + e^{-x}\right| + C$$

$$\therefore \qquad \int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \log\left|e^x + e^{-x}\right| + C$$

$$\left[\because \left(e^x - e^{-x}\right) \text{ is the derivative of } \left(e^x + e^{-x}\right)\right]$$

CHECK YOUR PROGRESS 30.4

1. Evaluate :

(a) $\int \frac{x}{3x^2 - 2} dx$ (b) $\int \frac{2x + 1}{x^2 + x + 1} dx$ (c) $\int \frac{2x + 9}{x^2 + 9x + 30} dx$

(d) $\int \frac{x^2+1}{x^3+3x+3}dx$ (e) $\int \frac{2x+1}{x^2+x-5}dx$ (f) $\int \frac{dx}{\sqrt{x}\left(5+\sqrt{x}\right)}$

(g) $\int \frac{dx}{x\left(8+\log x\right)}$

2. Evaluate :

(a) $\int \frac{e^x}{2+be^x}dx$ (b) $\int \frac{dx}{e^x-e^{-x}}$

30.5.3 INTEGRATION BY SUBSTITUTION

(i) $\int \tan x \, dx$ (ii) $\int \sec x \, dx$

Solution :

(i) $\int \tan x \, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx$

$$= -\log|\cos x| + C \quad (\because -\sin x \text{ is derivative of } \cos x)$$

$$= \log\left|\frac{1}{\cos x}\right| + C \quad \text{or} \quad = \log|\sec x| + C$$

$\therefore$ $\int \tan x \, dx = \log|\sec x| + C$

Alternatively,

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{-\sin x \, dx}{\cos x}$$

Put $\cos x = t.$

Then $-\sin x \, dx = dt$

$\therefore$ $\int \tan x \, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$

$$= \log\left|\frac{1}{\cos x}\right| + C = \log|\sec x| + C$$

(ii) $\int \sec x \, dx$

sec x can not be integrated as such because sec x by itself is not derivative of any function. But this is not the case with $\sec^2 x$ and sec x tan x. Now $\int \sec x \, dx$ can be written as

$$\int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}dx$$

$$= \int \frac{\left(\sec^2 x + \sec x \tan x\right)}{\sec x + \tan x} dx$$

Put $\sec x + \tan x = t.$

Then $\left(\sec x \tan x + \sec^2 x\right) dx = dt$

$$\therefore \quad \int \sec x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$$

Example 30.13 Evaluate $\int \frac{1}{a^2 - x^2} dx$

Solution : Put $x = a \sin\theta \quad \Rightarrow \quad dx = a\cos\theta \, d\theta$

$$\therefore \quad \int \frac{1}{a^2 - x^2} dx = \int \frac{a\cos\theta}{a^2 - a^2\sin^2\theta} d\theta$$

$$= \frac{1}{a}\int \frac{\cos\theta}{1 - \sin^2\theta} d\theta = \frac{1}{a}\int \frac{1}{\cos\theta} d\theta = \frac{1}{a}\int \sec\theta \, d\theta$$

$$= \frac{1}{a}\log|\sec\theta + \tan\theta| + C = \frac{1}{a}\log\left|\frac{1 + \sin\theta}{\cos\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1 + \frac{x}{a}}{\sqrt{1 - \frac{x^2}{a^2}}}\right| + C = \frac{1}{a}\log\left|\frac{a + x}{\sqrt{a^2 - x^2}}\right| + C = \frac{1}{a}\log\left|\frac{\sqrt{a + x}}{\sqrt{a - x}}\right| + C$$

$$= \frac{1}{a}\log\left|\left(\frac{a + x}{a - x}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + C$$

Example 30.14 Evaluate : $\int \frac{1}{x^2 - a^2} dx$

Solution : Put $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta \tan\theta \, d\theta$

$$\therefore \quad \int \frac{1}{x^2 - a^2} dx = \int \frac{a\sec\theta \tan\theta \, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \frac{1}{a}\int \frac{\sec\theta \tan\theta}{\tan^2\theta} d\theta \qquad \left(\tan^2\theta = \sec^2\theta - 1\right)$$

$$= \frac{1}{a} \int \frac{\sec\theta}{\tan\theta} d\theta = \frac{1}{a} \int \frac{1}{\sin\theta} d\theta = \frac{1}{a} \int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{a} \log \left| \operatorname{cosec}\theta - \cot\theta \right| + C = \frac{1}{a} \log \left| \frac{1 - \cos\theta}{\sin\theta} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{1 - \frac{a}{x}}{\sqrt{1 - \frac{a^2}{x^2}}} \right| + C = \frac{1}{a} \log \left| \frac{x - a}{\sqrt{x^2 - a^2}} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{\sqrt{x - a}}{\sqrt{x + a}} \right| + C = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C$$

Example 30.15 $\int \frac{1}{a^2 + x^2} dx$

Solution : Put $x = a \tan\theta \Rightarrow dx = a \sec^2\theta \, d\theta$

$$\therefore \qquad \int \frac{1}{a^2 + x^2} dx = \int \frac{a \sec^2\theta}{a^2 \left(1 + \tan^2\theta\right)} d\theta$$

$$= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + C \qquad \left(\frac{x}{a} = \tan\theta \Rightarrow \tan^{-1}\frac{x}{a} = \theta \right)$$

$$= \frac{1}{a} \tan^{-1}\frac{x}{a} + C$$

Example 30.16 $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Put $\quad x = a \sin\theta \quad \Rightarrow \quad dx = a \cos\theta \, d\theta$

$$\therefore \qquad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a \cos\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} d\theta$$

$$= \int \frac{a \cos\theta}{a \cos\theta} d\theta = \int d\theta = \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Example 30.17 $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

Solution : Let $\quad x = a \sec\theta \quad \Rightarrow \quad dx = a \sec\theta \tan\theta \, d\theta$

$$\therefore \qquad \int \frac{1}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta}{a\sqrt{\sec^2\theta - 1}}\, d\theta$$

$$= \int \sec\theta d\theta = \log|\sec\theta + \tan\theta| + C$$

$$= \log\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right| + C$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Example 30.18 $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$

Solution : $\quad$ Put $x = a \tan\theta \quad \Rightarrow \quad dx = a \sec^2\theta \, d\theta$

$$= \int \sec\theta \, d\theta$$

$$= \log|\sec\theta + \tan\theta| + C = \log\left|\frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{a^2 + x^2} + x\right| + C$$

Example 30.19 $\int \frac{x^2 + 1}{x^4 + 1}\, dx$

Solution : Since x^2 is not the derivative of $x^4 + 1$, therefore, we write the given integral as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$$

Let $\quad x - \frac{1}{x} = t$. Then $\quad \therefore \quad \left(1 + \frac{1}{x^2}\right) dx = dt$

Also $\quad x^2 - 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 + 2$

$$\therefore \quad \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2+2} = \int \frac{dt}{(t)^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C \quad = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

Example 30.20 $\int \frac{x^2-1}{x^4+1}\,dx$

Solution : $\int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

Put $x+\frac{1}{x} = t$. Then $\left(1-\frac{1}{x^2}\right)dx = dt$

Also $x^2+2+\frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2} = t^2-2$

$$\therefore \quad \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2-2} = \int \frac{dt}{(t)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

Example 30.21 $\int \frac{x^2}{x^4+1}\,dx$

Solution : In order to solve it, we will reduce the given integral to the integrals given in Examples 11.19 and 11.20.

i.e., $\int \frac{x^2}{x^4+1}\,dx = \frac{1}{2}\int\left[\frac{x^2+1}{x^4+1}+\frac{x^2-1}{x^4+1}\right]dx$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.22 $\int \frac{1}{x^4+1}dx$

Solution : We can reduce the given integral to the following form

$$\frac{1}{2}\int \frac{\left(x^2+1\right)-\left(x^2-1\right)}{x^4+1}dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.23 (a) $\int \frac{1}{x^2-x+1}dx$ (b) $\int \frac{x^2-1}{x^4+x^2+1}dx$

Solution : (a) $\int \frac{1}{x^2-x+1}dx = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}dx$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

(b) $$\int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

Put $\quad x+\frac{1}{x}=t. \quad \Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt$

Also $\quad x^2+2+\frac{1}{x^2}=t^2 \quad \Rightarrow \quad x^2+1+\frac{1}{x^2}=t^2-1$

$$\therefore \quad \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$$

Example 30.24 $\int \sqrt{\tan x}\,dx$

Solution : Let $\quad \tan x = t^2 \quad \Rightarrow \quad \sec^2 x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t}{\sec^2 x}\,dt = \frac{2t}{1+t^4}\,dt$$

$$\therefore \quad \int \sqrt{\tan x}\,dx = \int t\left(\frac{2t}{1+t^4}\right)dt = \int \frac{2t^2}{1+t^4}\,dt$$

$$= \int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}\right)dt = \int \frac{t^2+1}{t^4+1}\,dt + \int \frac{t^2-1}{t^4+1}\,dt$$

Example 30.25 $\int \sqrt{\cot x}\,dx$

Solution : Let $\cot x = t^2 \quad \Rightarrow \quad -\operatorname{cosec}^2 x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{-2t}{\operatorname{cosec}^2 x}\,dt = -\frac{2t}{t^4+1}\,dt$$

$$\therefore \quad \int \sqrt{\cot x}\,dx = -\int t\left(\frac{2t}{t^4+1}\right)dt$$

$$= -\int \frac{2t^2}{t^4+1}\,dt \;=\; -\int\left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}\right)dt$$

Proceed according to Examples 11.19 and 11.20 solved before.

Example 30.26 $\int\left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$

Let $\quad \sin x - \cos x = t \;\Rightarrow\; (\cos x + \sin x)\,dx = dt$

Also $\quad 1 - 2\sin x \cos x = t^2 \;\Rightarrow\; \quad 1 - t^2 = 2\sin x\,\cos x$

$$\Rightarrow \quad \frac{1-t^2}{2} = \sin x\,\cos x$$

$$\therefore \quad \int \frac{\sin x - \cos x}{\sqrt{\cos x \sin x}}\,dx = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\,\sin^{-1}[\sin x - \cos x] + C$$

(Using the result of Example 26.25)

Example 30.27 Evaluate :

(a) $\int \frac{dx}{\sqrt{8+3x-x^2}}$ (b) $\int \frac{dx}{x(1-2x)}$

Solution :

(a) $$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-\left(x^2-3x\right)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left[\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{41}}{2}}\right] + C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$$

(b) $$\int \frac{dx}{x(1-2x)} = \int \frac{dx}{\sqrt{x-2x^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{x}{2} - x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{16} - \left[x^2 - \frac{x}{2} + \frac{1}{16}\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} = \frac{1}{\sqrt{2}} \sin^{-1} \left\{ \frac{\left(x - \frac{1}{4}\right)}{\left(\frac{1}{4}\right)} \right\} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(4x - 1) + C$$

CHECK YOUR PROGRESS 30.5

1. Evaluate :

(a) $\int \frac{x^2}{x^2 - 9} dx$ (b) $\int \frac{e^x}{e^{2x} + 1} dx$ (c) $\int \frac{x}{1 + x^4} dx$

(d) $\int \frac{dx}{\sqrt{16 - 9x^2}}$ (e) $\int \frac{dx}{1 + 3\sin^2 x}$ (f) $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

(g) $\int \frac{dx}{3x^2 + 6x + 21}$ (h) $\int \frac{dx}{\sqrt{5 - 4x - x^2}}$ (i) $\int \frac{dx}{x\sqrt{3x^2 - 12}}$

(j) $\int \frac{d\theta}{\sin^4 \theta + \cos^4 \theta}$ (k) $\int \frac{e^x dx}{\sqrt{1 + e^{2x}}}$ (l) $\int \sqrt{\frac{1 + x}{1 - x}} dx$

(m) $\int \frac{dx}{\sqrt{2ax - x^2}}$ (n) $\int \frac{3x^2}{\sqrt{9 - 16x^6}} dx$ (o) $\int \frac{(x + 1)}{\sqrt{x^2 + 1}} dx$

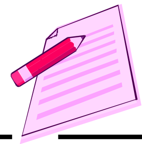
(p) $\int \frac{dx}{\sqrt{9 + 4x^2}}$ (q) $\int \frac{\sin \theta}{\sqrt{4\cos^2 \theta - 1}} d\theta$ (r) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 4}} dx$

(s) $\int \frac{1}{(x + 2)^2 + 1} dx$ (t) $\int \frac{1}{\sqrt{16x^2 + 25}} dx$

30.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$



or $$f\frac{d}{dx}(g)=\frac{d}{dx}(fg)-g\frac{d}{dx}(f) \qquad (1)$$

Also you know that $\int\frac{d}{dx}(fg)\,dx = fg$

Integrating (1). we have

$$\int f\frac{d}{dx}(g)\,dx=\int\frac{d}{dx}(fg)\,dx-\int g\frac{d}{dx}(f)\,dx$$

$$= fg-\int g\frac{d}{dx}(f)\,dx$$

if we take $$f=u(x):\frac{d}{dx}(g)=v(x).$$

(2) become $\int u(x)v(x)\,dx$

$$=u(x)\cdot\int v(x)\,dx-\int\left[\frac{d}{dx}(u(x))\int v(x)\,dx\right]dx$$

=I function $\times$ integral of II function $-\int$ [differential coefficient of function $\times$ integralof II function]dx

A B

Here the important factor is the choice of I and II function in the product of two functions because either can be I or II function. For that the indicator will be part 'B' of the result above.

The first function is to be chosen such that it reduces to a next lower term or to a constant term after subseqent differentiations.

In questions of integration like

$$x\sin x,\, x\cos^2 x,\, x^2e^x$$

(i) algebraic function should be taken as the first function

(ii) If there is no algebraic function then look for a function which simplifies the product in 'B' as above; the choice can be in order of preference like choosin first function

(i) an inverse function (ii) a logarithmic function

(iii) a trigonometric function (iv) an exponential function.

The following examples will give a practice to the concept of choosing first function.

		I function	**II function**
1.	$\int x\cos x\,dx$	x (being algebraic)	cos x

2.	$\int x^2 e^x dx$	x^2 (being algebraic)	e^x
3.	$\int x^2 \log dx$	log x	x^2
4.	$\int \frac{\log x}{\left(1+x^2\right)} dx$	log x	$\frac{1}{(1+x)^2}$
5.	$\int x \sin^{-1} x\, dx$	$\sin^{-1} x$	x
6.	$\int \log x\, dx$	log x	1 (In single function of logarithm and inverse trigonometric we take unity as II function)
7.	$\sin^{-1} x\, dx$	$\sin^{-1} x$	1

Example 30.28 Evaluate :

$$\int x^2 \sin x\, dx$$

Solution: Taking algebraic function x^2 as function and sin x as II function, we herv.

$$\int_{I}^{x^2} \sin x\, dx = x^2 \int \sin x - \int \left[\frac{d}{dx}\left(x^2\right) \int \sin x\, dx \right] dx$$

$$= -x^2 \cos x - 2\int x\left(-\cos x\right) dx$$

$$= -x^2 \cos x + 2\int x \cos x\, dx \qquad (1)$$

again $\int x \cos x\, dx = x \sin x + \cos x + c$ (2)

Substituting (2) in (1), we have

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2[x \sin x + \cos x] + C$$

$$= -x^2 \cos x + 2x \sin x + \cos x + C$$

Example 30.29 Evaluate :

$$\int x^2 \log x\, dx$$

Solution: In order of preference log x is to be taken as I function.

$$\therefore \quad \int \underset{\text{I}}{\log x}\, \underset{\text{II}}{x^2}\, dx = \frac{x^3}{3}\log x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\log x - \frac{1}{-3}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

Example 30.30 Evaluate :

$$\int \sin^{-1} x\, dx$$

Solution: $\int \sin^{-1} x\, dx = \int \sin^{-1} x \cdot 1 \cdot dx$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $\quad 1 - x^2 = t \quad \Rightarrow \quad -2x\,dx = dt \quad \Rightarrow \quad x\,dx = \frac{-1}{2}dt$

$$\therefore \quad \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{1-x^2} + C$$

$$\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

CHECK YOUR PROGRESS 30.6

Evaluate:

1. (a) $\int x\sin x\,dx$ (b) $\int (1+x^2)\cos 2x\,dx$ (c) $\int x\sin 2x\,dx$

2. (a) $\int x\tan^2 x\,dx$ (b) $\int x^2\sin^2 x\,dx$

3. (a) $\int x^3\log 2x\,dx$ (b) $(1-x^2)\log x\,dx$ (c) $\int (\log x)^2\,dx$

4. (a) $\int \frac{\log x}{x^n}\,dx$ (b) $\int \frac{\log(\log x)}{x}\,dx$

5. (a) $\int x^2 e^{3x} dx$ (b) $\int x\, e^{3x} dx$

6. (a) $\int x(\log x)^2\, dx$

7. (a) $\int \sec^{-1} x\, dx$ (b) $\int x \cot^{-1} x\, dx$

30.7 INTEGRAL OF THE FORM

$$\int e^x \left[f(x) + f'(x) \right] dx$$

where f' (x) is the differentiation of f (x). In such type of integration while integrating by parts the solution will be $e^x \left(f(x) \right) + C.$

For example, consider

$$\int e^x \left[\tan x + \log \sec x \right] dx$$

Let $\int(x) = \log \sec x,$ then $f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

So (1) can be rewritten as

$$\int e^x \left[f'(x) + f(x) \right] dx = e^x \left(f(x) \right) + C - e^x \log \sec x + C$$

Alternatively, you can evaluate it as under:

$$\int e^x \left[\tan x + \log \sec x \right] dx = \int e^x \tan x\, dx + \int e^x \log \sec x\, dx$$

I II

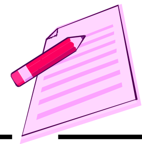
$$= e^x \log \sec x - \int e^x \log \sec x\, dx + \int e^x \log \sec x\, dx$$

$$= e^x \log \sec x + C$$

Example 30.31 Evaluate the following:

(a) $\int e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2} \right) dx$ (b) $\int e^x \left(\dfrac{1 + x \log x}{x} \right) dx$

(c) $\int \dfrac{x e^x}{(x+1)^2} dx$ (d) $\int e^x \left[\dfrac{1 + \sin x}{1 + \cos x} \right] dx$



Solution:

(a) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\int e^x\left[\frac{1}{x}+\frac{d}{dx}\left(\frac{1}{x}\right)\right]dx=e^x\left(\frac{1}{x}\right)$

(b) $\int e^x\left(\frac{1+x\log x}{x}\right)dx=\int e^x\left(\frac{1}{x}+\log x\right)dx$

$$=\int e^x\left(\log x+\frac{d}{dx}(\log x)\right)dx \quad = e^x\log x + C$$

(c) $\int \frac{x e^x}{(x+1)^2}dx=\int \frac{x+1-1}{(x+1)^2}e^x\,dx \;=\int e^x\left(\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx$

$$=\int e^x\left(\frac{1}{x+1}-\frac{d}{dx}\left(\frac{1}{(x+1)}\right)\right)dx$$

$$=e^x\left(\frac{1}{x+1}\right)+C$$

(d) $=\int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx=\int e^x\left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$

$$=\int e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]dx$$

$$=\int e^x\left[\tan\frac{x}{2}+\frac{d}{x}\left(\tan\frac{x}{2}\right)\right]dx$$

$$=e^x\tan\frac{x}{2}+C$$

Example 30.32 Evaluate the following:

(a) $\int \sec^3 x\,dx$ (b) $\int e^x \sin x\,dx$

Solution:

(a) $\int \sec^3 x \, dx$

Let $I = \int \sec x \cdot \sec^2 x \, dx$

$= \sec x \cdot \tan x - \int \sec x \tan x \cdot \tan x \, dx$

$\therefore$ $I = \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx$ $\left(\because \tan^2 x = \sec^2 x - 1\right)$

or $I = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

or $2I = \sec x \tan x + \int \sec x \, dx$

or $I = \sec x \tan x + \log\left|\sec x + \tan x\right| + C_1$

or $I = \dfrac{1}{2}\left[\sec x \tan x + \log\left|\sec x + \tan x\right|\right] + C$

(b) $\int e^x \sin x \, dx$

Let $I = \int e^x \sin x \, dx$

$= e^x\left(-\cos x\right) - \int e^x\left(-\cos x\right) dx \; = -e^x \cos x + \int e^x \cos x \, dx$

$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)$

$\therefore$ $I = -e^x \cos x + e^x \sin x - 1$

or

$2I = -e^x \cos x + e^x \sin x$

or $I = \dfrac{e^x}{2}\left(\sin x - \cos x\right) + C$

Example 30.33 Evaluate:

$$\int \sqrt{a^2 - x^2} \, dx$$

Solution:

Let $I = \int \sqrt{a^2 - x^2} \, dx = \int \sqrt{a^2 - x^2} \cdot 1 \, dx$

Integrating by parts only and taking 1 as the second function, we have

$$I = \left(\sqrt{a^2 - x^2}\right)x - \int \frac{1}{2\sqrt{a^2 - x^2}}(-2x)\cdot x\,dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}dx \;= x\sqrt{a^2 - x^2} + \int \frac{a^2 - \left(a^2 - x^2\right)}{\sqrt{a^2 - x^2}}dx$$

$$= x\sqrt{a^2 - x^2} + a^2\int \frac{1}{\sqrt{a^2 - x^2}}dx - \int \sqrt{a^2 - x^2}\,dx$$

$\therefore$ $$I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) - 1$$

or $$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

or $$I = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$$

Similarly, $$\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$\therefore$ $$\int \sqrt{a^2 + x^2}\,dx = \frac{x\sqrt{a^2 + x^2}}{2} - \frac{a^2}{2}\log\left|x + x\sqrt{a^2 + x^2}\right| + C$$

Example 30.34 Evaluate:

(a) $\int \sqrt{16x^2 + 25}dx$ (b) $\int \sqrt{16 - x^2}\,dx$ (c) $\int \sqrt{1 + x - 2x^2}\,dx$

Solution:

(a) $$\int \sqrt{16x^2 + 25}dx = 4\int \sqrt{x^2 + \frac{25}{16}}dx = 4\int \sqrt{x^2 + \left(\frac{5}{4}\right)^2}\,dx$$

Using the formula for $\int \sqrt{\left(x^2 + a^2\right)}dx$ we get,

$$\int \sqrt{16x^2 + 25}dx = \left[\frac{x}{2}\sqrt{x^2 + \frac{25}{16}} + \frac{25}{32}\log\left|x + \sqrt{x^2 + \frac{25}{16}}\right|\right] + C$$

$$= \frac{x}{8}\sqrt{16x^2 + 25} + \frac{25}{8}\log\left|4x + \sqrt{16x^2 + 25}\right| + C$$

(b) Using the formula for $\int \sqrt{\left(a^2 - x^2\right)}\, dx$ we get,

$$\int \sqrt{16 - x^2}\, dx = \int \sqrt{(4)^2 - x^2}\, dx = \frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4} + C$$

(c)
$$\int \sqrt{1 + x - 2x^2}\, dx = \sqrt{2}\int \sqrt{\frac{1}{2} + \frac{x}{2} - x^2}\, dx$$

$$= \sqrt{2}\int \sqrt{\left[\frac{1}{2} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right) + \frac{1}{16}\right]}\, dx$$

$$= \sqrt{2}\int \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{x - \frac{1}{4}}{2}\sqrt{\frac{9}{16} - \left(x - \frac{1}{4}\right)^2} + \frac{9}{16 \times 2}\sin^{-1}\frac{x - \frac{1}{4}}{\frac{3}{4}}\right] + C$$

$$= \sqrt{2}\left[\frac{4x - 1}{8} \cdot \frac{1}{\sqrt{2}}\sqrt{1 + x - 2x^2} + \frac{9}{32}\sin^{-1}\frac{4x - 1}{3}\right] + C$$

$$= \frac{4x - 1}{8}\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\frac{4x - 1}{3} + C$$

CHECK YOUR PROGRESS 30.7

Evaluate:

1. (a) $\int e^x \sec x\left[1 + \tan x\right] dx$ (b) $\int e^x \left[\sec x + \log\left|\sec x + \tan x\right|\right] \mathrm{dx}$

2. (a) $\int \frac{x - 1}{x^2} e^x\, dx$ (b) $\int e^x \left(\sin^{-1} x - \frac{1}{\sqrt{1 - x^2}}\right) dx$

3. $\int e^x \frac{(x - 1)}{(x + 1)^3}\, dx$

4. $\int \frac{x e^x}{(x + 1)^2}\, dx$

5. $\int \frac{x + \sin x}{1 + \cos x}\, dx$

6. $\int e^x \sin 2x\, dx$

30.8 INTEGRATION BY USING PARTIAL FRACTIONS

By now we are equipped with the various techniques of integration.

But there still may be a case like $\frac{4x+5}{x^2+x-6}$, where the substitution or the integration by parts may not be of much help. In this case, we take the help of another technique called **techmique of integrayion using partial functions.**

Any proper rational fraction $\frac{p(x)}{q(x)}$ can be expressed as the sum of rational functions, each having a single factor of q(x). Each such fraction is known as **partial fraction** and the process of obtaining them is called decomposition or resolving of the given fraction into partial fractions.

For example, $$\frac{3}{x+2}+\frac{5}{x-1}=\frac{8x+7}{(x+2)(x-1)}=\frac{8x+7}{x^2+x-2}$$

Here $\frac{3}{x+2},\frac{5}{x-1}$ are called partial fractions of $\frac{8x+7}{x^2+x-2}$.

If $\frac{f(x)}{g(x)}$ is a proper fraction and g (x)can be resolved into real factors then,

(a) corresponding to each non repeated linear factor ax + b, there is a partial fraction of the form

(b) for $(ax+b)^2$ we take the sum of two partial fractions $\cdot$ as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

For $(ax+b)^3$ we take the sum of three partial fractions as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

and so on.

(c) For non-fractorisable quadratic polynomial ax^2+bx+c there is a partial fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

Therefore, if g (x) is a proper fraction $\frac{f(x)}{g(x)}$ and can be resolved into real factors, $\frac{f(x)}{g(x)}$ can be written in the follwoing form:

Factor in the denominator	corresponding partial fraction
ax+b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C,D are arbitary constants.

The rational functions which we shall consider for integration will be those whose denominators can be fracted into linear and quadratic factors.

Example 30.35 Evaluate:

$$\int \frac{2x+5}{x^2-x-2}dx$$

Solution: $$\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$$

Let $$\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$

Multiplyping both sides by $(x-2)(x+1)$,we have

$$2x+5=A(x+1)+B(x-2)$$

Putting x = 2, weget 9 =3A or A = 3

Putting x = -1, we get 3 = -3B or B = -1

substituting these values in(1),we have

$$\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{1}{x+1}$$

$$\Rightarrow \quad \int \frac{2x+5}{x^2-x-2}dx = \int \frac{3}{x-2}dx - \int \frac{1}{x+1}dx$$

$$= 3\log|x-2| - \log|x+1| + C$$

Example 30.36 Evaluate:

$$\int \frac{x^3+x+1}{x^2-1}dx$$

Solution: $I = \int \frac{x^3+x+1}{x^2-1}dx$

Now $\quad \frac{x^2+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} = x + \frac{2x+1}{(x+1)(x-1)}$

$$\therefore \quad I = \int \left(x + \frac{2x+1}{(x+1)(x-1)} \right) dx$$

Let $\quad \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \qquad (2)$

$$\Rightarrow \quad 2x+1 = A(x-1) + B(x+1)$$

Putting $\quad$ x = 1, we get $B = \frac{3}{2}$

Putting $\quad$ x = -1, we get $A = \frac{1}{2}$

Substituting the values of A and B in (2) and integrating, we have

$$\int \frac{2x+1}{(x^2-1)}dx = \frac{1}{2}\int \frac{1}{(x^2+1)}dx + \frac{3}{2}\int \frac{1}{x-1}dx$$

$$= \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| \qquad (3)$$

$\therefore$ From (1) and (3), we have

$$I = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Example 30.37 Evaluate:

$$\int \frac{8}{(x-2)(x^2+4)}dx$$

Solution:

(a) $$\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

(As x^2+4 is not factorisable into linear factors)

Multiplying both sides by $(x+2)(x^2+4)$, we have

$$8 = A(x^2+4) + (Bx+C)(x+2)$$

On comparing the corresponding coeffcients of powers of x on both sides, we get

$$\left.\begin{aligned} 0 &= A+B \\ 0 &= 2B+C \\ 8 &= 4A+2C \end{aligned}\right\} \Rightarrow A=1, B=-1, C=2$$

$$\therefore \quad \int \frac{8}{(x+2)(x^2+4)}dx = \int \frac{1}{x+2}dx - \int \frac{x-2}{x^2-4}dx$$

$$= \int \frac{1}{x+2}dx = \frac{1}{2}\int \frac{2x}{x^2+4}dx + 2\int \frac{dx}{x^4+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + C$$

Example 30.38 Evaluate:

$$\int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}d\theta$$

Solution:

Let $$\text{I} = \int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}d\theta = \int \frac{(4\sin\theta - 1)\cos\theta\, d\theta}{3+\sin^2\theta - 4\sin\theta}$$

Let $\sin\theta = t$, then $\cos\theta\, d\theta = dt$

$$\therefore \qquad I = \int \frac{4t-1}{3+t^2-4t}\,dt$$

Let $\dfrac{4t-1}{3-t^2-4t} = \dfrac{A}{t-3} + \dfrac{B}{t-1}$ Thus $4t-1 = A(t-1) + b(t-3)$

Put $t = 1$ then $B = -\dfrac{3}{2}$ Put $t = 3$ then $A = \dfrac{11}{2}$

$$\therefore \qquad I = \frac{11}{2}\int\left(\frac{1}{t-3}\right)dt - \frac{3}{2}\int\frac{dt}{t-1} = \frac{11}{2}\log|t-3| - \frac{3}{2}\log|t-1| + C$$

$$= \frac{11}{2}\log|\sin\theta - 3| - \frac{3}{2}\log|\sin\theta - 1| + C$$

$$= -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{6}\int\frac{(2t-1)\,dt}{t^2-t+1} + \frac{1}{2}\int\frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log\left|t^2-t+1\right| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log\left|t^2-t+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log\left|\tan^2\theta - \tan\theta + 1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$$

CHECK YOUR PROGRESS 30.8

Evaluate the following:

1. (a) $\int\sqrt{4x^2-5}\,dx$ (b) $\int\sqrt{x^2+3x}\,dx$ (c) $\sqrt{3-2x-2x^2}\,dx$

2. (a) $\int\dfrac{x+1}{(x-2)(x-3)}\,dx$ (b) $\int\dfrac{x}{x^2-16}\,dx$

3. (a) $\int\dfrac{x^2}{x^2-4}\,dx$ (b) $\int\dfrac{2x^2+x+1}{(x-1)^2(x+2)}\,dx$

4. $\int \frac{x^2+x+1}{(x-1)^3}\,dx$

5. (a) $\int \frac{\sin x}{\sin 4x}\,dx$ (b) $\int \frac{1-\cos x}{\cos x(1+\cos x)}\,dx$

LET US SUM UP

- Integration is the inverse of differentiation
- Standard form of some inddefinite integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1}+C\,(n\neq -1)$

(b) $\int \frac{1}{x}dx = \log|x|+C$

(c) $\int \sin x\, ax = -\cos x + C$

(d) $\int \cos x\; dx = \sin x + C$

(e) $\int \sec^2 x\, dx = \tan x + C$

(f) $\int \cos ec^2 x\, dx = -\cot x + C$

(g) $\int \sec x \tan x dx = \sec x + C$

(h) $\int \cos ecx \cot x\, dx = -\cos ecx + C$

(i) $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$

(j) $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$

(k) $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$

(l) $\int e^x dx = e^x + C$

(m) $\int a^x\; dx = \frac{a^x}{\log}+C\,(a>0\, and\, a\neq 1)$

- Properties of indefinite integrals

(a) $\int [f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(b) $\int kf(x)\,dx \quad = k\int f(x)\,dx$

(i) $\int (ax+b)^n dx \quad = \dfrac{1}{a}\dfrac{(ax+b)^{n+1}}{n+1} + C \,(n \neq -1)$

(ii) $\int \dfrac{1}{ax+b}dx \quad = \dfrac{1}{a}\log|ax+b| + C$

(iii) $\int \sin(ax+b)\,dx \quad = \dfrac{-1}{a}\cos(ax+b) + C$

(iv) $\int \cos(ax+b)\,dx \quad = \dfrac{1}{a}\sin(ax+b) + C$

(v) $\int \sec^2(ax+b)\,dx \quad = \dfrac{1}{a}\tan(ax+b) + C$

(vi) $\int \text{cosec}^2(ax+b)\,dx \quad = \dfrac{1}{a}\cot(ax+b) + C$

(vii) $\int \sec(ax+b)\tan(ax+b)\,dx \quad = \dfrac{1}{a}\sec(ax+b) + C$

(viii) $\int \text{cosec}(ax+b)\cot(ax+b)\,dx = \dfrac{1}{a}\text{cosec}(ax+b) + C$

(ix) $\int e^{ax+b}dx = \dfrac{1}{a}e^{ax+b} + C$

(i) $\int \tan x\,dx \quad = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\,dx \quad = \log|\sin x| + C$

(iii) $\int \sec x\,dx \quad = \log|\sec x + \tan x| + C$

(iv) $\int \text{cosec}\,x\,dx \quad = \log|\text{cosec}\,x - \cot x| + C$

- (i) $\int \dfrac{1}{a^2-x^2}dx = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$

(ii) $\int \dfrac{1}{a^2-x^2}dx = \dfrac{1}{2a}\log\left|\dfrac{a-x}{a+x}\right| + C$

(iii) $\int \dfrac{1}{a^2+x^2}dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

(iv) $\int \dfrac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\dfrac{x}{a} + C$

(v) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right|+C$

(vi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$

- Integral of the product of two functions

I function $\times$ Integral of II function $-\int$[Derivative of I function $\times$ Integral of II function]dx

- $$\int e^x\left[f(x)+f(x)\right]dx = e^x f(x)+C$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+C$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+C$$

- Rational fractions are of following two types:

(i) Proper, where degree of variable of numerator < denominator.

(ii) Improper, where degree of variable of numerator $\geq$ denominator.

- If g (x) is a proper fraction $\frac{f(x)}{g(x)}$ can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factors in denominator	**Corresponding partial fraction**
a x + b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C, D are arbitary constants.

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/index.shtml

http://en.wiktionary.org/wiki/integration

http://www.sosmath.com/calculus/integration/byparts/byparts....

TERMINAL EXERCISE

Integrate the following functions w.r.t.x:

1. $\dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$ 2. $\sqrt{1+\sin 2x}$ 3. $\dfrac{\cos 2x}{\cos^2 x\sin^2 x}$

4. $(\tan x-\cot x)^2$ 5. $\dfrac{4}{1+x^2}-\dfrac{1}{\sqrt{1-x^2}}$ 6. $\dfrac{2\sin^2 x}{1+\cos 2x}$

7. $\dfrac{2\cos^2 x}{1-\cos 2x}$ 8. $\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)^2$ 9. $\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)^2$

10. $\cos(7x-\pi)$ 11. $\sin(3x+4)$ 12. $\cos^2(2x+b)$

13. $\displaystyle\int\frac{dx}{\sin x-\cos x}$ 14. $\displaystyle\int\frac{1}{(1+x^2)\tan^{-1}x}dx$ 15. $\displaystyle\int\frac{\cos ec}{\log\left(\tan\frac{x}{2}\right)}dx$

16. $\displaystyle\int\frac{\cot}{3+4\log\sin x}dx$ 17. $\displaystyle\int\frac{dx}{\sin 2x\log\tan x}$ 18. $\displaystyle\int\frac{e^x+1}{e^x-1}dx$

19. $\displaystyle\int\sec^4 x\tan x\,dx$ 20. $\displaystyle\int e^2\sin e^x dx$ 21. $\displaystyle\int\frac{x\,dx}{\sqrt{2x^2+3}}$

22. $\displaystyle\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$ 23. $\displaystyle\int\sqrt{25-9x^2}\,dx$ 24. $\displaystyle\int\sqrt{2ax-x^2dx}$

25. $\displaystyle\int\sqrt{3x^2+4}dx$ 26. $\displaystyle\int\sqrt{1+9x^2}\,dx$ 27. $\displaystyle\int\frac{x^2dx}{\sqrt{x^2-a^2}}$

28. $\displaystyle\int\frac{dx}{\sin^2 x+4\cos^2 x}$ 29. $\displaystyle\int\frac{dx}{2+\cos x}$ 30. $\displaystyle\int\frac{dx}{x^2-6x+13}$

31. $\displaystyle\int\frac{dx}{1+3\sin^2 x}$ 32. $\displaystyle\int\frac{x^2}{x^2-a^2}dx$ 33. $\displaystyle\int\frac{dx}{x\sqrt{9+x^4}}$

34. $\displaystyle\int\frac{\sin}{\sin 3x}dx$ 35. $\displaystyle\int\frac{dx}{1-4\cos^2 x}$ 36. $\displaystyle\int\sec^2(ax+b)\,dx$

37. $\int \frac{dx}{x(2+\log x)}$ 38. $\int \frac{x^5}{1+x^6}dx$ 39. $\int \frac{\cos x - \sin x}{\sin x + \cos x}dx$

40. $\int \frac{\cot x}{\log \sin x}dx$ 41. $\int \frac{\sec^2 x}{a + b\tan x}dx$ 42. $\int \frac{\sin x}{1+\cos}dx$

43. $\int \cos^2 x\, dx$ 44. $\int \sin^3 x\, dx$ 45. $\int \sin 5x \sin 3x\, dx$

46. $\int \sin^2 x \cos^3 x\, dx$ 47. $\int \sin^4 x\, dx$ 48. $\int \frac{1}{1+\sin x}dx$

49. $\int \tan^3 x\, dx$ 50. $\int \frac{\cos x - \sin x}{1+\sin 2x}dx$ 51. $\int \frac{\operatorname{cosec}^2 x}{1+\cot x}dx$

52. $\int \frac{1+x+\cos 2x}{x^2+\sin 2x+2x}dx$ 53. $\int \frac{\sec\theta \operatorname{cosec}\theta\, d\theta}{\log \tan\theta}$, 54. $\int \frac{\cot\theta\, d\theta}{\log \sin\theta}$

55. $\int \frac{dx}{1+4x^2}$ 56. $\int \frac{1-\tan\theta}{1+\tan\theta}d\theta$ 57. $\int \frac{1}{x^2}e^{\frac{-1}{x}}dx$

58. $\int \frac{\sin x \cos x\, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$ 59. $\int \frac{dx}{\sin x + \cos x}$

60. $\int e^x \left(\cos^{-1} x - \frac{1}{\sqrt{1-x^2}}\right)dx$ 61. $\int e^x \left(\frac{\sin x + \cos x}{\cos^2 x}\right)dx$

62. $\int \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}dx$ 63. $\int \cos\left[2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]dx$

64. $\int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}dx$ 65. $\int \sqrt{x}\log x\, dx$

66. $\int e^x (1+x)\log(xe^x)dx$ 67. $\int \frac{\log x}{(1+x)^2}dx$

68. $\int e^x \sin^2 x\, dx$ 69. $\int \cos(\log x)dx$

70. $\int \log(x+1)dx$ 71. $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

72. $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}dx$ 73. $\int \frac{dx}{x(x^5+1)}$

74. $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}dx$ 75. $\int \frac{\log x}{x(1+\log x)(2+\log x)}dx$

76. $\int \frac{dx}{1-e^x}$

ANSWERS

CHECK YOUR PROGRESS 30.1

1. $\frac{2}{7}x^{\frac{7}{2}}+1, \frac{2}{7}x^{\frac{7}{2}}+2, \frac{2}{7}x^{\frac{7}{2}}+3, \frac{2}{7}x^{\frac{7}{2}}+4, \frac{2}{7}x^{\frac{7}{2}}+5$

2. (a) $\frac{x^6}{6}+C$ (b) $\sin x+C$ (c) 0

3. (a) $\frac{x^7}{7}+C$ (b) $\frac{1}{6x^6}+C$ (c) $\log|x|+C$

   (d) $\frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)}+C$ (e) $\frac{3}{4}x^{\frac{4}{3}}+C$ (f) $\frac{-1}{8x^8}+C$

   (g) $2\sqrt{x}+C$ (h) $9x^{\frac{1}{9}}+C$

4. (a) $-\cos ec\theta+C$ (b) $\sec\theta$+C

   (c) $\tan\theta$+C (d) $-\cot\theta+C$

CHECK YOUR PROGRESS 30.2

1. (a) $\frac{x^2}{2}+\frac{1}{2}x+C$ (b) $-x+\tan^{-1}x+C$

   (c) $x^{10}-\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+C$ (d) $-\frac{1}{x^5}-\frac{3}{4x^4}+\frac{2}{3x^3}+\frac{7}{x}-8x+C$

   (e) $\frac{x^3}{3}-x-\tan^{-1}x+C$ (f) $\frac{x^2}{2}+4x+4\log x+C$

2. (a) $\frac{1}{2}\tan x+C$ (b) $\tan x-x+C$

   (c) $-2\cos ec\, x+C$ (d) $-\frac{1}{2}\cot x+C$

   (e) $-\sec x+C$ (f) $-\cot x+\cos ec\, x+C$

3. (a) $\sqrt{2}\sin x + C$ (b) $-\sqrt{2}\cos x + C$

(c) $-\frac{1}{2}\cot x + C$

4. (a) $\frac{2}{3}(x+2)^{\frac{3}{2}} + C$

CHECK YOUR PROGRESS 30.3

1. (a) $\frac{1}{5}\cos(4-5x) + C$ (b) $\frac{1}{3}\tan(2+3x) + C$

(c) $\log\left|\sec\left(x+\frac{\pi}{4}\right) + \tan\left(x+\frac{\pi}{4}\right)\right| + C$

(d) $\frac{1}{4}\sin(4x+5) + C$ (e) $\frac{1}{3}\sec(3x+5) + C$

(f) $-\frac{1}{5}\operatorname{cos}ec(3+5x) + C$

2. (a) $\frac{1}{12(3-4x)^3} + C$ (b) $\frac{1}{5}(x+1)^5 + C$

(c) $-\frac{1}{77}(4-7x)^{11} + C$ (d) $\frac{1}{16}(4x-5)^4 + C$

(e) $\frac{1}{3}\log|3x-5| + C$ (f) $-\frac{2}{9}\sqrt{5-9x} + C$

(g) $\frac{1}{6}(2x+1)^3 + C$ (h) $\log|x+1| + C$

3. (a) $\frac{1}{2}e^{2x+1} + C$ (b) $-\frac{1}{8}e^{3-8x} + C$

(c) $-\frac{1}{4e^{(7+4x)}} + C$

4. (a) $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$ (b) $\frac{1}{32}\left(-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right) + C$

(c) $\frac{1}{2}\left(-\frac{\cos 7x}{7} - \cos x\right) + C$ (d) $\frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + C$

CHECK YOUR PROGRESS 30.4

1. (a) $\frac{1}{6}\log\left|3x^2-2\right|+C$ (b) $\log\left|x^2+x+1\right|+C$

(c) $\log\left|x^2+9x+30\right|+C$ (d) $\frac{1}{3}\log\left|x^3+3x+3\right|+C$

(e) $\log\left|x^2+x-5\right|+C$ (f) $2\log\left|5+\sqrt{x}\right|+C$

(g) $\log\left|8+\log x\right|+C$

2. (a) $\frac{1}{b}\log\left|a+be^x\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

CHECK YOUR PROGRESS 30.5

1. (a) $x+\frac{3}{2}\log\left|\frac{x-3}{x+3}\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

(c) $\frac{1}{2}\tan^{-1}\left(x^2\right)+C$ (d) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

(e) $\frac{1}{2}\tan^{-1}\left(2\tan x\right)+C$ (f) $\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(g) $\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$ (h) $\sin^{-1}\left(\frac{x+2}{3}\right)+C$

(i) $\frac{1}{2\sqrt{3}}\sec^{-1}\frac{x}{2}+C$ (j) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)+C$

(k) $\log\left|e^x+\sqrt{1+e^{2x}}\right|+C$ (l) $\sin^{-1}x-\sqrt{1-x^2}+C$

(m) $\sin^{-1}\left(\frac{x-a}{a}\right)+C$ (n) $\frac{1}{4}\sin^{-1}\left(\frac{4}{3}x^3\right)+C$

(o) $\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|+C$

(p) $\frac{1}{2}\log\left|\frac{2x+\sqrt{9+4x^2}}{2}\right|+C$

(q) $-\frac{1}{2}\log\left|2\cos\theta+\sqrt{4\cos^2\theta-1}\right|+C$

(r) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(s) $\tan^{-1}\left(\frac{x+2}{1}\right)+C$ (t) $\frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|+C$

CHECK YOUR PROGRESS 30.6

1. (a) $-x\cos x+\sin+C$

(b) $\frac{1}{2}\left(1+x^2\right)\sin 2x+\frac{x\cos 2x}{2}-\frac{\sin 2x}{4}+C$

(c) $\frac{-x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+C$

2. (a) $x\tan x-\log\left|\sec x\right|-x+C$

(b) $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

3. (a) $\frac{x^4\log 2x}{4}-\frac{x^4}{16}+C$ (b) $\left(x-\frac{x^3}{3}\right)\log x-x+\frac{x^3}{9}+C$

(c) $x\left(\log x\right)^2-2x\log x+2x+C$

4. (a) $\frac{x^{1-n}}{1-n}\log x-\frac{x^{1-n}}{\left(1-n\right)^2}+C$ (b) $\log x.\left[\log\left(\log x\right)-1\right]+C$

5. (a) $e^{3x}\left[\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right]+C$ (b) $x\frac{e^{4x}}{4}-x\frac{e^{4x}}{16}+C$

6. (a) $\frac{x^2}{2}\left[\left(\log x\right)^2-\log x+\frac{1}{2}\right]+C$

7. (a) $x\sec^{-1}x-\log\left|x+\sqrt{x^2-1}\right|+C$

(b) $\frac{x^2}{2}\cot^{-1}x+\frac{x}{2}+\frac{1}{2}\cot^{-1}x+C$

CHECK YOUR PROGRESS 30.7

1. (a) $e^x \sec x0 + C$ (b) $e^x \log\left|\sec x + \tan x\right| + C$

2. (a) $\frac{1}{x} e^x + C$ (b) $e^x \sin^{-1} x + C$

3. $\frac{e^x}{(1+x)^2} + C$

4. $\frac{e^X}{1+x} + C$

5. $x \tan\frac{x}{2} + C$

6. $\frac{1}{5} e^x \left(\sin 2x - 2\cos 2x\right) + C$

CHECK YOUR PROGRESS 30.8

1. (a) $x\sqrt{x^2 - \frac{5}{4}} - \frac{5}{4}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right| + C$

   (b) $\frac{(2x+3)}{4}\sqrt{x^2 + 3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^{2+}3x}\right| + C$

   (c) $\frac{1}{4}(2x+1)\sqrt{3-2x-2x^2} + \frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

2. (a) $4\log|x-3| - 3\log|x-2| + C$

   (b) $\frac{1}{2}\log|x-4| + \log|x+4| + C$

3. (a) $\frac{x^2}{2} - 2\left[\log|x-2|| + \log||x+2|\right] + c$

   (b) $\frac{11}{9}\log|x-1| + \frac{7}{9}\log(x+2) - \frac{4}{3(x-1)} + C$

4. $\log|x-1| - \frac{3}{(x-1)} - \frac{3}{2(x-1)^2} + C$

5. (a) $\frac{1}{8}\log|1-\sin x| - \frac{1}{8}|1+\sin x|$

   $-\frac{1}{4\sqrt{2}}\log\left|1-\sqrt{2}\sin x\right| + \frac{1}{4\sqrt{2}}\log\left|1+\sqrt{2}\sin x\right| + C$

(b) $\log|\sec x + \tan x| - 2\tan\dfrac{x}{2} + C$

TERMINAL EXERCISE

1. $\sec x - \operatorname{cosec} x + C$
2. $\sin x - \cos x + C$
3. $-\cot x - \tan x + C$
4. $\tan x - \cot x - 4x + C$
5. $4\tan^{-1} x - \sin^{-1} x + C$
6. $\tan x - x + C$
7. $-\cot x - x + C$
8. $x - \cos x + C$
9. $x + \cos x + C$
10. $\dfrac{\sin(7x-\pi)}{7} + C$
11. $\dfrac{-\cos(3x+4)}{3} + C$
12. $\dfrac{\tan(2x+b)}{2} + C$
13. $\dfrac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(x-\dfrac{\pi}{4}\right) - \cot\left(x-\dfrac{\pi}{4}\right)\right| + C$
14. $\log\left|\tan^{-1} x\right| + C$
15. $\log\left|\log\tan\dfrac{x}{2}\right| + C$
16. $\dfrac{1}{4}\log|3+4\log\sin x| + C$
17. $\dfrac{1}{2}\log|\log\tan x| + C$
18. $2\log\left|e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right| + C$
19. $\dfrac{1}{4}\sec^4 x + C$
20. $-\cos e^x + C$
21. $\dfrac{\sqrt{2x^2+3}}{2} + C$
22. $2\sqrt{\tan x} + C$
23. $\dfrac{1}{6}x\sqrt{(25-9x^2)} + \dfrac{25}{6}\sin^{-1}\left(\dfrac{3}{5}x\right) + C$
24. $\dfrac{1}{2}(x-a)\sqrt{2ax-x^2} + \dfrac{1}{2}a^2\sin^{-1}\left(\dfrac{x-a}{a}\right) + C$
25. $\dfrac{x\sqrt{3x^2+4}}{2} + \dfrac{2}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}x+\sqrt{x^2+4}}{2}\right| + C$

26. $\dfrac{x\sqrt{9x^2+1}}{2}+\dfrac{1}{6}\log\left|3x+\sqrt{1+9x^2}\right|+C$

27. $\left[\dfrac{1}{2}x\sqrt{x^2-a^2}+\dfrac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|\right]+C$

28. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right)+C$

29. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{\tan\left(\dfrac{x}{2}\right)}{\sqrt{3}}\right]+C$

30. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right)+C$

31. $\dfrac{1}{2}\tan^{-1}(2\tan x)+C$

32. $x+\dfrac{a}{2}\log\left|\dfrac{x-a}{x+a}\right|+C$

33. $\dfrac{1}{12}\log\left|\dfrac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right|+C$

34. $\dfrac{2}{2\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$

35. $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right|+C$

36. $\dfrac{1}{a}\tan(ax+b)+C$

37. $\log\left|(2+\log x)\right|+C$

38. $\dfrac{1}{6}\log(1+x^6)+C$

39. $\log\left|\sin x+\cos x\right|+C$

40. $\log\left|\log(\sin x)\right|+C$

41. $\dfrac{1}{b}\log\left|a+b\tan x\right|+C$

42. $-\log\left|1+\cos x\right|+C$

43. $\dfrac{1}{2}\dfrac{\sin 2x}{2}+\dfrac{1}{2}x+C$

44. $-\cos x+\dfrac{\cos^3 x}{3}+C$

45. $\dfrac{1}{2}\dfrac{\sin 2x}{2}-\dfrac{1}{2}\dfrac{\sin 8x}{8}+C$

46. $\dfrac{1}{3}\sin^3 x-\dfrac{\sin^5 x}{5}+C$

47. $\dfrac{1}{32}[12x-8\sin 2x+\sin 4x]+C$

48. $\tan x-\sec x+C$

49 $\dfrac{\tan^2 x}{2}+\log\left|\cos x\right|+C$.

50. $\dfrac{-1}{\cos x+\sin x}+C$

51. $\log\left|\dfrac{1}{1+\cot x}\right|+C$

52. $\frac{1}{2}\log\left|x^2+\sin 2x+2x\right|+C$

53 $\log\left|\tan\theta\right|+C$.

54. $\log\left|\log\sin\theta\right|+C$

55. $\frac{1}{2}\tan^{-1}2x$

56. $\log\left|\cos\theta+\sin\theta\right|+C$

57. $e^{\frac{1}{x}}+C$

58. $\frac{1}{2\left(a^2-b^2\right)}\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+C$

59. $\frac{1}{\sqrt{2}}\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+C$

60. $e^x\cos^{-1}x+C$

61. $e^x\sec x+C$

62. $\frac{1}{4}x^2+C$

63. $-\frac{1}{2}x^2+C$

64. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\frac{1}{2}\log\left|1-x^2\right|+C$

65. $\frac{2}{3}x^{\frac{3}{2}}\left(\log x-\frac{2}{3}\right)+C$

66. $xe^x\left[\log\left(xe^x\right)-1\right]+C$

67. $-\frac{1}{1+x}\log|x|+\log|x|-\log|x+1|+c$

68. $\frac{1}{2}e^x-\frac{e^x}{10}\left(2\sin 2x+\cos 2x\right)+C$

69. $\frac{x}{2}\left[\cos\left(\log x\right)+\sin\left(\log x\right)\right]+C$

70. $x\log|x+1|-x+\log|x+1|+C$

71. $\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$

72. $-\frac{2}{3}\log\left|\cos\theta-2\right|-\frac{1}{3}\log\left|\cos\theta+1\right|+C$

73. $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+C$

74. $\frac{1}{3\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\tan^{-1}\left(\sqrt{2}x\right)\right]+C$

75. $\log\left|\frac{(2+\log x)^2}{1+\log x}\right|+C$

76. $\log\left|\frac{e^x}{1-e^x}\right|+C$

31

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DEFINITE INTEGRALS

In the previous lesson we have discussed the anti-derivative, i.e., integration of a function.The very word integration means to have some sort of summation or combining of results.

Now the question arises : Why do we study this branch of Mathematics? In fact the integration helps to find the areas under various laminas when we have definite limits of it. Further we will see that this branch finds applications in a variety of other problems in Statistics, Physics, Biology, Commerce and many more.

In this lesson, we will define and interpret definite integrals geometrically, evaluate definite integrals using properties and apply definite integrals to find area of a bounded region.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (ii) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(iii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if f is an even function of x

= 0 if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of integration
- Area of a bounded region

31.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 31.1)

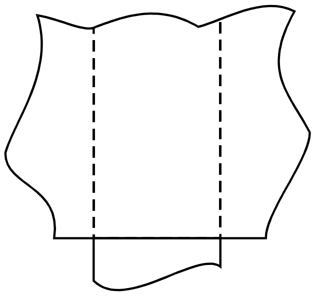


Fig. 31.1

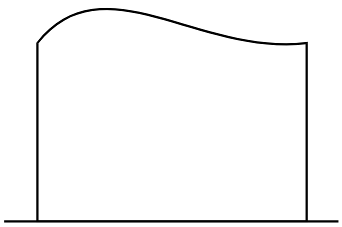


Fig. 31.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 31.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 31.1)

Now, let f (x) be a continuous function defined on the closed interval [a, b]. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.31.3).

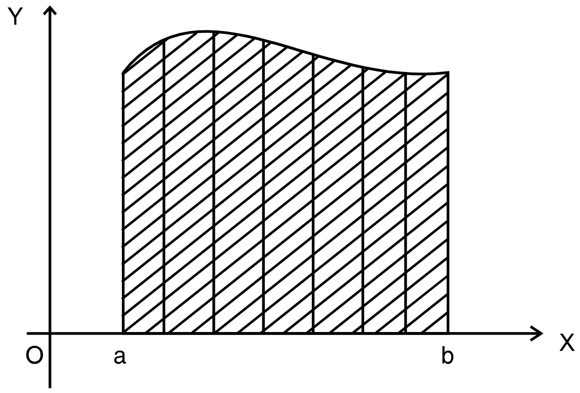


Fig. 31.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.31.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region , triangular region and trapezoidal region.

The area of these regions $=$ base $\times$ average height

In general for any function $f(x)$ on $[a, b]$

Area of the bounded region (shaded region in Fig. 31.3) = base $\times$ average height

The base is the length of the domain interval [a, b]. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in [a, b]. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in [a,b].

31.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in [a,b], we can easily get the average value by the formula.

$$\text{Average value of f in } [a, b] = \frac{\text{Sum of the values of f in } [a, b]}{\text{Numbers of values}}$$

But in our problem, there are infinite number of values taken by f in [a, b]. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at 'a' only. The value of f at a is f (a). We take this value, namely f (a), as a rough estimate of the average value of f in [a,b].

Average value of f in [a, b] (first estimate) = f (a) (i)

Second Estimate : Divide [a, b] into two equal parts or sub-intervals.

Let the length of each sub-interval be h, $h = \dfrac{b - a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are $f(a)$ and $f(a + h)$ (Fig. 31.4)

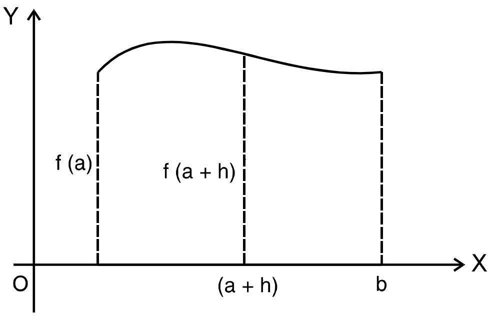


Fig. 31.4

Take the average of these two values as the average of f in [a, b].

Average value of f in [a, b] (Second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \quad h = \frac{b-a}{2} \qquad \text{(ii)}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval [a, b] into n subintervals of length h (Fig. 31.5), $h = \frac{b-a}{n}$

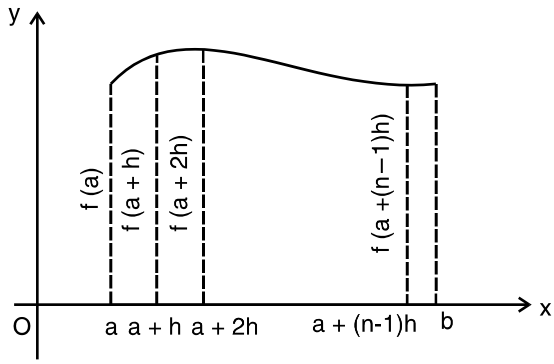


Fig. 31.5

Take the values of f at the left end points of the n subintervals.

The values are f (a), f (a + h),......,f [a + (n-1) h]. Take the average of these n values of f in [a, b].

Average value of f in [a, b] (nth estimate)

$$= \frac{f(a) + f(a+h) + ... + f(a+(n-1)h)}{n}, \quad h = \frac{b-a}{n} \qquad \text{(iii)}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in [a, b]

Thus, we get the following sequence of estimates for the average value of f in [a, b]:

$$f(a)$$

$$\frac{1}{2}\left[f(a)+f(a+h)\right], \qquad h=\frac{b-a}{2}$$

$$\frac{1}{3}\left[f(a)+f(a+h)+f(a+2h)\right], \qquad h=\frac{b-a}{3}$$

....

....

$$\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right], \quad h=\frac{b-a}{n}$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in [a, b]. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in [a, b]. In other words,

Average value of f in [a, b]

$$\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+......+f\left[a+(n-1)h\right]\right\},$$

$$h=\frac{b-a}{n} \tag{iv}$$

It can be proved that this limit exists for all continuous functions f on a closed interval [a, b].

Now, we have the formula to find the area of the shaded region in Fig. 31.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve f (x), x-axis, the ordinates x = a and x = b

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+......+f\left[a+(n-1)h\right]\right\},$$

$$\lim_{n\to 0}\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right], h=\frac{b-a}{n} \tag{v}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\,dx$$

read as integral of f (x) from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\,dx$ are called respectively the lower and upper limits of integration, and f (x) is called the integrand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

Why not right end points of the subintervals? We can as well take the right end points of the subintervals throughout and in that case we get

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a+h)+f(a+2h)+......+f(b)\},\quad h=\frac{b-a}{n}$$

$$=\lim_{h\to 0} h\,[f(a+h)+f(a+2h)+......+f(b)] \qquad \text{(vi)}$$

Example 31.1 Find $\int_1^2 x\;dx$ as the limit of sum.

Solution : By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+........+f\{a+(n-1)h\}],$$

$$h=\frac{b-a}{n}$$

Here a = 1, b = 2, f (x) = x and $h=\frac{1}{n}$.

$$\therefore \qquad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+........+f\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)........+\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{1+1+......+1}_{\text{n times}}+\left(\frac{1}{n}+\frac{2}{n}+........+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}(1+2+......+(n-1))\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{(n-1).n}{n.2}\right]$$

$$\left[\text{Since } 1+2+3+....+(n-1)=\frac{(n-1).n}{2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]=\frac{3}{2}$$

Example 31.2 Find $\int_0^2 e^x dx$ as limit of sum.

Solutions : By definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a+(n-1)h\}\right]$$

where $\quad h = \dfrac{b-a}{n}$

Here $a = 0, b = 2, f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \qquad \int_0^2 e^x dx = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + \ldots\ldots + f(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[e^0 + e^h + e^{2h} + \ldots\ldots + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[e^0\left(\frac{\left(e^h\right)^n - 1}{e^h - 1}\right)\right]$$

$$\left[\text{Since} \quad a + ar + ar^2 + \ldots\ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right] = \lim_{h\to 0} \frac{h}{h}\left[\frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)}\right] \qquad (\because nh = 2)$$

$$= \lim_{h\to 0} \frac{e^2 - 1}{\frac{e^h - 1}{h}} = \frac{e^2 - 1}{1}$$

$$= e^2 - 1 \qquad \left[\because \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

In examples 31.1 and 31.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in [a, b] and F is an antiderivative of f in [a, b] then

$$\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)} \qquad \textbf{......(1)}$$

The difference F (b) – F (a) is commonly denoted by $\left[F(x)\right]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \text{ or } \left[F(x)\right]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\,dx = \text{(Value of antiderivative at the upper limit b)}$$
$$- \text{(Value of the same antiderivative at the lower limit a)}$$

Example 31.3 Evaluate the following

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\,dx$ (b) $\displaystyle\int_0^2 e^{2x}\,dx$

Solution : We know that

$$\int \cos x\,dx = \sin x + c$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $$\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2, \qquad \left[\because \int e^x dx = e^x\right]$$

$$= \left(\frac{e^4 - 1}{2}\right)$$

Theorem 2 : If f and g are continuous functions defined in [a, b] and c is a constant then,

(i) $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

(ii) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iii) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example 31.4 Evaluate $\displaystyle\int_0^2 \left(4x^2 - 5x + 7\right)dx$

Solution : $\displaystyle\int_0^2 \left(4x^2 - 5x + 7\right)dx = \int_0^2 4x^2 dx - \int_0^2 5x\,dx + \int_0^2 7\,dx$

$$= 4\int_0^2 x^2 dx - 5\int_0^2 x dx + 7\int_0^2 1\, dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}$$

CHECK YOUR PROGRESS 31.1

1. Find $\int_0^5 (x+1)\, dx$ as the limit of sum.

2. Find $\int_{-1}^1 e^x\, dx$ as the limit of sum.

3. Evaluate (a) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ (b) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$

   (c) $\int_0^1 \frac{1}{1+x^2} dx$ (d) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

31.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2} dx, \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 31.5 Evaluate the following :

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\, dx = -dt$

When $x = 0$, $t = 1$ and $x = \frac{\pi}{2}, t = 0$. As x varies from 0 to $\frac{\pi}{2}$, t varies from 1 to 0.

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

(b) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\left(\sin^2\theta+\cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1-2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta\, d\theta}{1-2\sin^2\theta\left(1-\sin^2\theta\right)}$$

Let $\sin^2\theta = t$

Then $2\sin\theta\cos\theta\, d\theta = dt$ i.e. $\sin 2\theta\, d\theta = dt$

When $\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$$\therefore \qquad I = \int_0^1 \frac{1}{1 - 2t(1-t)}\,dt = \int_0^1 \frac{1}{2t^2 - 2t + 1}\,dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{4} + \frac{1}{4}}\,dt \qquad I = \frac{1}{2}\int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 = \left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{5 + \dfrac{4\left(1 - \tan^2\left(\frac{x}{2}\right)\right)}{\left(1 + \tan^2\left(\frac{x}{2}\right)\right)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{9 + \tan^2\left(\frac{x}{2}\right)}\,dx \qquad (1)$$

Let $\qquad \tan\frac{x}{2} = t$

Then $\quad \sec^2\frac{x}{2}\,dx = 2dt$ when $x = 0,\ \ t = 0$, when $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x}\,dx = 2\int_0^1 \frac{1}{9 + t^2}\,dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{1}{3}\right]$$

31.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of f (x) between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ Where } \frac{d}{dx}[F(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where a<c<b.

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(vi) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vii) $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

(viii) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \end{cases}$$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 31.6 Show that

(a) $\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$ (b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

Solution : (a) Let $\quad I = \int_0^{\frac{\pi}{2}} \log|\tan x|\,dx$ ....(i)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx = \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1}\,dx = -\int_0^{\frac{\pi}{2}} \log \tan x\,dx$$

$$= -I \qquad \text{[Using (i)]}$$

$\therefore \quad 2I = 0$

i.e. $\quad I = 0 \quad$ or $\quad \int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$

(b) $\quad \int_0^{\pi} \frac{x}{1+\sin x}\,dx$

Let $\quad I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx \qquad$ (i)

$$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

or $$2I = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \pi\int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right)dx$$

$$= \pi \left[\tan x - \sec x \right]_0^{\pi}$$

$$= \pi \left[\left(\tan \pi - \sec \pi \right) - \left(\tan 0 - \sec 0 \right) \right]$$

$$= \pi \left[0 - (-1) - (0 - 1) \right]$$

$$= 2\pi$$

$\therefore \qquad I = \pi$

Example 31.7 Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$

Solution : (a) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ...(i)

Also

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

(Using the property $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$).

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\frac{\pi}{2}} 1.dx$$

$$= \left[x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \qquad I = \dfrac{\pi}{4}$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$$

(b) Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx \qquad \text{(i)}$$

Then $$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \sin x}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$= 0$$

$\therefore$ $I = 0$

Example 31.8 Evaluate (a) $\displaystyle\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\,dx$ (b) $\displaystyle\int_{-3}^{3} |x+1|\,dx$

Solution : (a) Here $f(x) = \dfrac{xe^{x^2}}{1+x^2}$ $\therefore$ $f(-x) = -\dfrac{xe^{x^2}}{1+x^2}$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function of x.

$\therefore$ $$\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\,dx = 0$$

(b) $\int_{-3}^{3} |x+1|\,dx$

$$|x+1| = \begin{cases} x+1, \text{ if } x \geq -1 \\ -x-1, \text{ if } x < -1 \end{cases}$$

$$\therefore \int_{-3}^{3} |x+1|\,dx = \int_{-3}^{-1} |x+1|\,dx + \int_{-1}^{3} |x+1|\,dx, \text{ using property (iii)}$$

$$= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 31.9 Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$

Solution : Let $I = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$ (i)

Also $I = \int_0^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx,$ [Using property (iv)]

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} [\log(\sin x) + \log(\cos x)]\,dx = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log(2)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2 \qquad \text{(iii)}$$

Again, let $\quad I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

Put $2x = t \quad \Rightarrow \quad dx = \frac{1}{2}dt$

When $x = 0$, $t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$$\therefore \qquad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt, \qquad \text{[using property (vi)]}$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dt \qquad \text{[using property (i)]}$$

$\therefore \qquad I_1 = I, \qquad$ [from (i)] $\qquad$ .....(iv)

Putting this value in (iii), we get

$$2I = I - \frac{\pi}{2}\log 2 \qquad \Rightarrow \qquad I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$

CHECK YOUR PROGRESS 31.2

Evaluate the following integrals :

1. $\int_0^1 xe^{x^2}dx$ $\qquad$ 2. $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x}$ $\qquad$ 3. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

4. $\int_{-5}^{5} |x+2|\,dx$ $\qquad$ 5. $\int_0^2 x\sqrt{2-x}\,dx$ $\qquad$ 6. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}dx$

7. $\int_0^{\frac{\pi}{2}} \log \cos x \, dx$ 8. $\int_{-a}^{a} \frac{x^3 e^{x^4}}{1+x^2} dx$ 9. $\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x \, dx$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x} dx$

31.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $f(x) \le g(x)$ for $x \in [a,\ b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) over [a, b]. Now the question is how to find the area of the region bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Again what happens when the upper curve y = f (x) intersects the lower curve y = g (x) at either the left hand boundary x = a , the right hand boundary x = b or both?

31.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve y = f (x) and CA, DB the two ordinates at x = a and x = b respectively. Suppose y = f (x) is an increasing function of x in the interval $a \le x \le b$.

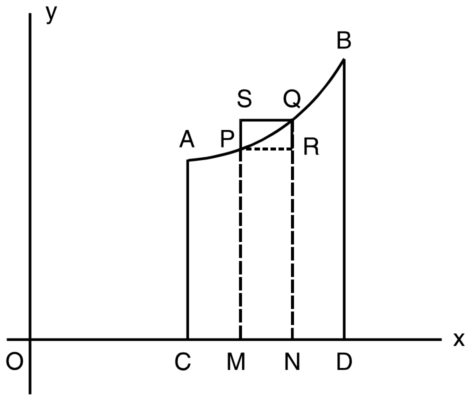


Fig. 31.6

Let P (x, y) be any point on the curve and $Q(x+\delta x,\ y+\delta y)$ a neighbouring point on it. Draw their ordinates PM and QN.

Here we observe that as x changes the area (ACMP) also changes. Let

A=Area (ACMP)

Then the area (ACNQ) $= A + \delta A$.

The area (PMNQ)=Area (ACNQ) – Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectangles PMNR and SMNQ, that is

δA lies between $y\ \delta x$ and $(y+\delta y)\delta x$

$\Rightarrow$ $\frac{\delta A}{\delta x}$ lies between y and $(y+\delta y)$

In the limiting case when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$.

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta A}{\delta x} \text{ lies between } y \text{ and } \lim_{\delta y \to 0} (y + \delta y)$$

$$\therefore \quad \frac{dA}{dx} = y$$

Integrating both sides with respect to x, from x = a to x = b, we have

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx} \cdot dx = [A]_a^b$$

$$= (\text{Area when } x = b) - (\text{Area when } x = a)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}.$$

Hence $\quad$ Area (ACDB) $= \int_a^b f(x)\,dx$

The area bounded by the curve y = f (x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where y = f (x) is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example31.10 Find the area bounded by the curve y = x, x-axis and the lines x =0, x = 2.

Solution : The given curve is y = x

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0$, $x = 2$ (as shown in Fig.31.7)

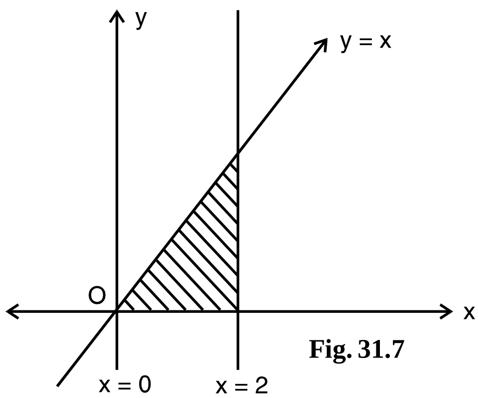


Fig. 31.7

is $\quad \int_0^2 x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$= 2 - 0 = 2$ square units

Example 31.11 Find the area enclosed by the circle $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates x = 0 and x = a.

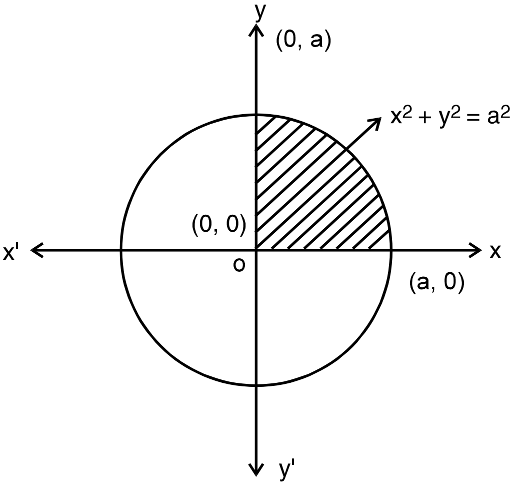


Fig. 31.8

$\therefore$ Required area $= \int_0^a y\,dx$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} \quad \left(\because \sin^{-1}1 = \frac{\pi}{2}, \sin^{-1}0 = 0\right)$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

CHECK YOUR PROGRESS 31.3

1. Find the area bounded by the curve $y = x^2$, x-axis and the lines $x = 0, x = 2$.
2. Find the area bounded by the curve y = 3 x, x-axis and the lines x = 0 and x = 3.

31.4.2. Area Bounded by the Curve x = f (y) between y-axis and the Lines y = c, y = d

Let AB be the curve x = f (y) and let CA, DB be the abscissae at y = c, y = d respectively.

Let P (x, y) be any point on the curve and let $Q(x + \delta x,\ y + \delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

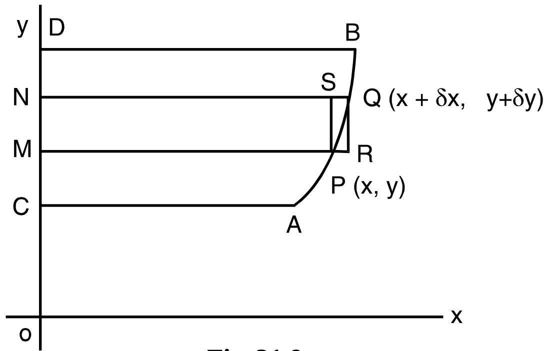


Fig. 31.9

The area (PMNQ) = Area (ACNQ) – Area (ACMP) $= A + \delta A - A = \delta A$.

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x\,\delta y$ and $(x+\delta x)\,\delta y$

$$\Rightarrow \qquad \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } x+\delta x$$

In the limiting position when $Q \to P,\ \delta x \to 0$ and $\therefore$.

$$\therefore \qquad \lim_{\delta y \to 0} \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } \lim_{\delta x \to 0}(x+\delta x)$$

$$\Rightarrow \qquad \Rightarrow \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\,dy = \int_c^d \frac{dA}{dy}\cdot dy$$

$$= \;=[A]_c^d$$

= (Area when y = d) – (Area when y = c)
= Area (ACDB) – 0
= Area (ACDB)

Hence area $\quad$ (ACDB) $= \int_c^d x\,dy = \int_c^d f(y)\,dy$

The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$

where $x = f(y)$ is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 31.12 Find the area bounded by the curve x = y, y-axis and the lines y = 0, y = 3.

Solution : The given curve is x = y.

$\therefore$ Required area bounded by the curve, y-axis and the lines y =0, y = 3 is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

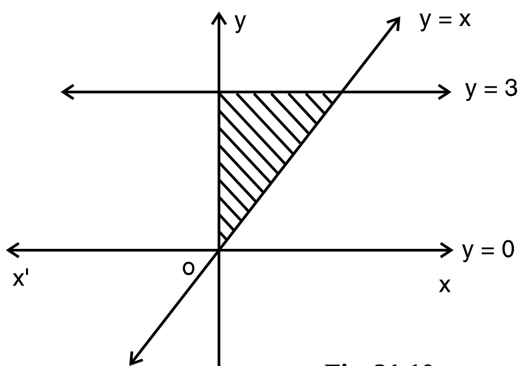


Fig. 31.10

$$= \frac{9}{2} \text{ square units}$$

Example 31.13 Find the area enclosed by the circle $x^2 + y^2 = a^2$ and y-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae y = 0, y = a.

$$\therefore \quad \text{Required area} = \int_0^a x\, dy$$

$$= \int_0^a \sqrt{a^2 - y^2}\, dy$$

(because x is positive in first quadrant)

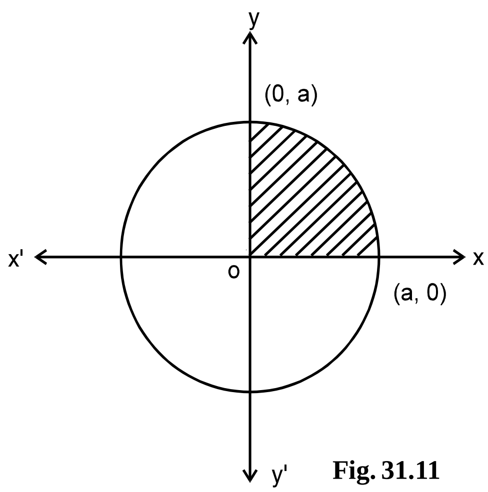


Fig. 31.11

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{\pi a^2}{4} \text{ square units} \qquad \left(\because \sin^{-1} 0 = 0, \sin^{-1} 1 = \frac{\pi}{2} \right)$$

Note : The area is same as in Example 31.11, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method.

Example 31.14 Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution : The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

Area of circle = 4 × area of OAB

$$= 4 \times \frac{\pi a^2}{4} \text{(From Example 12.11 and 12.13)} = \pi a^2$$

square units

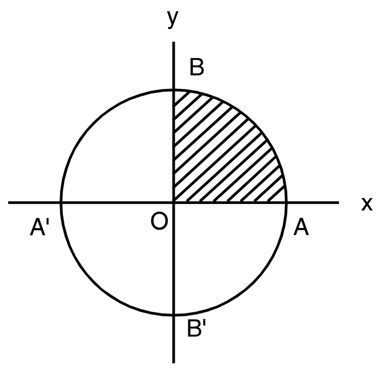
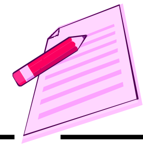


Fig. 31.12



Example 31.15 Find the whole area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Solution : The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,

Whole area of the ellipse $= 4 \times$ area (OAB)

In the first quadrant,

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \text{or} \quad y=\frac{b}{a}\sqrt{a^2-x^2}$$

Now for the area (OAB), x varies from 0 to a

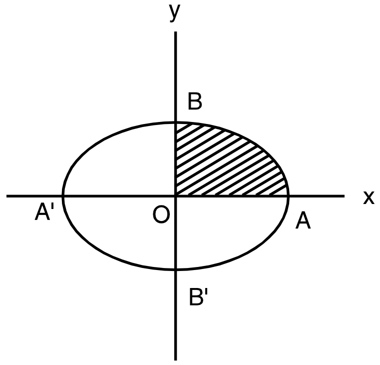


Fig. 31.13

$$\therefore \qquad \text{Area (OAB)} = \int_0^a y\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{b}{a}\left[0+\frac{a^2}{2}\sin^{-1}1-0-\frac{a^2}{2}\sin^{-1}0\right]$$

$$=\frac{ab\pi}{4}$$

Hence the whole area of the ellipse

$$=4\times\frac{ab\pi}{4}$$

$$=\pi ab. \text{ square units}$$

31.4.3 Area between two Curves

Suppose that f (x) and g (x) are two continuous and non-negative functions on an interval [a, b] such that $f(x) \geq g(x)$ for all $x \in [a, b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) for $x \in [a, b]$. We want to find the area bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Let A= [Area under y = f (x)] – [Area under y = g (x)] .....(1)

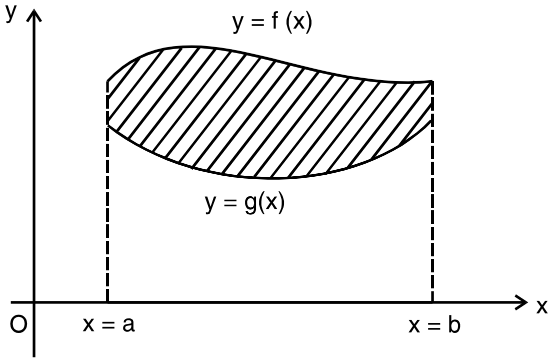


Fig. 31.14

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates x = a and x = b, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad .....(2)$$

Similarly,Area under $y = g(x) = \int_a^b g(x)\,dx$ .....(3)

Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad .....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves f (x) and g (x) upwards until both are above the x-axis. To do this let-m be the minimum value of g (x) on [a, b] (see Fig. 31.15).

Since $\qquad g(x) \geq -m \qquad \Rightarrow \qquad g(x) + m \geq 0$

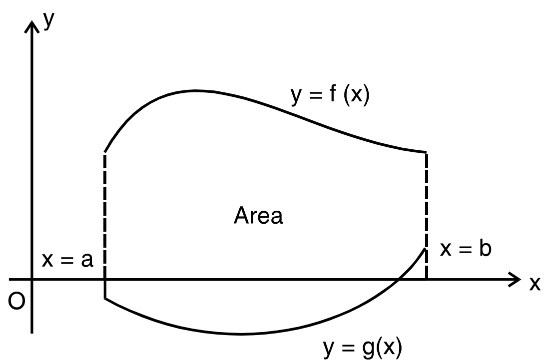


Fig. 31.15

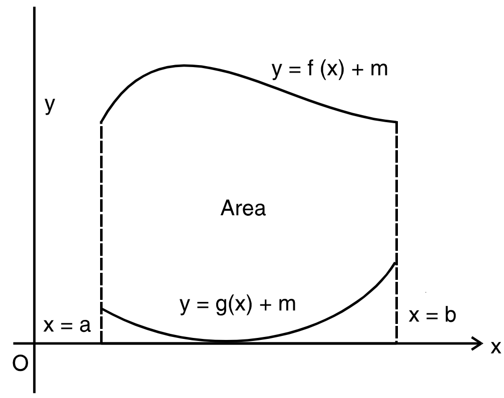


Fig. 31.16

Now, the functions $g(x) + m$ and $f(x) + m$ are non-negative on [a, b] (see Fig. 31.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x) + m$ and $f(x) + m$. Thus,

$A = [\text{area under } y = [f(x)+m]] - [\text{area under } y = [g(x)+m]]$ .....(5)

Now using the definitions for the area bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b, we have

$$\text{Area under } y = f(x)+m = \int_a^b [f(x)+m]\,dx \quad .....(6)$$

and

$$\text{Area under } y = g(x)+m = \int_a^b [g(x)+m]\,dx \quad (7)$$

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x)+m]\,dx - \int_a^b [g(x)+m]\,dx$$

$$= \int_a^b [f(x)-g(x)]\,dx$$

which is same as (4) Thus,

If f (x) and g (x) are continuous functions on the interval [a, b], and $f(x) \geq g(x), \forall x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$= \int_a^b [f(x)-g(x)]\,dx$$

$$= \frac{34}{3} \text{ square units}$$

If the curves intersect then the sides of the region where the upper and lower curves intersect reduces to a point, rather than a vertical line segment.

Example 31.16 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and y = x + 6 is the equation of the straight line. (See Fig. 31.17).

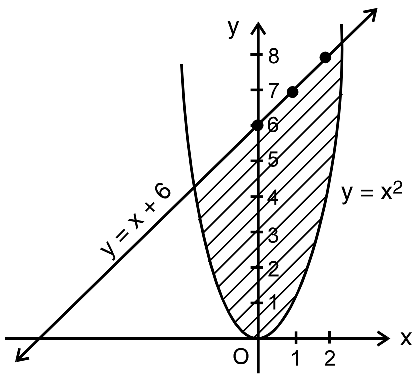


Fig. 31.17

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is y = x +6. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, -2$$

When x = 3, y = 9 and when x = –2, y = 4

$$\therefore \text{ The required area } = \int_{-2}^{3} \left[(x+6) - x^2 \right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3} \right)$$

$$= \frac{125}{6} \text{ square units}$$

Example 31.17 Find the area bounded by the curves $y^2 = 4x$ and y = x.

Solution : We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex. y = x is the equation of the straight line (see Fig. 31.18).

A sketch of the region shows that the lower boundary is y = x and the upper boundary is $y^2 = 4x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0, 4$$

When y = 0, x = 0 and when y = 4, x = 4.

Here $f(x) = (4x)^{\frac{1}{2}}, g(x) = x, a = 0, b = 4$

Therefore, the required area is

$$= \int_0^4 \left(2x^{\frac{1}{2}} - x \right) dx$$

$$= \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{x^2}{2} \right]_0^4$$

$$= \frac{32}{3} - 8$$

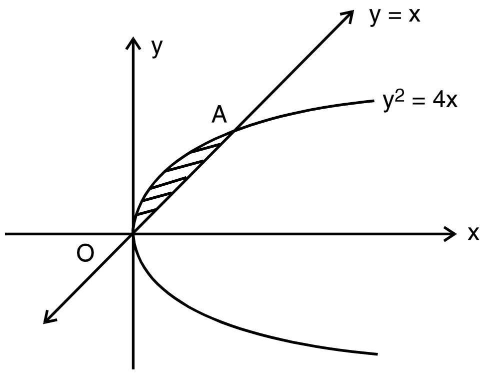


Fig. 31.18

$$= \frac{8}{3} \quad \text{square units}$$

Example 31.18 Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution : We know that $y^2 = 4ax$ and $x^2 = 4ay$ are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 31.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \qquad x\left(x^3 - 64a^3\right) = 0$$

$$\Rightarrow \qquad x = 0, 4a$$

Hence the two parabolas intersect at point (0, 0) and (4a, 4a).

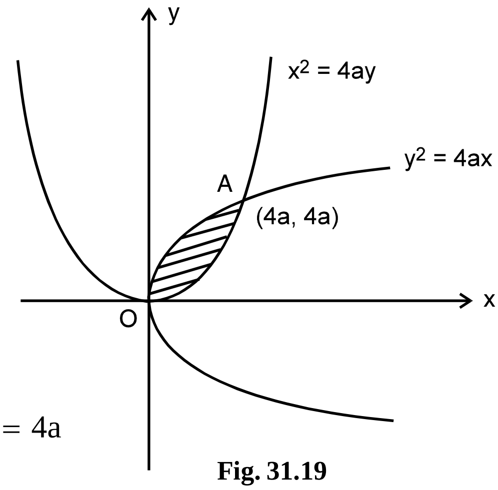


Fig. 31.19

Here $\quad f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}, a = 0 \text{ and } b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[\frac{2.2\sqrt{a}\, x^{\frac{3}{2}}}{3} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3} a^2 \text{ square units}$$

CHECK YOUR PROGRESS 31.4

1. Find the area of the circle $x^2 + y^2 = 9$

2. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \frac{x^2}{4a}$

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$

LET US SUM UP

- If f is continuous in [a, b] and F is an anti derivative of f in [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If f and g are continuous in [a, b] and c is a constant, then

(i) $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

(ii) $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iii) $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

- The area bounded by the curve y = f (x), the x-axis and the ordinates

$x = a, x = b$ is $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$

- If f (x) and g (x) are continuous functions on the interval [a, b] and $f(x) \ge g(x)$, for all $x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$\int_a^b [f(x) - g(x)]\,dx$$

SUPPORTIVE WEB SITES

http://mathworld.wolfram.com/DefiniteIntegral.html

http://www.mathsisfun.com/calculus/integration-definite.html

TERMINAL EXERCISE

Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$
2. $\int_a^b x^2\,dx$
3. $\int_0^2 \left(x^2 + 1\right) dx$

Evaluate the following integrals (4 to 20)

4. $\int_0^2 \sqrt{a^2 - x^2}\,dx$
5. $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$
6. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$
7. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
8. $\int_0^1 \sin^{-1} x\,dx$
9. $\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx$
10. $\int_3^4 \frac{1}{x^2 - 4}\,dx$
11. $\int_0^{\pi} \frac{1}{5 + 3\cos\theta}\,d\theta$
12. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$
13. $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$
14. $\int_0^2 x\sqrt{x + 2}\,dx$
15. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\cos^5\theta\,d\theta$
16. $\int_0^{\pi} x \log \sin x\,dx$
17. $\int_0^{\pi} \log(1 + \cos x)\,dx$
18. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$

19. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

20. $\displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

21. Find the area bounded by the curve $x = y^2$, y – axis and lines y=0, y=2.

22. Find the area of the region bounded by the curve $y = x^2$ and y=x.

23. Find the area bounded by the curve $y^2 = 4x$ and straight line x=3.

24. Find the area of triangular region whose vertices have coordinates (1,0), (2,2) and (3.1)

25. Find the area of the smaller region bounded by the cllipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the straight line $\dfrac{x}{3} + \dfrac{y}{2} = 1$

26. Find the area of the region bounded by the paralal $y = x^2$ and the curve $y = |x|$

Notes

ANSWERS

CHECK YOUR PROGRESS 31.1

1. $\frac{35}{2}$ 2. $e - \frac{1}{e}$

3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

CHECK YOUR PROGRESS 31.2

1. $\frac{e-1}{2}$ 2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

4. 29 5. $\frac{24\sqrt{2}}{15}$ 6. $\frac{\pi}{4}$

7. $-\frac{\pi}{2}\log 2$ 8. 0 9. 0 10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

CHECK YOUR PROGRESS 31.3

1. $\frac{8}{3}$ sq. units 2. $\frac{27}{2}$ sq. units

CHECK YOUR PROGRESS 31.4

1. 9π sq. units 2. 6π sq. units 3. 20π sq. units

4. $\frac{16}{3}a^2$ sq. units 5. $\frac{16}{3}$ sq. units 6. $\frac{9}{2}$ sq. units

TERMINAL EXERCISE

1. $\frac{b^2 - a^2}{2}$ 2. $\frac{b^3 - a^3}{3}$ 3. $\frac{14}{3}$ 4. $\frac{\pi a^2}{4}$

5. 1 6. $\frac{1}{2}\log 2$ 7. $\frac{\pi}{4}$ 8. $\frac{\pi}{2} - 1$

9. $\frac{\pi}{2}$ 10. $\frac{1}{4}\log\frac{5}{3}$ 11. $\frac{\pi}{4}$ 12. $1 - \log 2$

Notes

13.	$\frac{2}{3}$	14.	$\frac{16}{15}\left(2+\sqrt{2}\right)$	15.	$\frac{64}{231}$
16.	$-\frac{\pi^2}{2}\log 2$	17.	$-\pi\log 2$	18.	$\frac{\pi^2}{4}$
19.	$\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$	20.	$\frac{\pi}{8}\log 2$	21.	$\frac{8}{3}$ square unit.
22.	$\frac{1}{6}$ Square unit	23.	$8\sqrt{3}$ Square unit		
24.	$\frac{3}{2}$ Square unit	25.	$\frac{3}{2}(\pi-2)$ Square unit		
26.	$\frac{1}{3}$ Square unit				

32

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DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9\ t^2$ where s is distance and t is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time t.

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.

OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

  (i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)\,g(y)$

  (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)\,y = Q(x)$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

32.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

32.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation.

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 32.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2}+\left[1+\left(\frac{dy}{dx}\right)^2\right]=0 \quad \text{or} \quad \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1$$

Hence order of the diferential equation is 2 and the degree of the differential equation is 1.

Note : Degree of a differential equation is defiend if it is a palynomial equation in terms of its derivatives.

32.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2}+y=0 \quad \text{and} \quad \cos^2 x\frac{d^3y}{dx^3}+x^3\frac{dy}{dx}+y=0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2+\frac{y}{x}=\log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y\frac{d^2y}{dx^2}-4=x$ is non-linear because the dependent variable y and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

32.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx}=m \qquad .....(2)$$

From (1) and (2), we get

$$y = x\frac{dy}{dx} \qquad .....(3)$$

So $y = mx$ and $y = x\dfrac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

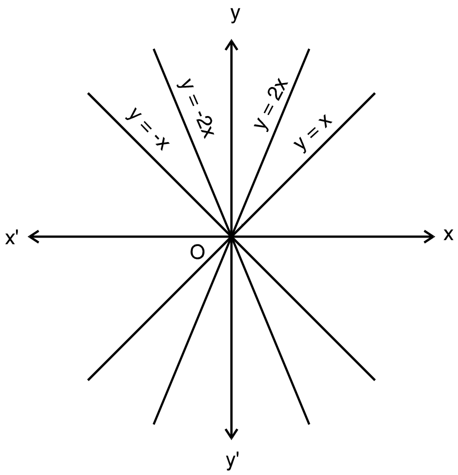


Fig. 32.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say *x* and *y* and some arbitrary constants, say, *a*, *b*, *c*, etc.

(i) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 32.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \qquad ......(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \qquad ......(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \qquad ......(3)$$

$$\Rightarrow \qquad a = \frac{1}{2}\frac{d^2y}{dx^2} \qquad ......(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \qquad b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad ......(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or $$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or $$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or $$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 32.3 Form the differential equation representing the family of curves

$$y = a\cos(x + b).$$

Solution : $$y = a\cos(x + b) \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x + b) \qquad ....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \qquad .....(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \qquad \text{or} \qquad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 32.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x-a)+2y\frac{dy}{dx}=0$$

or $$x-a+y\frac{dy}{dx}=0$$

or $$a=y\frac{dy}{dx}+x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x-y\frac{dy}{dx}-x\right)^2+y^2=\left(y\frac{dy}{dx}+x\right)^2$$

or $$\left(y\frac{dy}{dx}\right)^2+y^2=x^2+\left(y\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

or $$y^2=x^2+2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

or $$\frac{dr}{dt}=k$$

which is the required differential equation.

CHECK YOUR PROGRESS 32.1

1. Find the order and degree of the differential equation

$$y=x\frac{dy}{dx}+1$$

2. Write the order and degree of each of the following differential equations.

(a) $$\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$$

(b) $$\left(\frac{d^2s}{dt^2}\right)^2+3\left(\frac{ds}{dt}\right)^3+4=0$$

3. State whether the following differential equations are linear or non-linear.

(a) $\left(xy^2 - x\right)dx + \left(y - x^2y\right)dy = 0$ (b) $dx + dy = 0$

(c) $\dfrac{dy}{dx} = \cos x$ (d) $\dfrac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x - a)^2 + (y - b)^2 = r^2$ by eliminating 'a' and 'b'.

5. (a) Form the differential equation corresponding to

$y^2 = m\left(a^2 - x^2\right)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

32.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

(1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.

(2) Solution of a differential equation is one which satisfies the differential equation.

Example 32.5 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \qquad .....(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \qquad .....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or $$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 32.6 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ .....(1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 32.7 If $y = 3x^2 + C$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when y = 3, x = 2.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \qquad \text{....(1)}$$

Now on substituting y = 3, x = 2 in the above equation , we get

$$3 = 12 + C \qquad \text{or} \qquad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

32.6 TECHNIQUES OF SOLVING IN G A DIFFERENTIAL EQUATION

32.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $\qquad dy = f(x)\,dx$

On integrating both sides, we get

$$\int dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx + c$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 32.8 Solve

$$(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\quad \dfrac{dy}{dx} = \dfrac{x^2 + 4x - 5}{x+2}$ $\quad$ or $\quad \dfrac{dy}{dx} = \dfrac{x^2 + 4x + 4 - 4 - 5}{x+2}$

or $\quad \dfrac{dy}{dx} = \dfrac{(x+2)^2}{x+2} - \dfrac{9}{x+2}$ $\quad$ or $\quad \dfrac{dy}{dx} = x + 2 - \dfrac{9}{x+2}$

or $\quad dy = \left(x + 2 - \dfrac{9}{x+2} \right) dx$ $\quad$ .....(1)

On integrating both sides of (1), we have

$$\int dy = \int \left(x + 2 - \frac{9}{x+2} \right) dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9 \log|x+2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 32.9 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that y = 1 when x = 0

Solution : The given differential equation is $\dfrac{dy}{dx} = 2x^3 - x$

or $\quad dy = \left(2x^3 - x\right) dx$ $\quad$ ...(1)

On integrating both sides of (1), we get

$$\int dy = \int \left(2x^3 - x\right) dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $\quad y = \dfrac{x^4}{2} - \dfrac{x^2}{2} + C$ $\quad$ .....(2)

where C is an arbitrary constant.

Since y = 1 when x = 0, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \quad \Rightarrow \quad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}\left(x^4 - x^2\right) + 1 \quad \text{or} \quad y = \frac{1}{2}x^2\left(x^2 - 1\right) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\dfrac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or $$\frac{dy}{g(y)} = f\ (x)\ dx \qquad .....(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 32.10 Solve

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

Solution : The given differential equation

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

can be written as $$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{(Here variables have been seperated)}$$

On integrating both sides of (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 32.11 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2+1}$$

when y (0) = 3 (i.e. when x = 0, y = 3).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2+1} \text{ or } \quad \left(3y^2+1\right)dy = 2x\,dx \quad \text{(Variables separated)} \qquad ....(1)$$

If we integrate both sides of (1), we get

$$\int \left(3y^2+1\right)dy = \int 2x\,dx\,,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \qquad ....(2)$$

It is given that, $y(0) = 3$.

$\therefore$ on substituting $y = 3$ and $x = 0$ in (2), we get

$$27 + 3 = C$$

$$\therefore \qquad C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

32.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $\quad y^2 + x^2 \frac{dy}{dx} = xy\frac{dy}{dx}$ $\qquad$ (ii) $\quad \left(x^3 + y^3\right)dx - 3xy^2dy = 0$

(iii) $\quad \frac{dy}{dx} = \frac{x^3 + xy^2}{y^2x}$

In equation (i) above, we see that each term except $\frac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\frac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\frac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2x\frac{dy}{dx} = x^3 + x\,y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$\left(x^2 + 3yx\right)dx - \left(x^3 + x\right)dy = 0$$

is not a homogeneous equation as the degree of the function except $\frac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of 3yx is 2, of x^3 is 3, and of x is 1]

32.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(i) write one variable = v. (the other variable).
(i.e. either y = vx or x = vy)

(ii) reduce the equation to separable form

(iii) solve the equation as we had done earlier.

Example 32.12 Solve

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

Solution : The given differential equation is

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

or $$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \qquad .....(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (1), we have

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \quad \text{or} \quad v + x\frac{dv}{dx} = x^2\left[\frac{1 + 3v + v^2}{x^2}\right]$$

or $$v + x\frac{dv}{dx} = 1 + 3v + v^2 \quad \text{or} \quad x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

or $$x\frac{dv}{dx} = v^2 + 2v + 1 \quad \text{or} \quad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

or $$\frac{dv}{(v+1)^2} = \frac{dx}{x} \qquad .....(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|,$$ where C is an arbitrary constant.

On substituting the value of v, we get

$$\frac{x}{y+x} + \log|x| = C$$ which is the required solution.

Note: If the Homoqeneous differential equation is written in the form $\dfrac{dx}{qy} = \dfrac{P(x,y)}{Q(x,y)}$ then x=vy is substituted to find solution.

32.6.4 Differential Equation of the type $\frac{dy}{qx} + py = Q$, where P and Q are functions of x only.

Consider the equation

$$\frac{dy}{dx} + Py = Q \qquad .....(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int Pdx}$ (called integrating factor) and get

$$e^{\int Pdx}\frac{dy}{dx} + Py\, e^{\int Pdx} = Qe^{\int Pdx}$$

or $$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \qquad .....(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + Py.e^{\int Pdx}\right]$$

On integrating, we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C \qquad .....(3)$$

where C is an arbitrary constant,

or $$y = e^{-\int Pdx}\left[\int Qe^{\int Pdx}dx + C\right]$$

Note : $e^{\int Pdx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int Pdx}\right)$ after the equation has been multiplied by the factor $e^{\int Pdx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int Pdx} = \int Q\left(e^{\int Pdx}\right)dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\quad$ I.F. $= e^{\int P dy}$

and the solution is given by

$$x\,(\text{I.F.}) = \int Q.(\text{I.F.})dy + C$$

Example 32.13 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}, Q = e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x \qquad (x > 0)$

$\therefore$ Solution of the given equation is:

$$yx = \int xe^{-x}dx + C$$

where C is an arbitrary constant

or $\quad xy = -xe^{-x} + \int e^{-x}dx + C$

or $\quad xy = -xe^{-x} - e^{-x} + C$

or $\quad xy = -e^{-x}(x+1) + C$

or $\quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$

Note: In the solution $x > 0$.

Example 32.14 Solve :

$$\sin x \frac{dy}{dx} + y \cos x = 2\sin^2 x \cos x$$

Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y \cos x = 2\sin^2 x \cos x$$

or $\quad \frac{dy}{dx} + y \cot x = 2 \sin x \cos x \qquad .....(1)$

Here $\quad P = \cot x, Q = 2\sin x \cos x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

$\therefore$ Solution of the given equation is:

$$y \sin x = \int 2 \sin^2 x \cos x \, dx + C$$

where C is an arbitrary constant (sin x >0)

or $$y \sin x = \frac{2}{3} \sin^3 x + C,$$ which is the required solution.

Example 32.15 Solve $$\left(1 + y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

Solution : The given differential equation is

$$\left(1 + y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2} \qquad ....(1)$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ Solution of the given equation is:

$$x - e^{\tan^{-1} y} = \left(\frac{e^{\tan^{-1} y}}{1 + y^2}\right) e^{\tan^{-1} y} dy + c.$$

where C is an arbitrary constant let $t = \tan^{-1} y$ therefore $dt = \frac{1}{1 + y^2} dy$

or $$\left(e^{\tan^{-1} y}\right) x = \int e^t . t dt + C,$$

or $$\left(e^{\tan^{-1} y}\right) x = t e^t - \int e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = t e^t - e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \qquad (\text{on putting } t = \tan^{-1} y)$$

or $$x = \tan^{-1} y - 1 + C e^{-\tan^{-1} y}$$

CHECK YOUR PROGRESS 32.2

1. (i) Is $y = \sin x$, a solution of $\frac{d^2y}{dx^2} + y = 0$?

   (ii) Is $y = x^3$, a solution of $x\frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

   State which are particular solutions and which are general solutions.

   (i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$

   (iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$

3. State whether the following differential equations are homogeneous or not ?

   (i) $\frac{dy}{dx} = \frac{x^2}{1 + y^2}$ (ii) $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

   (iii) $(x + 2)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $\left(x^3 - yx^2\right)dy + \left(y^3 + x^3\right)dx = 0$

4. (a) Show that $y = a \sin 2x$ is a solution of $\frac{d^2y}{dx^2} + 4y = 0$

   (b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$

5. The general solution of the differential equation

   $\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.

   Find the particular solution when

   (a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$

6. Solve the following differential equations :

   (a) $\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

   (c) $\left(1 + x^2\right)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$

7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$, when $x = 0$

8. Solve the following differential equations :

(a) $\left(x^2 - yx^2\right)\dfrac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\dfrac{dy}{dx} = xy + x + y + 1$

(c) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ (d) $\dfrac{dy}{dx} = e^{x-y} + e^{-y}x^2$

9. Solve the following differential equations :

(a) $\left(x^2 + y^2\right)dx - 2xy\, dy = 0$ (b) $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$

(c) $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2 - y^2} + y}{x}$ (d) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

10. Solve : $\dfrac{dy}{dx} + y \sec x = \tan x$

11. Solve the following differential equations :

(a) $\left(1 + x^2\right)\dfrac{dy}{dx} + y = \tan^{-1} x$ (b) $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

(c) $x \log x \dfrac{dy}{dx} + y = 2 \log x,\ x > 1$

12. Solve the following differential equations:

(a) $(x + y + 1)\dfrac{dy}{dx} = 1$

[Hint: $\dfrac{dx}{dy} = x + y + 1$ or $\dfrac{dx}{dy} - x = y + 1$ which is of the form $\dfrac{dx}{dy} + Px = Q$]

(b) $\left(x + 2y^2\right)\dfrac{dy}{dx} = y,\ y > 0$ [Hint: $y\dfrac{dx}{dy} = x + 2y^2$ or $\dfrac{dx}{dy} - \dfrac{x}{y} = 2y$]

Example 32.16 Verify if $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - \frac{dy}{dx} - m^2 y = 0$$

Solution : We have,

$$y = e^{m \sin^{-1} x} \qquad .....(1)$$

Differentiating (1) w.r.t. x , we get

$$\frac{dy}{dx} = \frac{m e^{m \sin^{-1} x}}{\sqrt{1 - x^2}} = \frac{my}{\sqrt{1 - x^2}}$$

or $$\sqrt{1 - x^2}\,\frac{dy}{dx} = my$$

Squaring both sides, we get

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

Differentiating both sides, we get

$$-2x\left(\frac{dy}{dx}\right)^2 + 2\left(1-x^2\right)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2m^2y\frac{dy}{dx}$$

or $$-x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = m^2y$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Hence $y = e^{m\sin^{-1}x}$ is the solution of

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Example 32.17 Find the equation of the curve represented by

$$(y - yx)\,dx + (x + xy)\,dy = 0$$

and passing through the point $(1, 1)$.

Solution : The given differential equation is

$$(y - yx)\,dx + (x + xy)\,dy = 0$$

or $$(x + xy)\,dy = (yx - y)\,dx$$

or $$x(1 + y)\,dy = y(x - 1)\,dx$$

or $$\frac{(1+y)}{y}\,dy = \frac{x-1}{x}\,dx \qquad \text{.....(1)}$$

Integrating both sides of equation (1), we get

$$\int\left(\frac{1+y}{y}\right)dy = \int\left(\frac{x-1}{x}\right)dx$$

or $$\int\left(\frac{1}{y}+1\right)dy = \int\left(1-\frac{1}{x}\right)dx \qquad \text{.....(2)}$$

or $$\log y + y = x - \log x + C$$

Since the curve is passing through the point (1,1), therefore, substituting $x = 1$, $y = 1$ in equation (2), we get

$$1 = 1 + C$$

or $$C = 0$$

Thus, the equation of the required curve is

$$\log y + y = x - \log x$$

or $$\log(xy) = x - y$$

Example 32.18 Solve $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

Solution : We have $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

or $$\frac{dy}{dx} = \frac{3e^{3x}\left(e^{-x} + e^x\right)}{e^x + e^{-x}}$$ or $$\frac{dy}{dx} = 3e^{3x}$$

or $$dy = 3e^{3x}dx \quad .....(1)$$

Integrating both sides of (1), we get

$$y = \int 3e^{3x}dx + C$$

where C is an arbitrary constant.

or $$y = 3\frac{e^{3x}}{3} + C$$ or $$y = e^{3x} + C$$

which is required solution.

$$y(1 + ax)\left(1 - a^2\right) = x(1 - ay)\left(1 + a^2\right)$$

which is the required solution.

CHECK YOUR PROGRESS 32.3

1. (a) If $y = \tan^{-1} x$, prove that $\left(1 + x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

   (b) $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

2. (a) Find the equation of the curve represented by

   $\frac{dy}{dx} = xy + x + y + 1$ and passing through the point (2, 0)

   (b) Find the equation of the curve represented by

   $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$ and passing through the point $\left(\frac{\pi}{2}, 2\right)$

3. Solve : $\dfrac{dy}{dx} = \dfrac{4e^{3x} + 4e^{5x}}{e^{x} + e^{-x}}$

4. Solve the following differential equations :

(a) $dx + xdy = e^{-y} \sec^2 y\, dy$

(b) $\left(1 + x^2\right)\dfrac{dy}{dx} - 4x = 3\cot^{-1} x$

(c) $(1 + y)\, xy\, dy = \left(1 - x^2\right)(1 - y)\, dx$

LET US SUM UP

- A differential equation is an equation involving independent variable, dependent variable and the derivatives of dependent variable (and differentials) with respect to independent variable.
- The order of a differential equation is the order of the highest derivative occurring in it.
- The degree of a differential equation is the degree of the highest derivative.
- Degree of a differential equation exists, if it is a polynomial equation in terms of its derivatives.
- A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.
- A linear differential equation is always of the first degree.
- A general solution of a differential equation is that solution which contains as many as the number of arbitrary constants as the order of the differential equation.
- A general solution becomes a particular solution when particular values of the arbitrary constants are determined satisfying the given conditions.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x)$ is obtained by integrating both sides.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x)\, g(y)$ is obtained after separating the variables and integrating both sides.
- The differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous and are of the same degree.
- The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ and then separating the variables.
- The solution of the first order linear equation $\dfrac{dy}{dx} + Py = Q$ is

Notes

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right) dx + C, \quad \text{where C is an arbitrary constant.}$$

The expression $e^{\int P dx}$ is called the integrating factor of the differential equation and is written as I.F. in short.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=9Wfn-WWV1aY

http://www.youtube.com/watch?v=6YRGEsQWZzY

TERMINAL EXERCISE

1. Find the order and degree of the differential equation :

   (a) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4 = 0$ (b) $xdx + ydy = 0$

   (c) $\frac{d^4y}{dx^4} - 4\frac{dy}{dx} + 4y = 5\cos 3x$ (d) $\frac{dy}{dx} = \cos x$

   (e) $x^2\frac{d^2y}{dx^2} - xy\frac{dy}{dx} = y$ (f) $\frac{d^2y}{dx^2} + y = 0$

2. Find which of the following equations are linear and which are non-linear

   (a) $\frac{dy}{dx} = \cos x$ (b) $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x$

   (c) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ (d) $x\frac{dy}{dx} - 4 = x$

   (e) $dx + dy = 0$

3. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

4. Find the differential equation by eliminating a, b, c from

   $y = ax^2 + bx + c$. Write its order and degree.

5. How many constants are contained in the general solution of

   (a) Second order differential equation.

(b) Differential equation of order three.

(c) Differential equation of order five.

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

7. Solve the following differential equations:

(a) $\sin^2 x\dfrac{dy}{dx} = 3\cos x + 4$ (b) $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$

(c) $\dfrac{dy}{dx} + \dfrac{\cos x\sin y}{\cos y} = 0$ (d) $dy + xydx = xdx$

(e) $\dfrac{dy}{dx} + y\tan x = x^m\cos mx$ (f) $\left(1+y^2\right)\dfrac{dx}{dy} = \tan^{-1}y - x$

ANSWERS

CHECK YOUR PROGRESS 32.1

1. Order is 1 and degree is 1.
2. (a) Order 2, degree 1
   (b) Order 2, degree 2
3. (a) Non- linear (b) Linear
   (c) Linear (d) Non-linear
4. $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$
5. (a) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

   (b) $\left(x^2 - 2y^2\right)\left(\dfrac{dy}{dx}\right)^2 - 4xy\dfrac{dy}{dx} - x^2 = 0$

   (c) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$

   (d) $y = (x-3)\dfrac{dy}{dx} + 2$

   (e) $\left(x^2 - y^2\right)\dfrac{dy}{dx} - 2xy = 0$

CHECK YOUR PROGRESS 32.2

1. (i) Yes (ii) No
2. (i), (ii) and (iv) are particular solutions (iii) is the general solution
3. (ii), (iv) are homogeneous
5. (a) $y = \tan x$
   (b) $y = \tan x + \sqrt{3}$
6. (a) $y = \dfrac{1}{6}x^6 \tan^{-1}\left(x^3\right) - \dfrac{1}{6}x^3 + \dfrac{1}{6}\tan^{-1}\left(x^3\right) + C$

   (b) $y = \dfrac{1}{5}\cos^5 x - \dfrac{1}{3}\cos^3 x + (x-1)e^x + C$

(c) $y = \frac{1}{2}\log\left|x^2+1\right| + C$ (d) $y = \frac{1}{3}x^3 - \frac{1}{3}\cos 3x + C$

7. $y = -4e^{-x} + 7$

8. (a) $\log\left|\frac{x}{y}\right| = C + \frac{1}{x} + \frac{1}{y}$ (b) $\log|y+1| = x + \frac{x^2}{2} + C$

(c) $\tan x \tan y = C$ (d) $e^y = e^x + \frac{x^3}{3} + C$

9. (a) $x = C\left(x^2 - y^2\right)$ (b) $x + cy = y \log|x|$

(c) $\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + C$ (d) $\tan\frac{y}{2x} = Cx$

10. $y(\sec x + \tan x) = \sec x + \tan x - x + C$

11. (a) $y = \tan^{-1} x - 1 + Ce^{-\tan x}$ (b) $y = \tan x - 1 + Ce^{-\tan x}$

(c) $y = \log x + \frac{C}{\log x}$

12. (a) $x = Ce^y - (y+2)$ (b) $x = y^2 + Cy$

CHECK YOUR PROGRESS 32.3

2. (a) $\log(y+1) = \frac{1}{2}x^2 + x - 4$ (b) $y \sin x + 5e^{\cos x} = 7$

3. $y = \frac{4}{5}e^{5x} + C$

4. (a) $x = e^{-y}(C + \tan y)$ (b) $y = 2\log\left|1+x^2\right| - \frac{3}{2}\left(\cot^{-1} x\right)^2 + C$

(c) $\log x + 2\log|1-y| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$

TERMINAL EXERCISE

1. (a) Order 2, degree 3 (b) Order 1, degree 1

(c) Order 4, degree 1 (d) Order 1, degree 1

(e) Order 2, degree 1 (f) Order 2, degree 1

2. (a), (d), (e) are linear; (b), (c) are non-linear

3. $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\left(\frac{dy}{dx}\right) - x^2 = 0$

4. $\dfrac{d^3y}{dx^3} = 0$, Order 3, degree 1.

5. (a) Two (b) Three

   (c) Five

7. (a) $y + 3 \operatorname{cosec} x + 4 \cot x = C$ (b) $e^y = e^x + \dfrac{x^3}{3} + C$

   (c) $\sin y = Ce^{-\sin x}$ (d) $\log(1 - y) + \dfrac{x^2}{2} = C$

   (e) $y = \dfrac{x^{m+1}}{m+1} \cos x + C \cos x$ (f) $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$

33

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y).

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5cm and the position of the point where it strikes the ground is given by (5, 4), one way of describing the position of ball in space is with the help of these three numbers (5, 4, 2.5).

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

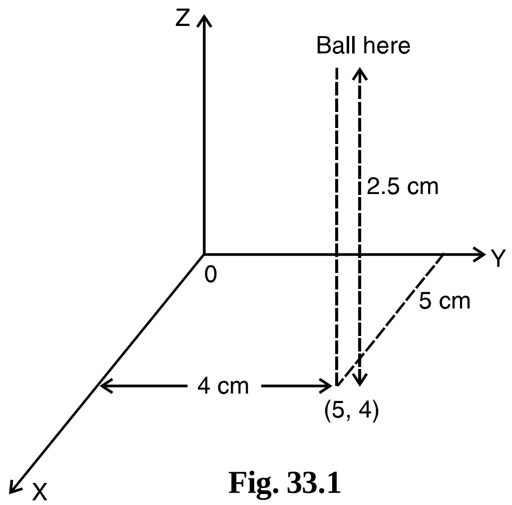


Fig. 33.1

OBJECTIVES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;
- find the coordinates of a point which divides the line segment joining twogiven points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;

MODULE - IX
Vectors and three dimensional Geometry

Notes

- find the projection of a line segment on another line; and
- find the condition of prependicularity and parallelism of two lines in space.

EXPECTED BACKGROUND KNOWLEDGE

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

33.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the co-ordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

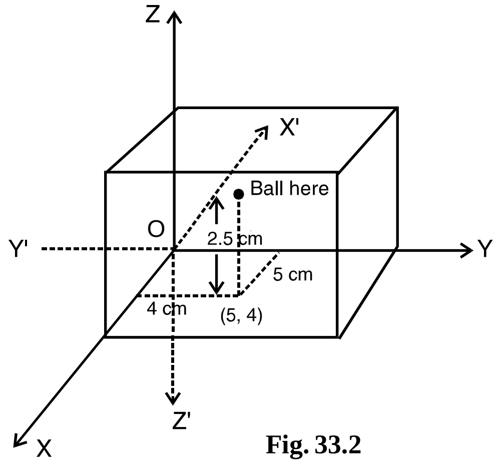


Fig. 33.2

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines X'OX, Y'OY and Z'OZ through O. The point O is called the origin of the co-ordinate system and the lines X'OX, Y'OY and Z'OZ are called the X-axis, the Y-axis and the Z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 33.2. The plane determined by the X-axis and the Y-axis is called XY-plane (XOY plane) and similarly, YZ-plane (YOZ-plane) and ZX-plane (ZOX-plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

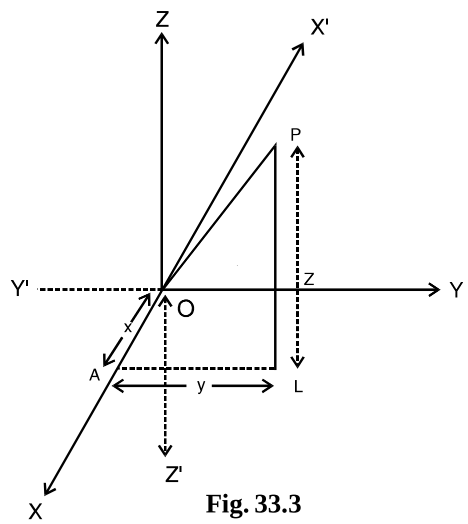


Fig. 33.3

Let P be any point is space. Through P draw perpendicular PL on XY-plane meeting this plane at L. Through L draw a line LA parallel to OY cutting OX in A. If we write OZ = x, AL = y and LP = z, then (x, y, z) are the co-ordinates of the point P.

Again, if we complete a reactangular parallelopiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

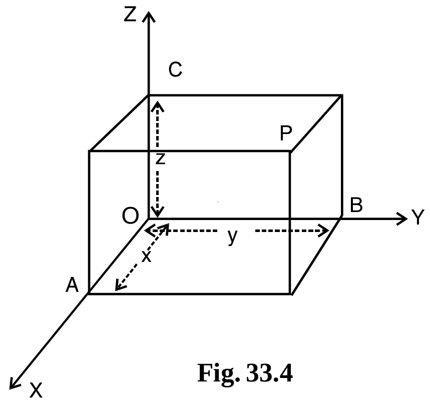


Fig. 33.4

Note : You may note that in Fig. 33.4

(i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.

(ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.

(iii) The x co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimentional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX'YZ, OXY'Z, OXYZ', OXY'Z', OX'YZ', OX'Y'Z and OX'Y'Z'.

2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are equal to those of P. If P be (a, b, c), the other points are (–a, b, c), (a, –b, c), (a, b, –c), (a, –b, –c), (–a, b, –c), (–a, –b, c) and (–a, –b, –c) respectively in order in the octants referred in (i).

3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form (a, b, 0), (0, b, c) and (a, 0, c) respectively.

4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.

5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector $\overrightarrow{OP}$.

Example 33.1 Name the octant wherein the given points lies :

(a) (2, 6, 8) (b) (–1, 2, 3) (c) (–2, –5, 1)

(d) (–3, 1, –2)   (e) (–6, –1, –2)

Solution :

(a) Since all the co-ordinates are positive, $\therefore$ (2, 6, 8) lies in the octant OXYZ.

(b) Since x is negative and y and z are positive, $\therefore$ (–1, 2, 3) lies in the octant OX'YZ.

(c) Since x and y both are negative and z is positive $\therefore$ (–2, –5, 1) lies in the octant OX'Y'Z.

(d) (–3, 1, –2) lies in octant OX'YZ'.

(e) Since x, y and z are all negative $\therefore$ (–6, –1, –2) lies in the octant OX'Y'Z'.

CHECK YOUR PROGRESS 33.1

1. Name the octant wherein the given points lies :

   (a) (–4, 2, 5)   (b) (4, 3, –6)   (c) (–2, 1, –3)

   (d) (1, –1, 1)   (f) (8, 9, –10)

33.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

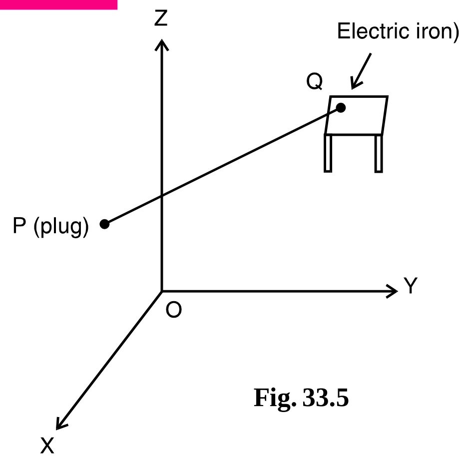


Fig. 33.5

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

PK is perpendicular to the line KQ.

$\therefore$ From the right-angled triangle PKQ, right angled at K,

We have $PQ^2 = PK^2 + KQ^2$

Again from the right angled triangle PKL right angled at L,

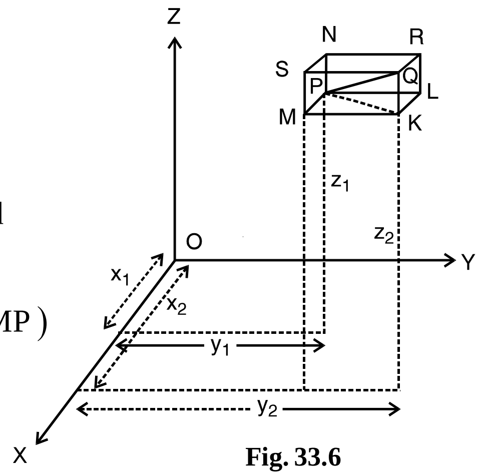


Fig. 33.6

$$PK^2 = KL^2 + PL^2 = MP^2 + PL^2 \qquad (\because KL = MP)$$

$$\therefore \qquad PQ^2 = MP^2 + PL^2 + KQ^2 \quad ...(i)$$

The edges MP, PL and KQ are parallel to the co-ordinate axes.

Now, the distance of the point P from the plane $YOZ = x_1$ and the distance of Q and M from

YOZ plane $= x_2$

$\therefore \qquad MP = |x_2 - x_1|$

Similarly, $\qquad PL = |y_2 - y_1|$ and $KQ = |z_2 - z_1|$

$\therefore \qquad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad$ ....[From (i)]

or $\qquad \mathbf{|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$

Corollary : Distance of a Point from the Origin

If the point $Q(x_2, y_2, z_2)$ coincides with the origin (0, 0, 0), then $x_2 = y_2 = z_2 = 0$

$\therefore$ The distance of P from the origin is

$$|OP| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

Example 33.2 Find the distance between the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let $P(2, 5, -4)$ and $Q(8, 2, -6)$ be the two given points.

$$\therefore \qquad |PQ| = \sqrt{(8-2)^2 + (2-5)^2 + (-6+4)^2}$$

$$= \sqrt{36 + 9 + 4}$$

$$= \sqrt{49}$$

$$= 7$$

$\therefore$ The distance between the given points is 7 units.

Example 33.3 Prove that the points $(-2, 4, -3)$, $(4, -3, -2)$ and $(-3, -2, 4)$ are the vertices of an equilateral triangle.

Solution : Let $A(-2, 4, -3)$, $B(4, -3, -2)$ and $C(-3, -2, 4)$ be the three given points.

Now $\qquad |AB| = \sqrt{(4+2)^2 + (-3-4)^2 + (-2+3)^2}$

$$= \sqrt{36 + 49 + 1} = \sqrt{86}$$

$$|BC| = \sqrt{(-3-4)^2 + (-2+3)^2 + (4+2)^2} = \sqrt{86}$$

$$|CA| = \sqrt{(-2+3)^2 + (4+2)^2 + (-3-4)^2} = \sqrt{86}$$

Since $|AB| = |BC| = |CA|$, $\Delta\ ABC$ is an equilateral triangle.

Example 33.4 Verify whether the following points form a triangle or not :

(a) $A(-1, 2, 3)$ $B(1, 4, 5)$ and $C(5, 4, 0)$

(b) $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$

Solution :

(a) $$|AB| = \sqrt{(1+1)^2 + (4-2)^2 + (5-3)^2}$$

$$= \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3} \quad = 3.464 \text{ (approx.)}$$

$$|BC| = \sqrt{(5-1)^2 + (4-4)^2 + (0-5)^2}$$

$$= \sqrt{16 + 0 + 25} = \sqrt{41} \quad = 6.4 \text{ (approx.)}$$

and $$|AC| = \sqrt{(5+1)^2 + (4-2)^2 + (0-3)^2}$$

$$= \sqrt{36 + 4 + 9} = 7$$

$\therefore$ $|AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|$, $|AB| + |AC| > |BC|$

and $|BC| + |AC| > |AB|$.

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$ be denoted by P, Q and R respectively,

then $$|PQ| = \sqrt{(1-2)^2 + (2+3)^2 + (4-3)^2}$$

$$= \sqrt{1 + 25 + 1} = 3\sqrt{3}$$

$$|QR| = \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2}$$

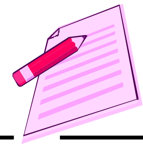
$$= \sqrt{4 + 100 + 4} = 6\sqrt{3}$$

$$|PR| = \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2}$$

$$= \sqrt{1 + 25 + 1}$$

$$= 3\sqrt{3}$$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$. Hence the given points do not form a triangle. In fact the points lie on a line.



Example 33.5 Show that the points $A(1, 2, -2)$, $B(2, 3, -4)$ and $C(3, 4, -3)$ form a right angled triangle.

Solution : $AB^2 = (2-1)^2 + (3-2)^2 + (-4+2)^2 = 1+1+4 = 6$

$BC^2 = (3-2)^2 + (4-3)^2 + (-3+4)^2 = 1+1+1 = 3$

and $AC^2 = (3-1)^2 + (4-2)^2 + (-3+2)^2 = 4+4+1 = 9$

We observe that $AB^2 + BC^2 = 6 + 3 = 9 = AC^2$

$\therefore$ ΔABC is a right angled triangle.

Hence the given points form a right angled triangle.

Example 33.6 Prove that the points $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$ and $D(2, 6, 2)$ are vertices of a square.

Solution : Here,
$$AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$$
$$= \sqrt{4+1+4} = 3 \text{ units}$$
$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$
$$= \sqrt{4+4+1} = 3 \text{ units}$$
$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$
$$= \sqrt{4+1+4} = 3 \text{ units}$$
and
$$DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$$
$$= \sqrt{4+4+1} = 3 \text{ units}$$
$$\therefore \quad AB = BC = CD = DA$$
Now
$$AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$$
$$= 16+1+1 = 18$$
$$\therefore \quad AB^2 + BC^2 = 3^2 + 3^2 = 18 = AC^2$$
$$\therefore \quad \angle B = 90^\circ$$

$\therefore$ In quadrilateral ABCD, $AB = BC = CD = DA$ and $\angle B = 90^\circ$

$\therefore$ ABCD is a square.

MODULE - IX
Vectors and three dimensional Geometry

Notes

CHECK YOUR PROGRESS 33.2

1. Find the distance between the following points :

   (a) $(4, 3, -6)$ and $(-2, 1, -3)$ (b) $(-3, 1, -2)$ and $(-3, -1, 2)$

   (c) $(0, 0, 0)$ and $(-1, 1, 1)$

2. Show that if the distance between the points $(5, -1, 7)$ and $(a, 5, 1)$ is 9 units, "a" must be either 2 or 8.

3. Show that the triangle formed by the points (a, b, c), (b, c, a) and (c, a, b) is equilateral.

4. Show that the the points $(-1, 0, -4)$, $(0, 1, -6)$ and $(1, 2, -5)$ form a right angled tringle.

5. Show that the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of an isosceles right-angled triangle.

6. Show that the points $(3, -1, 2)$, $(5, -2, -3)$, $(-2, 4, 1)$ and $(-4, 5, 6)$ form a parallelogram.

7. Show that the points $(2, 2, 2)$, $(-4, 8, 2)$, $(-2, 10, 10)$ and $(4, 4, 10)$ form a square.

8. Show that in each of the following cases the three points are collinear :

   (a) $(-3, 2, 4)$, $(-1, 5, 9)$ and $(1, 8, 14)$

   (b) $(5, 4, 2)$, $(6, 2, -1)$ and $(8, -2, -7)$

   (c) $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$

33.3 COORDINATES OF A POINT OF DIVISION OF A LINE SEGMENT

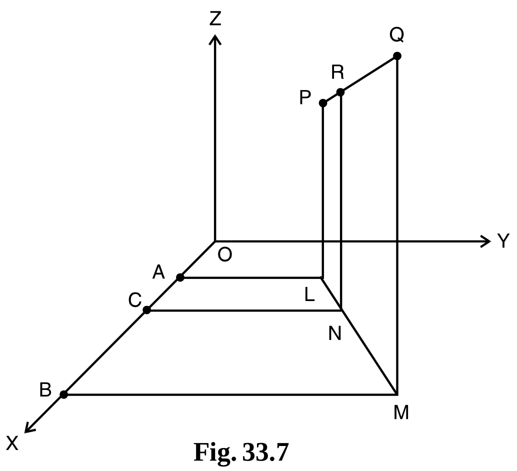


Fig. 33.7

Let the point R (x, y, z) divide PQ in the ratio $l : m$ internally.

Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2). From points P, R and Q, draw PL, RN and QM perpendiculars to the XY-plane.

Draw LA, NC and MB perpendiculars to OX.

Now $AC = OC - OA = x - x_1$ and $BC = OB - OC = x_2 - x$

Also we have, $\dfrac{AC}{BC} = \dfrac{LN}{NM} = \dfrac{PR}{RQ} = \dfrac{l}{m}$

$$\therefore \qquad \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

$$\text{or} \qquad mx - mx_1 = lx_2 - lx$$

$$\text{or} \qquad (l + m)x = lx_2 + mx_1$$

$$\text{or} \qquad x = \frac{lx_2 - mx_1}{l + m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

we get $y = \dfrac{ly_2 + my_1}{l + m}$ and $z = \dfrac{lz_2 + mz_1}{l + m}$

$$\therefore \qquad \text{R is the point } \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m}\right)$$

If $\lambda = \dfrac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right), \lambda + 1 \neq 0$$

It is clear that to every value of λ, there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ. If λ is postive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda : 1$.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio $l : m$ are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m}\right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 33.7 Find the co-ordinates of the point which divides the line segment joining the points $(2, -4, 3)$ and $(-4, 5, -6)$ in the ratio 2 : 1 internally.

Solution : Let $A(2, -4, 3)$, $B(-4, 5, -6)$ be the two points.

Let P (x, y, z) divides AB in the ratio 2 : 1.

$$\therefore \qquad x = \frac{2(-4)+1.2}{2+1} = -2, \quad y = \frac{2.5+1(-4)}{2+1} = 2$$

$$\text{and} \qquad z = \frac{2(-6)+1.3}{2+1} = -3$$

Thus, the co-ordinates of P are $(-2, 2, -3)$

Example 33.8 Find the point which divides the line segment joining the points $(-1, -3, 2)$ and $(1, -1, 2)$ externally in the ratio 2 : 3.

Solution : Let the points $(-1, -3, 2)$ and $(1, -1, 2)$ be denoted by P and Q respectively.

Let R (x, y, z) divide PQ externally in the ratio 2 : 3. Then

$$x = \frac{2(1)-3(-1)}{2-3} = -5, \quad y = \frac{2(-1)-3(-3)}{2-3} = -7$$

$$\text{and} \qquad z = \frac{2(2)-3(2)}{2-3} = 2$$

Thus, the co-ordinates of R are $(-5, -7, 2)$.

Example 33.9 Find the ratio in which the line segment joining the points $(2 -3, 5)$ and $(7, 1, 3)$ is divided by the XY-plane.

Solution : Let the required ratio in which the line segment is divided be $l : m$.

The co-ordinates of the point are $\left(\frac{7l+2m}{l+m}, \frac{l-3m}{l+m}, \frac{3l+5m}{l+m}\right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

$$\text{i.e.,} \qquad \frac{3l+5m}{l+m} = 0 \text{ or } \frac{l}{m} = -\frac{5}{3}$$

Hence the XY-plane divides the join of given points in the ratio 5 : 3 externally.

CHECK YOUR PROGRESS 33.3

1. Find the co-ordinates of the point which divides the line segment joining two points $(2, -5, 3)$ and $(-3, 5, -2)$ internally in the ratio 1 : 4.
2. Find the coordinates of points which divide the join of the points $(2, -3, 1)$ and $(3, 4, -5)$ internally and externally in the ratio 3 : 2.
3. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, 5, -4)$ is divided by the YZ-plane.
4. Show that the YZ-plane divides the line segment joining the points $(3, 5, -7)$ and $(-2, 1, 8)$ in the ration 3 : 2 at the point $\left(0, \frac{13}{5}, 2\right)$.
5. Show that the ratios in which the co-ordinate planes divide the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$ are 2 : 3, 4 : 5 (internally) and 7 : 8 (externally).
6. Find the co-ordinates of a point R which divides the line segment $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio 2 : 1. Verify that Q is the mid-point of PR.

LET US SUM UP

- For a given point P (x, y, z) in space with reference to reactangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then
  OA = x, OB = y and OC = z where O is the origin.
  Converswly, given any three numbers, x, y and z we can find a unique point in space whose co-ordinates are (x, y, z).
- The distance PQ between the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

  $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$.
- The co-ordinates of the point which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $l : m$

  (a) internally are $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m}\right)$

  (b) externally are $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m}\right)$

  In particular, the co-ordinates of the mid-point of PQ are

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

SUPPORTIVE WEB SITES

http://www.mathguru.com/level3/introduction-to-three-dimensional-geometry-

http://www.goiit.com/posts/show/0/content-3-d-geometry-804299.htm

http://www.askiitians.com/iit-jee-3d-geometry

TERMINAL EXERCISE

1. Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right-angled triangle.

2. Prove that the points P, Q and R, whose co-ordinates are respectively $(3, 2, -4)$, (5, 4, –6) and (9, 8, –10) are collinear and find the ratio in which Q divides PR.

3. Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

4. Show that the points (4, 7, 8), (2, 3, 4), (–1, –2, 1) and (1, 2, 5) are the vertices of a parallelogram.

5. Three vertices of a parallelogram ABCD are A (3, –4, 7), B (5, 3, –2) and C (1, 2, –3). Find the fourth vertex D.

ANSWERS

CHECK YOUR PROGRESS 33.1

1. (a) OX' YZ (b) OXYZ' (c) OX' YZ'
   (d) OXY'Z (e) OXYZ'

CHECK YOUR PROGRESS 33.2

1. (a) 7 (b) $2\sqrt{5}$ (c) $\sqrt{3}$

CHECK YOUR PROGRESS 33.3

1. $(1, -3, 2)$ 2. $\left(\frac{13}{5}, \frac{6}{5}, -\frac{13}{5}\right)$; $(5, 18, -17)$

3. $-2:3$ 6. $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

TERMINAL EXERCISE

2. $1:2$ 5. $(-1, -5, 6)$

34

VECTORS

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In day to day life situations, we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively. We also come across physical quantities such as dispacement, velocity, acceleration, momentum etc. which are of a difficult type.

Let us consider the following situation. Let A, B, C and D be four points equidistant (say 5 km each) from a fixed point P. If you are asked to travel 5 km from the fixed point P, you may reach either A, B, C, or D. Therefore, only starting (fixed point) and distance covered are not sufficient to describe the destination. We need to specify end point (terminal point) also. This idea of terminal point from the fixed point gives rise to the need for direction.

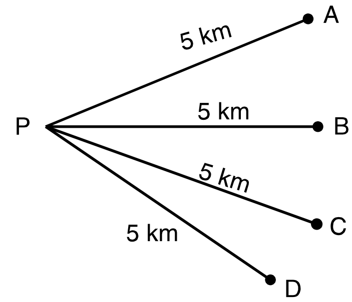


Fig. 34.1

Consider another example of a moving ball. If we wish to predict the position of the ball at any time what are the basics we must know to make such a prediction?

Let the ball be initially at a certain point A. If it were known that the ball travels in a straight line at a speed of 5cm/sec, can we predict its position after 3 seconds ? Obviously not. Perhaps we may conclude that the ball would be 15 cm away from the point A and therefore it will be at some point on the circle with A as its centre and radius 15 cms. So, the mere knowledge of speed and time taken are not sufficient to predict the position of the ball. However, if we know that the ball moves in a direction due east from A at a speed of 5cm/sec., then we shall be able to say that after 3 seconds, the ball must be precisely at the point P which is 15 cms in the direction east of A.

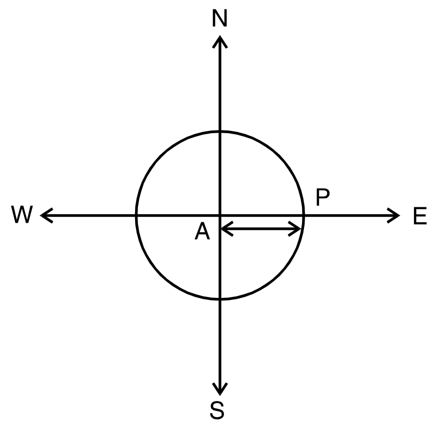


Fig. 34.2

Thus, to study the displacement of a ball after time t (3 seconds), we need to know the magnitude of its speed (i.e. 5 cm/sec) and also its direction (east of A)

In this lesson we will be dealing with quantities which have magnitude only, called scalars and the quantities which have both magnitude and direction, called vectors. We will represent vectors as directed line segments and determine their magnitudes and directions. We will study about various types of vectors and perform operations on vectors with properties thereof. We will also acquaint ourselves with position vector of a point w.r.t. some origin of reference. We will find out the resolved parts of a vector, in two and three dimensions, along two and three mutually perpendicular directions respectively. We will also derive section formula and apply that to problems. We will also define scalar and vector products of two vectors.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the need of mentioning direction;
- define a scalar and a vector;
- distinguish between scalar and vactor;
- represent vectors as directed line segment;
- determine the magnitude and direction of a vector;
- classify different types of vectors-null and unit vectors;
- define equality of two vectors;
- define the position vector of a point;
- add and subtract vectors;
- multiply a given vector by a scalar;
- state and use the properties of various operations on vectors;
- comprehend the three dimensional space;
- resolve a vector along two or three mutually prependicular axes;
- derive and use section formula; and
- define scalar (dot) and vector (cross) product of two vectors.
- define and understand direction cosines and direction ratios of a vector.
- define triple product of vectors.
- understand scalar triple product of vectors and apply it to find volume of a rectangular parallelopiped.
- understand coplanarity of four points.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of plane and coordinate geometry.
- Knowledge of Trigonometry.

34.1 SCALARS AND VECTORS

A physical quantity which can be represented by a number only is known as a scalar i.e, quantities which have only magnitude. Time, mass, length, speed, temperature, volume, quantity of heat, work done etc. are all ***scalars***.

The physical quantities which have magnitude as well as direction are known as vectors. Displacement, velocity, acceleration, force, weight etc. are all examples of ***vectors***.

34.2 VECTOR AS A DIRECTED LINE SEGMENT

You may recall that a line segment is a portion of a given line with two end points. Take any line l (called a support). The portion of L with end points A and B is called a line segment. The line segment AB along with direction from A to B is written as $\overrightarrow{AB}$ and is called a directed line segment. A and B are respectively called the initial point and terminal point of the vector $\overrightarrow{AB}$.

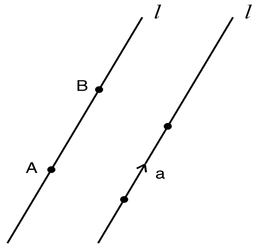


Fig. 34.3

The length AB is called the ***magnitude*** or ***modulus*** of $\overrightarrow{AB}$ and is denoted by $|\overrightarrow{AB}|$. In other words the length $AB = |\overrightarrow{AB}|$.

Scalars are usually represented by a, b, c etc. whereas vectors are usually denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. Magnitude of a vector $\vec{a}$ i.e., $|\vec{a}|$ is usually denoted by 'a'.

34.3 CLASSIFICATION OF VECTORS

34.3.1 Zero Vector (Null Vector)

A vector whose magnitude is zero is called a zero vector or ***null vector***. Zero vector has not definite direction. $\overrightarrow{AA}$, $\overrightarrow{BB}$ are zero vectors. Zero vectors is also denoted by $\vec{0}$ to distinguish it from the scalar 0.

34.3.2 Unit Vector

A vector whose magnitude is unity is called a ***unit vector***. So for a unit vector $\vec{a}$, $|\vec{a}| = 1$. A unit vector is usually denoted by $\hat{a}$. Thus, $\vec{a} = |\vec{a}|\,\hat{a}$.

34.3.3 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude. i.e., $|\vec{a}| = |\vec{b}|$ and the same direction as shown in Fig. 14.4. Symbolically, it is denoted by $\vec{a} = \vec{b}$.

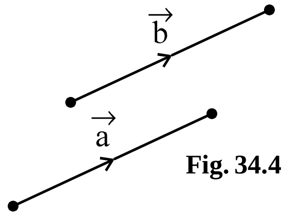


Fig. 34.4

Remark : Two vectors may be equal even if they have different parallel lines of support.

34.3.4 Like Vectors

Vectors are said to be like if they have same direction whatever be their magnitudes. In the adjoining Fig. 14.5, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are like vectors, although their magnitudes are not same.

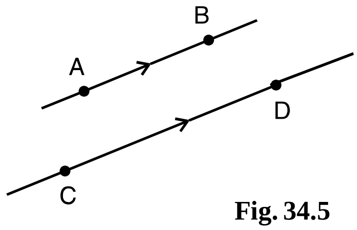


Fig. 34.5

34.3.5 Negative of a Vector

$\overrightarrow{BA}$ is called the ***negative of the vector*** $\overrightarrow{AB}$, when they have the same magnitude but opposite directions.

i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$

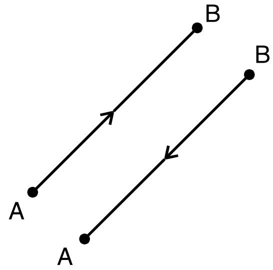


Fig. 34.6

34.3.6 Co-initial Vectors

Two or more vectors having the same initial point are called ***Co-initial vectors***.

In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are co-initial vectors with the same initial point A.

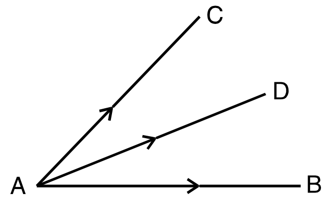


Fig. 34.7

34.3.7 Collinear Vectors

Vectors are said to be collinear when they are parallel to the same line whatever be their magnitudes. In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are collinear vectors. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are also ***collinear***.

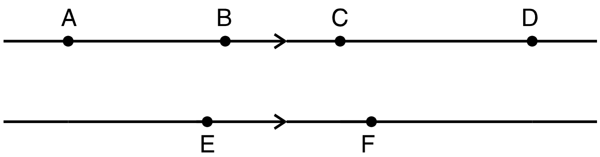


Fig. 34.8

34.3.8 Co-planar Vectors

Vectors are said to be co-planar when they are parallel to the same plane. In the adjoining figure $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-planar. Whereas $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane, $\vec{d}$ is parallel to the plane of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

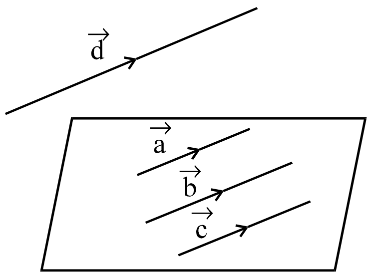


Fig. 34.9

Note : (i) A zero vector can be made to be collinear with any vector.

(ii) Any two vectors are always co-planar.

Example 34.1 State which of the following are scalars and which are vectors. Give reasons.

(a)	Mass	(b)	Weight	(c)	Momentum
(d)	Temperature	(e)	Force	(f)	Density

Solution : (a), (d) and (f) are scalars because these have only magnitude while (b), (c) and (e) are vectors as these have magnitude and direction as well.

Example 34.2 Represent graphically

(a) a force 40N in a direction 60° north of east.

(b) a force of 30N in a direction 40° east of north.

Solution :

(a) (b)

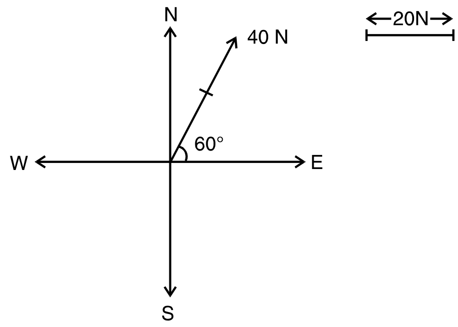


Fig. 34.10

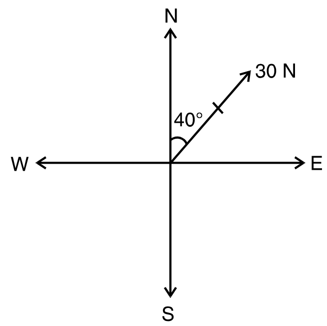


Fig. 34.11

CHECK YOUR PROGRESS 34.1

1. Which of the following is a scalar quantity ?

   (a) Displacement (b) Velocity (c) Force (d) Length.

2. Which of the following is a vector quantity ?

   (a) Mass (b) force (c) time (d) tempertaure

3. You are given a displacement vector of 5 cm due east. Show by a diagram the corresponding negative vector.

4. Distinguish between like and equal vectors.

5. Represent graphically

   (a) a force 60 Newton is a direction 60° west of north.

   (b) a force 100 Newton in a direction 45° north of west.

34.4 ADDITION OF VECTORS

Recall that you have learnt four fundamental operations viz. addition, subtraction, multiplication and division on numbers. The addition (subtraction) of vectors is different from that of numbers (scalars).

In fact, there is the concept of resultant of two vectors (these could be two velocities, two forces etc.) We illustrate this with the help of the following example :

Let us take the case of a boat-man trying to cross a river in a boat and reach a place directly in the line of start. Even if he starts in a direction perpendicular to the bank, the water current carries him to a place different from the place he desired., which is an example of the effect of two velocities resulting in a third one called the resultant velocity.

Thus, two vectors with magnitudes 3 and 4 may not result, on addition, in a vector with magnitude 7. It will depend on the direction of the two vectors i.e., on the angle between them. The addition of vectors is done in accordance with the triangle law of addition of vectors.

34.4.1 Triangle Law of Addition of Vectors

A vector whose effect is equal to the resultant (or combined) effect of two vectors is defined as the resultant or sum of these vectors. This is done by the triangle law of addition of vectors.

In the adjoining Fig. 32.12 vector $\overrightarrow{OB}$ is the resultant or sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ and is written as

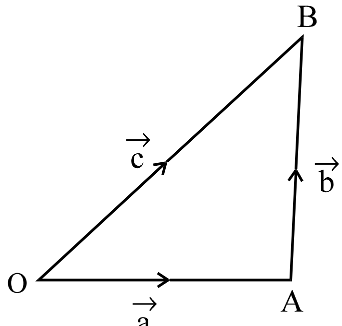


Fig. 34.12

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

i.e. $$\vec{a} + \vec{b} = \overrightarrow{OB} = \vec{c}$$

You may note that the terminal point of vector $\vec{a}$ is the initial point of vector $\vec{b}$ and the initial point of $\vec{a} + \vec{b}$ is the initial point of $\vec{a}$ and its terminal point is the terminal point of $\vec{b}$.

34.4.2 Addition of more than two Vectors

Addition of more then two vectors is shown in the adjoining figure

$$\vec{a} + \vec{b} + \vec{c} + \vec{d}$$
$$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AD} + \overrightarrow{DE}$$
$$= \overrightarrow{AE}$$

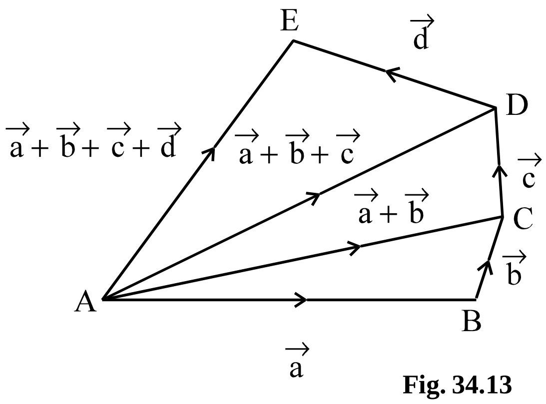


Fig. 34.13

The vector $\overrightarrow{AE}$ is called the sum or the resultant vector of the given vectors.

34.4.3 Parallelogram Law of Addition of Vectors

Recall that two vectors are equal when their magnitude and direction are the same. But they could be parallel [refer to Fig. 14.14].

See the parallelogram OABC in the adjoining figure :

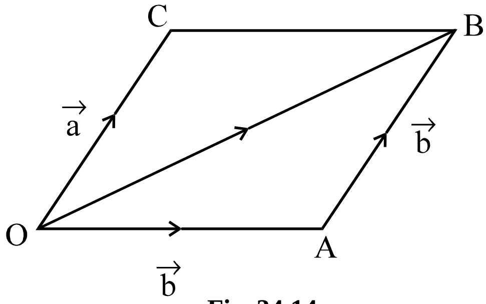


Fig. 34.14

We have, $$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

But $$\overrightarrow{AB} = \overrightarrow{OC}$$

$$\therefore \quad \overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$$

which is the parallelogram law of addition of vectors. **If two vectors are represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal through the common point of the adjacent sides.**

34.4.4 Negative of a Vector

For any vector $\vec{a} = \overrightarrow{OA}$, the negative of $\vec{a}$ is represented by $\overrightarrow{AO}$. The negative of $\overrightarrow{AO}$ is the same as $\overrightarrow{OA}$. Thus, $|\overrightarrow{OA}| = |\overrightarrow{AO}| = |\vec{a}|$ and $\overrightarrow{OA} = -\overrightarrow{AO}$. It follows from definition that for any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = \vec{0}$.

34.4.5 The Difference of Two Given Vectors

For two given vectors $\vec{a}$ and $\vec{b}$, the difference $\vec{a} - \vec{b}$ is defined as the sum of $\vec{a}$ and the negative of the vector $\vec{b}$. i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

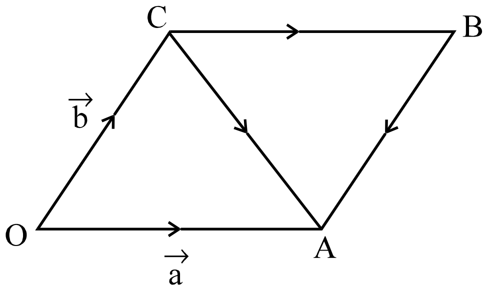


Fig. 34.15

In the adjoining figure if $\overrightarrow{OA} = \vec{a}$ then, in the parallelogram OABC, $\overrightarrow{CB} = \vec{a}$

and $\qquad \overrightarrow{BA} = -\vec{b}$

$$\therefore \quad \overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = \vec{a} - \vec{b}$$

Example 34.3 When is the sum of two non-zero vectors zero ?

Solution : The sum of two non-zero vectors is zero when they have the same magnitude but opposite direction.

Example 34.4 Show by a diagram $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Solution : From the adjoining figure, resultant

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \vec{a} + \vec{b} \qquad \text{....(i)}$$

Complete the parallelogram OABC

$$\overrightarrow{OC} = \overrightarrow{AB} = \vec{b},\ \overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$$

$$\therefore \quad \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$$

$$= \vec{b} + \vec{a} \qquad \text{....(ii)}$$

$$\therefore \quad \vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{[From (i) and (ii)]}$$

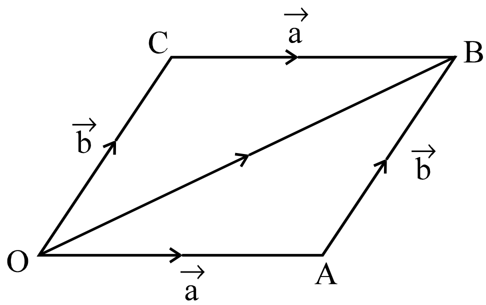


Fig. 34.16

CHECK YOUR PROGRESS 34.2

1. The diagonals of the parallelogram ABCD intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

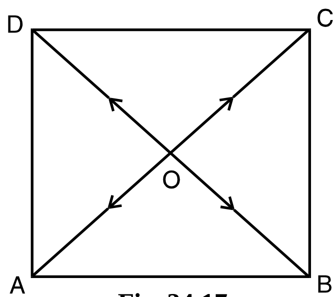


Fig. 34.17

2. The medians of the triangle ABC intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

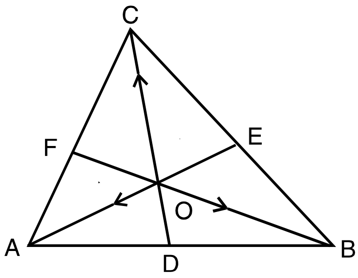


Fig. 34.18

34.5 POSITION VECTOR OF A POINT

We fix an arbitrary point O in space. Given any point P in space, we join it to O to get the vector $\overrightarrow{OP}$. This is called the position vector of the point P with respect to O, called the ***origin of reference***. Thus, to each given point in space there corresponds a unique position vector with respect to a given origin of reference. Conversely, given an origin of reference O, to each vector with the initial point O, corresponds a point namely, its terminal point in space.

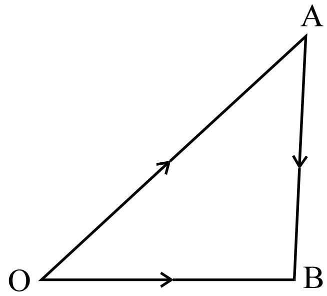


Fig. 34.19

Consider a vector AB. Let O be the origin of reference.

Then $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

or $\overrightarrow{AB}$ = (Position vector of terminal point B)–(Position vector of initial point A)

34.6 MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a non-zero vector $\vec{a}$ by the scalar $x \neq 0$ is a vector whose length is equal to $|x||\vec{a}|$ and whose direction is the same as that of $\vec{a}$ if $x > 0$ and opposite to that of $\vec{a}$ if $x < 0$.

The product of the vector $\vec{a}$ by the scalar x is denoted by $x\vec{a}$.

The product of vector $\vec{a}$ by the scalar 0 is the vector $\vec{\mathbf{0}}$.

By the definition it follows that the product of a zero vector by any non-zero scalar is the zero vector i.e., $x\vec{0} = \vec{0}$; also $0\vec{a} = \vec{0}$.

Laws of multiplication of vectors : If $\vec{a}$ and $\vec{b}$ are vectors and x, y are scalars, then

(i) $x(y\vec{a}) = (xy)\vec{a}$

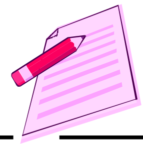
(ii) $x\vec{a} + y\vec{a} = (x+y)\vec{a}$

(iii) $x\vec{a} + x\vec{b} = x(\vec{a} + \vec{b})$

(iv) $0\vec{a} + x\vec{0} = \vec{0}$

Recall that two collinear vectors have the same direction but may have different magnitudes. This implies that $\vec{a}$ is collinear with a non-zero vector $\vec{b}$ if and only if there exists a number (scalar) x such that

$$\vec{a} = x\vec{b}$$



Theorem A necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ to be collinear is that there exist scalars x and y (not both zero simultaneously) such that $x\,\vec{a} + y\,\vec{b} = \vec{0}$.

The Condition is necessary

Proof : Let $\vec{a}$ and $\vec{b}$ be collinear. Then there exists a scalar l such that $\vec{a} = l\,\vec{b}$

i.e., $\qquad \vec{a} + (-l)\,\vec{b} = \vec{0}$

$\therefore$ We are able to find scalars x (= 1) and $y\ (= -l)$ such that $x\,\vec{a} + y\,\vec{b} = \vec{0}$

Note that the scalar 1 is non-zero.

The Condition is sufficient

It is now given that $\qquad x\,\vec{a} + y\,\vec{b} = \vec{0}$ and $x \neq 0$ and $y \neq 0$ simultaneously.

We may assume that $y \neq 0$

$$\therefore \qquad y\,\vec{b} = -x\,\vec{a} \quad \Rightarrow \quad \vec{b} = -\frac{x}{y}\vec{a} \quad \text{i.e., } \vec{b} \text{ and } \vec{a} \text{ are collinear.}$$

Corollary : Two vectors $\vec{a}$ and $\vec{b}$ are non-collinear if and only if every relation of the form

$x\,\vec{a} + y\,\vec{b} = \vec{0}$ given as x = 0 and y = 0.

[**Hint :** If any one of x and y is non-zero say y, then we get $\vec{b} = -\frac{x}{y}\vec{a}$ which is a contradiction]

Example 34.5 Find the number x by which the non-zero vector $\vec{a}$ be multiplied to get

(i) $\hat{a}$ $\qquad$ (ii) $-\hat{a}$

Solution : (i) $x\,\vec{a} = \hat{a}$ $\qquad$ i.e., $\qquad x\,|\vec{a}|\,\hat{a} = \hat{a}$

$$\Rightarrow \qquad x = \frac{1}{|\vec{a}|}$$

(ii) $\qquad x\,\vec{a} = -\hat{a}$ $\quad$ i.e., $\qquad x\,|\vec{a}|\,\hat{a} = -\hat{a}$

$$\Rightarrow \qquad x = -\frac{1}{|\vec{a}|}$$

Example 34.6 The vectors $\vec{a}$ and $\vec{b}$ are not collinear. Find x such that the vector

$$\vec{c} = (x-2)\,\vec{a} + \vec{b} \text{ and } \vec{d} = (2x+1)\,\vec{a} - \vec{b}$$

Solution : $\vec{c}$ is non-zero since the co-efficient of $\vec{b}$ is non-zero.

$\therefore$ There exists a number y such that $\vec{d} = y\,\vec{c}$

i.e. $\qquad (2x+1)\,\vec{a} - \vec{b} = y\,(x-2)\,\vec{a} + y\,\vec{b}$

$\therefore \qquad (yx - 2y - 2x - 1)\,\vec{a} + (y+1)\,\vec{b} = 0$

As $\vec{a}$ and $\vec{b}$ are non-collinear.

$\therefore \qquad yx - 2y - 2x - 1 = 0$ and $y + 1 = 0$

Solving these we get $y = -1$ and $x = \dfrac{1}{3}$

Thus $\qquad \vec{c} = -\dfrac{5}{3}\vec{a} + \vec{b}$ and $\vec{d} = \dfrac{5}{3}\vec{a} - \vec{b}$

We can see that $\vec{c}$ and $\vec{d}$ are opposite vectors and hence are collinear.

Example 34.7 The position vectors of two points A and B are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + \vec{b}$ respectively. Find $\overrightarrow{AB}$.

Solution : Let O be the origin of reference.

Then $\qquad \overrightarrow{AB}$ = Position vector of B — Position vector of A

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\vec{a} + \vec{b}) - (2\vec{a} + 3\vec{b})$$

$$= (3 - 2)\vec{a} + (1 - 3)\vec{b} = \vec{a} - 2\vec{b}$$

Example 34.8 Show that the points P, Q and R with position vectors $\vec{a} - 2\vec{b}$, $2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

Solution : $\overrightarrow{PQ}$ = Position vector of Q — Position vector of P

$$= (2\vec{a} + 3\vec{b}) - (\vec{a} - 2\vec{b})$$

$$= \vec{a} + 5\vec{b} \qquad \text{....(i)}$$

and $\overrightarrow{QR}$ = Position vector of R — Position vector of Q

$$= -7\vec{b} - (2\vec{a} + 3\vec{b})$$

$$= -7\vec{b} - 2\vec{a} - 3\vec{b}$$

$$= -2\vec{a} - 10\vec{b}$$

$$= -2(\vec{a} + 5\vec{b}) \qquad \text{....(ii)}$$

From (i) and (ii) we get $\overrightarrow{PQ} = -2\,\overrightarrow{QR}$, a scalar multiple of $\overrightarrow{QR}$

$\therefore \qquad \overrightarrow{PQ} \;||\; \overrightarrow{QR}$

But Q is a common point

$\therefore \qquad \overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear. Hence points P, Q and R are collinear.

CHECK YOUR PROGRESS 34.3

1. The position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ respectively with respect to a given origin of reference. Find $\overrightarrow{AB}$.
2. Interpret each of the following :

   (i) $3\vec{a}$ (ii) $-5\vec{b}$
3. The position vectors of points A, B, C and D are respectively $2\vec{a}$, $3\vec{b}$, $4\vec{a}+3\vec{b}$ and $\vec{a}+2\vec{b}$. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.
4. Find the magnitude of the product of a vector $\vec{n}$ by a scalar y.
5. State whether the product of a vector by a scalar is a scalar or a vector.
6. State the condition of collinearity of two vectors $\vec{p}$ and $\vec{q}$.
7. Show that the points with position vectors $5\vec{a}+6\vec{b}$, $7\vec{a}-8\vec{b}$ and $3\vec{a}+20\vec{b}$ are collinear.

34.7 CO-PLANARITY OF VECTORS

Given any two non-collinear vectors $\vec{a}$ and $\vec{b}$, they can be made to lie in one plane. There (in the plane), the vectors will be intersecting. We take their common point as O and let the two vectors be $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Given a third vector $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, we can choose its initial point also as O. Let C be its terminal point. With $\overrightarrow{OC}$ as diagonal complete the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides.

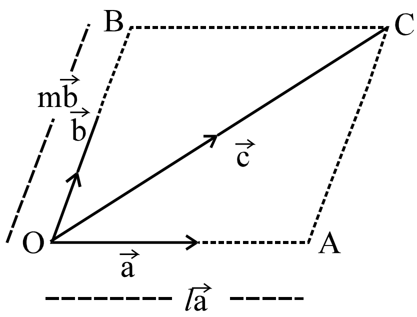


Fig. 34.20

$$\therefore \qquad \vec{c} = l\,\vec{a} + m\,\vec{b}$$

Thus, any $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, is expressible as a linear combination of $\vec{a}$ and $\vec{b}$.

i.e. $$\vec{c} = l\,\vec{a} + m\,\vec{b}.$$

34.8 RESOLUTION OF A VECTOR ALONG TWO PER PERPEN DICULAR AXES

Consider two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ along two mutually perpendicular axes OX and OY. We have seen above that any vector $\vec{r}$ in the plane of $\hat{i}$ and $\hat{j}$, can be written in the form $\vec{r} = x\hat{i} + y\hat{j}$

If O is the initial point of $\vec{r}$, then OM = x and ON = y and $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are called the component vectors of $\vec{r}$ along x-axis and y-axis.

$\overrightarrow{OM}$ and $\overrightarrow{ON}$, in this special case, are also called the ***resolved parts*** of $\vec{r}$

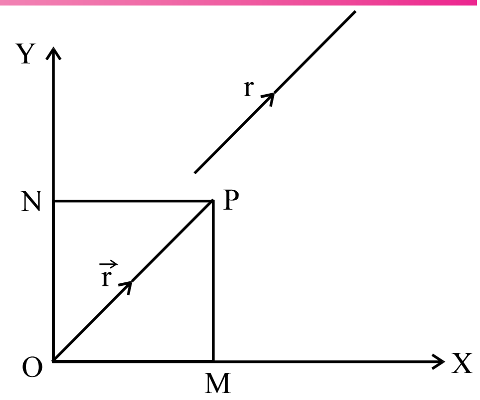


Fig. 34.21

34.9 RESOLUTION OF A VECTOR IN THREE DIMENSIONS ALONG THREE MUTUALLY PERPENDICULAR AXES

The concept of resolution of a vector in three dimensions along three mutually perpendicular axes is an extension of the resolution of a vector in a plane along two mutually perpendicular axes.

Any vector $\vec{r}$ in space can be expressed as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ as is shown in the adjoining Fig. 14.22.

We complete the rectangular parallelopiped with $\overrightarrow{OP} = \vec{r}$ as its diagonal :

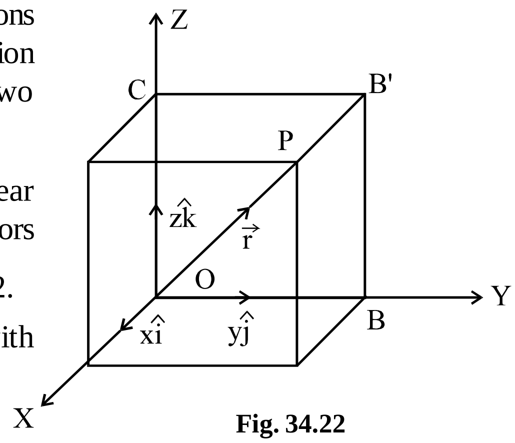


Fig. 34.22

then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the resolved parts of $\vec{r}$ along three mutually perpendicular axes.

Thus any vector $\vec{r}$ in space is expressible as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Refer to Fig. 34.21 in which $OP^2 = OM^2 + ON^2$ (Two dimensions)

or $\vec{r}^2 = x^2 + y^2$ ......(i)

and in Fig. 34.22

$$OP^2 = OA^2 + OB^2 + OC^2$$

$$\vec{r}^2 = x^2 + y^2 + z^2 \qquad ....(ii)$$

Magnitude of $\vec{r} = |\vec{r}|$ in case of (i) is $\sqrt{x^2 + y^2}$

and (ii) is $\sqrt{x^2 + y^2 + z^2}$

Note : Given any three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (not necessarily mutually perpendicular unit vectors) any vector $\vec{d}$ is expressible as a linear combination of

$$\vec{a},\ \vec{b} \text{ and } \vec{c}, \text{ i.e., } \vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$$

Example 34.9 A vector of 10 Newton is 30° north of east. Find its components along east and north directions.

Solution : Let $\hat{i}$ and $\hat{j}$ be the unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ (East and North respectively)

Resolve OP in the direction OX and OY.

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$= 10\cos 30°\ \hat{i} + 10\sin 30°\ \hat{j}$$

$$= 10.\frac{\sqrt{3}}{2}\hat{i} + 10.\frac{1}{2}\hat{j}$$

$$= 5\sqrt{3}\ \hat{i} + 5\ \hat{j}$$

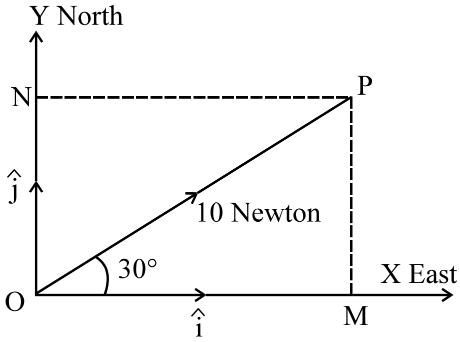


Fig. 34.23

$\therefore$ Component along (i) East $= 5\sqrt{3}$ Newton

(ii) North = 5 Newton

Example 34.10 Show that the following vectors are coplanar :

$$\vec{a} - 2\vec{b},\ 3\vec{a} + \vec{b} \text{ and } \vec{a} + 4\vec{b}$$

Solution : The vectors will be coplanar if there exists scalars x and y such that

$$\vec{a} + 4\vec{b} = x(\vec{a} - 2\vec{b}) + y(3\vec{a} + \vec{b})$$

$$= (x + 3y)\vec{a} + (-2x + y)\vec{b} \qquad .....(i)$$

Comparing the co-efficients of $\vec{a}$ and $\vec{b}$ on both sides of (i), we get

$$x + 3y = 1 \text{ and } -2x + y = 4$$

which on solving, gives $x = -\frac{11}{7}$ and $y = \frac{6}{7}$

As $\vec{a} + 4\vec{b}$ is expressible in terms of $\vec{a} - 2\vec{b}$ and $3\vec{a} + \vec{b}$, hence the three vectors are coplanar.

Example 34.11 Given $\vec{r_1} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{r_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, find the magnitudes of

(a) $\vec{r_1}$ (b) $\vec{r_2}$ (c) $\vec{r_1} + \vec{r_2}$ (d) $\vec{r_1} - \vec{r_2}$

Solution :

(a) $|\vec{r_1}| = |\hat{i} - \hat{j} + \hat{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

(b) $|\vec{r}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

(c) $\vec{r}_1 + \vec{r}_2 = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r}_1 + \vec{r}_2| = |3\hat{i} - 5\hat{j} - 2\hat{k}| = \sqrt{3^2 + (-5)^2 + (-2)^2} = \sqrt{38}$

(d) $\vec{r}_1 - \vec{r}_2 = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \quad |\vec{r}_1 - \vec{r}_2| = |-\hat{i} + 3\hat{j} + 4\hat{k}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$

Example 34.12 Determine the unit vector parallel to the resultant of two vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

Solution : The resultant vector $\vec{R} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + (\hat{i} + \hat{j} + 2\hat{k})$

$$= 4\hat{i} + 3\hat{j} - 2\hat{k}$$

Magnitude of the resultant vector $\vec{R}$ is $|\vec{R}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore$ The unit vector parallel to the resultant vector

$$\frac{\vec{R}}{|\vec{R}|} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 34.13 Find a unit vector in the direction of $\vec{r} - \vec{s}$

where $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{s} = 2\hat{i} - \hat{j} + 2\hat{k}$

Solution : $\vec{r} - \vec{s} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$\therefore \quad |\vec{r} - \vec{s}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{35}$

$\therefore$ Unit vector in the direction of $(\vec{r} - \vec{s})$

$$= \frac{1}{\sqrt{35}}(-\hat{i} + 3\hat{j} - 5\hat{k}) = -\frac{1}{\sqrt{35}}\hat{i} + \frac{3}{\sqrt{35}}\hat{j} - \frac{5}{\sqrt{35}}\hat{k}$$

Example 34.14 Find a unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$.

Solution : $2\vec{a} + 3\vec{b} = 2(\hat{i} + 3\hat{j} + \hat{k}) + 3(3\hat{i} - 2\hat{j} - \hat{k})$

$$= (2\hat{i} + 6\hat{j} + 2\hat{k}) + (9\hat{i} - 6\hat{j} - 3\hat{k})$$

$$= 11\hat{i} - \hat{k}.$$

$\therefore \qquad |2\vec{a}+3\vec{b}| = \sqrt{(11)^2+(-1)^2} = \sqrt{122}$

$\therefore$ Unit vector in the direction of $(2\vec{a}+3\vec{b})$ is $\dfrac{11}{\sqrt{122}}\hat{i} - \dfrac{1}{\sqrt{122}}\hat{k}$.

Example 34.15 Show that the following vectors are coplanar :

$4\vec{a}-2\vec{b}-2\vec{c}$, $-2\vec{a}+4\vec{b}-2\vec{c}$ and $-2\vec{a}-2\vec{b}+4\vec{c}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

Solution : If these vectors be co-planar, it will be possible to express one of them as a linear combination of other two.

Let $\quad -2\vec{a}-2\vec{b}+4\vec{c} = x\,(4\vec{a}-2\vec{b}-2\vec{c}) + y\,(-2\vec{a}+4\vec{b}-2\vec{c})$

where x and y are scalars,

Comparing the co-efficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both sides, we get

$$4x-2y=-2,\ -2x+4y=-2 \text{ and } -2x-2y=4$$

These three equations are satisfied by $x=-1,\ y=-1$ Thus,

$$-2\vec{a}-2\vec{b}+4\vec{c} = (-1)\,(4\vec{a}-2\vec{b}-2\vec{c}) + (-1)\,(-2\vec{a}+4\vec{b}-2\vec{c})$$

Hence the three given vectors are co-planar.

CHECK YOUR PROGRESS 34.4

1. Write the condition that $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planar.

2. Determine the resultant vector $\vec{r}$ whose components along two rectangular Cartesian co-ordinate axes are 3 and 4 units respectively.

3. In the adjoining figure :

   $|OA| = 4,\ |OB| = 3$ and

   $|OC| = 5$. Express OP in terms of its component vectors.

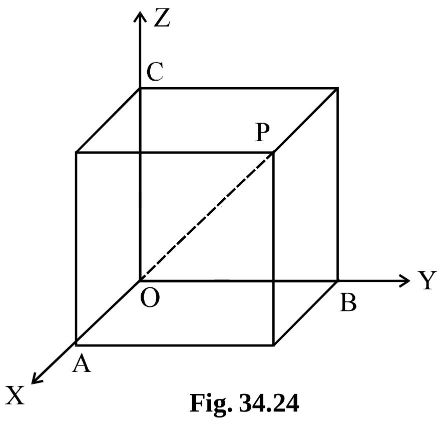


Fig. 34.24

4. If $\vec{r_1} = 4\hat{i}+\hat{j}-4\hat{k}$, $\vec{r_2} = -2\hat{i}+2\hat{j}+3\hat{k}$ and

   $\vec{r_3} = \hat{i}+3\hat{j}-\hat{k}$ then show that

   $|\vec{r_1}+\vec{r_2}+\vec{r_3}| = 7\cdot$

5. Determine the unit vector parallel to the resultant of vectors :

   $\vec{a} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

6. Find a unit vector in the direction of vector $3\vec{a} - 2\vec{b}$ where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

7. Show that the following vectors are co-planar :

$3\vec{a} - 7\vec{b} - 4\vec{c}$, $3\vec{a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

34.10 SECTION FORMULA

Recall that the position vector of a point P is space with respect to an origin of reference O is $\vec{r} = \overrightarrow{OP}$.

In the following, we try to find the position vector of a point dividing a line segment joining two points in the ratio m : n internally.

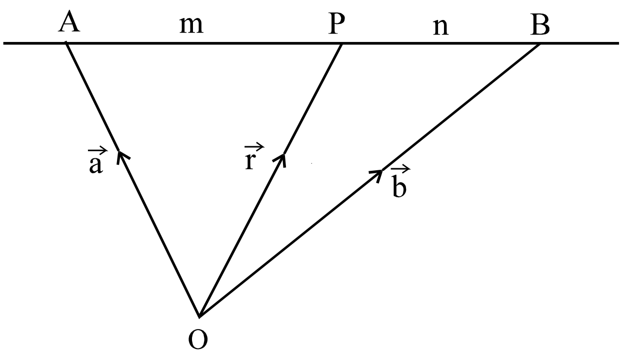


Fig. 34.25

Let A and B be two points and $\vec{a}$ and $\vec{b}$ be their position vectors w.r.t. the origin of reference O, so that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P divide AB in the ratio m : n so that

$$\frac{AP}{PB} = \frac{m}{n} \quad \text{or,} \quad n\overrightarrow{AP} = m\overrightarrow{PB} \qquad ....(i)$$

Since $\quad n\overrightarrow{AP} = m\overrightarrow{PB}$, it follows that

$$n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

or $\quad (m+n)\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA}$

or $$\overrightarrow{OP} = \frac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$$

or $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

where $\vec{r}$ is the position vector of P with respect to O.

Corollary 1 : If $\frac{m}{n} = 1 \Rightarrow$ m = n, then P becomes mid-point of AB.

$\therefore$ The position vector of the mid-point of the join of two given points, whose position vectors are $\vec{a}$ and $\vec{b}$, is given by $\frac{1}{2}(\vec{a} + \vec{b})$.

Corollary 2 : The position vector P can also be written as

$$\vec{r} = \frac{\vec{a} + \frac{m}{n}\vec{b}}{1 + \frac{m}{n}} = \frac{\vec{a} + k\,\vec{b}}{1 + k}, \qquad \text{.....(ii)}$$

where $\quad k = \frac{m}{n},\ k \neq -1.$

(ii) represents the position vector of a point which divides the join of two points with position vectors $\vec{a}$ and $\vec{b}$, in the ratio k : 1.

Corollary 3 : The position vector of a point P which divides AB in the ratio m : n externally is

$$\vec{r} = \frac{n\,\vec{a} - m\,\vec{b}}{n - m}$$ [**Hint :** This division is in the ratio –m : n]

Example 34.16 Find the position vector of a point which divides the join of two points whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

Solution : Let $\vec{r}$ be the position vector of the point.

$$\therefore \qquad \vec{r} = \frac{3\,\vec{x} + 2\,\vec{y}}{3 + 2} = \frac{1}{5}(3\,\vec{x} + 2\,\vec{y}).$$

Example 34.17 Find the position vector of mid-point of the line segment AB, if the position vectors of A and B are respectively, $\vec{x} + 2\,\vec{y}$ and $2\,\vec{x} - \vec{y}$.

Solution : Position vector of mid-point of AB

$$= \frac{(\vec{x} + 2\,\vec{y}) + (2\,\vec{x} - \vec{y})}{2}$$

$$= \frac{3}{2}\vec{x} + \frac{1}{2}\vec{y}$$

Example 34.18 The position vectors of vertices A, B and C of ΔABC are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find the position vector of the centroid of ΔABC.

Solution : Let D be the mid-point of side BC of ΔABC.

Let G be the centroid of ΔABC. Then G divides AD in the ratio 2 : 1 i.e. AG : GD = 2 : 1.

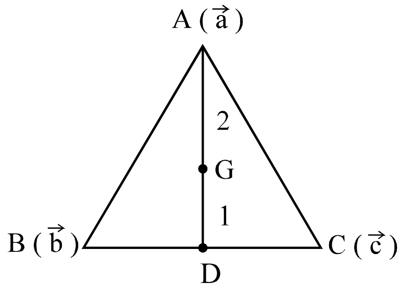


Fig. 34.26

Now position vector of D is $\dfrac{\vec{b}+\vec{c}}{2}$

$\therefore$ Position vector of G is $\dfrac{2\cdot\dfrac{\vec{b}+\vec{c}}{2}+1\cdot\vec{a}}{2+1}$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

CHECK YOUR PROGRESS 34.5

1. Find the position vector of the point C if it divides AB in the ratio (i) $\dfrac{1}{2} : \dfrac{1}{3}$

   (ii) $2 : -3$, given that the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively.
2. Find the point which divides the join of $P(\vec{p})$ and $Q(\vec{q})$ internally in the ratio 3 : 4.
3. CD is trisected at points P and Q. Find the position vectors of points of trisection, if the position vectors of C and D are $\vec{c}$ and $\vec{d}$ respectively
4. Using vectors, prove that the medians of a triangle are concurrent.
5. Using vectors, prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

34.11 DIRECTION COSINES OF A VECTOR

In the adjoining figure $\overrightarrow{AB}$ is a vector in the space and $\overrightarrow{OP}$ is the position vector of the point P(x, y, z) such that $\overrightarrow{OP} \parallel \overrightarrow{AB}$. Let $\overrightarrow{OP}$ makes angles α, β and γ respectively with the positive directions of x, y and z axis respectively. α, β and γ are called direction angles of vector $\overrightarrow{OP}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines.

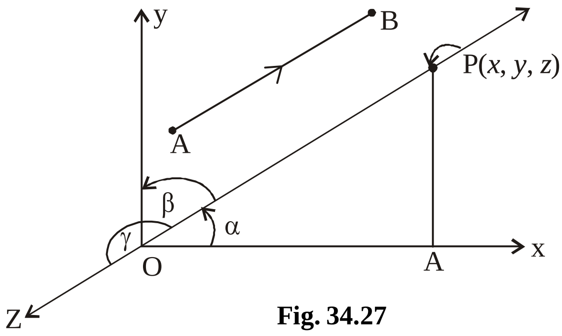


Fig. 34.27

Since $\overrightarrow{OP} \parallel \overrightarrow{AB}$, therefore $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of vector $\overrightarrow{AB}$ also.

Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of x, y and z axes respectively.

- By reversing the direction, we observe that $\overrightarrow{PO}$ makes angles $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$ with the positive directions of x, y and z axes respectively. So $\cos(\pi - \alpha) = -\cos\alpha$, $\cos(\pi - \beta) = -\cos\beta$ and $\cos(\pi - \gamma) = -\cos\gamma$ are the direction cosines of $\overrightarrow{PO}$. In fact any vector in space can be extended in two directions so it has two sets of direction cosines. If $(\cos\alpha, \cos\beta, \cos\gamma)$ is one set of direction cosines then $(-\cos\alpha, -\cos\beta, -\cos\gamma)$ is the other set. It is enough to mention any one set of direction cosines of a vector.
- Direction cosines of a vector are usually denoted by l, m and n. In other words $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.
- Since $\overrightarrow{OX}$ makes angles 0°, 90° and 90° with $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ respectively. Therefore cos0°, cos90°, cos90° i.e. 1, 0, 0 are the direction cosines of x-axis. Similarly direction cosines of y and z axes are (0, 1, 0) and (0, 0, 1) respectively.

  In the figure, 1 let $|\overrightarrow{OP}| = r$. and PA $\perp$ OX.

  Now in right angled ΔOAP, $\dfrac{OA}{OP} = \cos\alpha$

  i.e. $\quad OA = OP\cos\alpha$

  i.e. $\quad x = r.l \Rightarrow \quad x = l\,r$

  Similarly by dropping perpendiculars to y and z axes respectively we get $y = mr$ and $z = nr$.

  Now $\quad x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2) \quad$ ...(i)

  But $\quad |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

  or $\quad |\overrightarrow{OP}|^2 = x^2 + y^2 + z^2$

  $\quad = r^2$

  therefore from (i) $l^2 + m^2 + n^2 = 1$

  Again $l = \dfrac{x}{r}$, $m = \dfrac{y}{r}$, $n = \dfrac{z}{r}$

  i.e. $\quad l = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $m = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, $n = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Hence, if P(x, y, z) is a point in the space, then direction cosines of $\overrightarrow{OP}$ are $\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, $\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

34.11.1 DIRECTION COSINES OF A VECTOR JOINING TWO POINTS :

In the adjoining figure $\overrightarrow{PQ}$ is a vector joining points P(x_1 y_1 z) and Q(x_2 y_2 z_2). If we shift the origin to the point P($x_1 y_1 z_1$) without changing the direction of coordinate axes. The coordinates of point Q becomes ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$) therefore direction cosines of

$\overrightarrow{PQ}$ are $\dfrac{x_2 - x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$, $\dfrac{y_2 - y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$,

$\dfrac{z_2 - z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$,

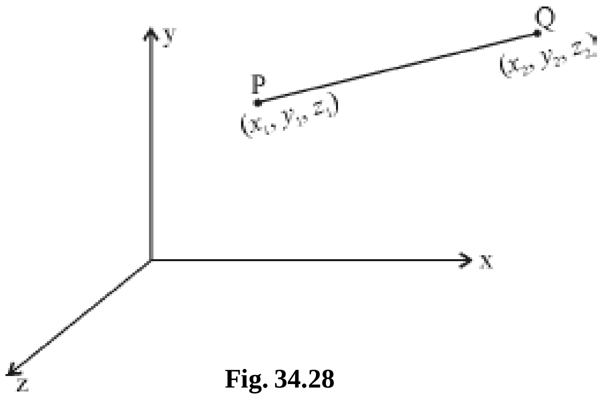


Fig. 34.28

34.11.2 DIRECTION RATIOS OF A VECTOR :

Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector. Let *l*, *m*, *n* be the direction cosines of a vector and *a*, *b*, *c* be the direction ratios.

then, $\dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n} = \lambda$ (say)

$\Rightarrow \quad a = \lambda l, b = \lambda m, c = \lambda n$

$\therefore \quad a^2 + b^2 + c^2 = \lambda^2(l^2 + m^2 + n^2)$

$\Rightarrow \quad \lambda^2 = a^2 + b^2 + c^2$

i.e. $\lambda = \pm\sqrt{a^2 + b^2 + c^2}$

$\therefore \quad l = \pm\dfrac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\dfrac{b}{\sqrt{a^2+b^2+c^2}}, n = \dfrac{\pm c}{\sqrt{a^2+b^2+c^2}}$

- If a, b, c are direction ratios of a vector then for every $\lambda \neq 0$, λa, λb, λc are also its direction ratios. Thus a vector can have infinite number of direction ratios.
- If P(x,y,z) is a point in the space, then the direction ratios of $\overrightarrow{OP}$ are x, y, z.
- If P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are two points in the space then the direction ratios of $\overrightarrow{PQ}$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ in general.

Example 34.19 Let P be a point in space such that OP $= \sqrt{3}$ and $\overrightarrow{OP}$ makes angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ with positve directions of, x, y and z axes respectively. Find coordinates of point P.

Solution : d.c.s of $\overrightarrow{OP}$ are $\cos\frac{\pi}{3}, \cos\frac{\pi}{4}, \cos\frac{\pi}{3}$ i.e. $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

$\therefore$ coordinates of point are $x = lr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$

$$y = mr = \frac{1}{\sqrt{2}} \times \sqrt{3} = \frac{\sqrt{3}}{\sqrt{2}}$$

and $$z = nr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$$

Example 34.20 If P(1, 2, –3) is a point in the space, find the direction cosines of vector $\overrightarrow{OP}$.

Solution :

$$l = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{\sqrt{14}}$$

$$m = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{2}{\sqrt{14}}$$

$$n = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{-3}{\sqrt{14}}$$

Example 34.21 Can $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ be direction cosines of a vector.

Solution : $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{4}{3} \neq 1$

$\therefore$ $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ can not be direction cosines of a vector.

Example 34.22 If P(2, 3, –6) and Q(3, –4, 5) are two points in the space. Find the direction cosines of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$, where O is the origin.

Solution : D.C.'S of $\overrightarrow{OP}$ are $\dfrac{2}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{3}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{-6}{\sqrt{2^2+3^2+(-6)^2}}$

i.e. $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$.

Similarly d.c.'s of $\overrightarrow{QO}$ are $\dfrac{-3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{-5}{\sqrt{2}}$

D.C.'s of $\overrightarrow{PQ}$ are : $\dfrac{3-2}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}, \dfrac{-4-3}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

$\dfrac{5+6}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

i.e. $\dfrac{1}{\sqrt{171}}, \dfrac{-7}{\sqrt{171}}, \dfrac{11}{\sqrt{171}}$

Example 34.23 Find the direction cosines of a vector which makes equal angles with the axes.

Solution : Suppose the given vector makes angle α with each of the $\overrightarrow{OX}, \overrightarrow{OY}$ and $\overrightarrow{OZ}$. Therefore $\cos\alpha$, $\cos\alpha$, $\cos\alpha$ are the direction cosines of the vector.

Now, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

i.e. $\cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ d.c.'s of the vector are $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Example 34.24 If P(1, 2, –3) and Q(4, 3, 5) are two points in space, find the direction ratios of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$

Solution : d.r.'s of $\overrightarrow{OP}$ are 1, 2, –3

d.r.'s of $\overrightarrow{QO}$ are (–4, –3, –5) or (4, 3, 5)

d.r.'s of $\overrightarrow{PQ}$ are 4 –1, 3 –2, 5 – (–3)

i.e. 3, 1, 8.

CHECK YOUR PROGRESS 34.6

1. Fill in the blanks:
   (i) Direction cosines of y-axis are...
   (ii) If l, m, n are direction cosines of a vector, then $l^2 + m^2 + n^2 = ...$

(iii) If a, b, c are direction ratios of a vector, then $a^2 + b^2 + c^2$ is .... to 1

(iv) The direction cosines of a vector which makes equal angles with the coordinate axes are...

(v) If two vectors are parallel to each other then their direction ratios are...

(vi) (1, –1, 1) are not direction cosines of any vector because...

(vii) The number of direction ratios of a vector are... (finite/infinite)

2. If P(3, 4, –5) is a point in the space. Find the direction cosines of $\overrightarrow{OP}$.
3. Find the direction cosines of $\overrightarrow{AB}$ where A(–2, 4, –5) and B(1, 2, 3) are two points in the space.
4. If a vector makes angles 90°, 135° and 45° with the positive directions of *x*, *y* and *z* axis respectively, find its direction ratios.

34.12 PRODUCT OF VECTORS

In Section 34.9, you have multiplied a vector by a scalar. The product of vector with a scalar gives us a vector quantity. In this section we shall take the case when a vector is multiplied by another vector. There are two cases :

(i) When the product of two vectors is a scalar, we call it a scalar product, also known as dot product corresponding to the symbol '$\bullet$' used for this product.

(ii) When the product of two vectors is a vector, we call it a vector product, also known as cross product corresponding to the symbol '$\times$' used for this product.

34.13 SCALAR PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ two vectors and θ be the angle between them. The scalar product, denoted by $\vec{a}$, $\vec{b}$, is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Clearly, $\vec{a}\cdot\vec{b}$ is a scalar as $|\vec{a}|$, $|\vec{b}|$ and $\cos\theta$ are all scalars.

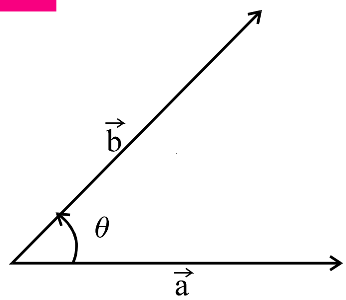


Fig. 34.29

Remarks

1. If $\vec{a}$ and $\vec{b}$ are like vectors, then $\vec{a}\cdot\vec{b} = ab\cos\theta = ab$, where a and b are magnitudes of $\vec{a}$ and $\vec{b}$.
2. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then $\vec{a}\cdot\vec{b} = ab\cos\pi = -ab$
4. Angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$
5. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ and $\vec{a}\cdot(\vec{b}+\vec{c}) = (\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})$.
6. $n(\vec{a}\cdot\vec{b}) = (n\vec{a})\cdot\vec{b} = \vec{a}\cdot(n\vec{b})$ where n is any real number.
7. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ as $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors.

MODULE - IX
Vectors and three dimensional Geometry

Notes

Example 34.25 If $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, find $\vec{a} \cdot \vec{b}$. Also find angle between $\vec{a}$ and $\vec{b}$.

Solution : $\vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} - 6\hat{k}) \cdot (4\hat{i} - 3\hat{j} + \hat{k})$

$$= 3 \times 4 + 2 \times (-3) + (-6) \times 1$$

$$\left[\because \ \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ and } \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0\right]$$

$$= 12 - 6 - 6 = 0$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

Then $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = 0$$

$\therefore$ $$\theta = \frac{\pi}{2}.$$

34.14 VECTOR PRODUCT OF TWO VECTORS

Before we define vector product of two vectors, we discuss below right handed and left handed screw and associate it with corresponding vector triad.

34.14.1 Right Handed Screw

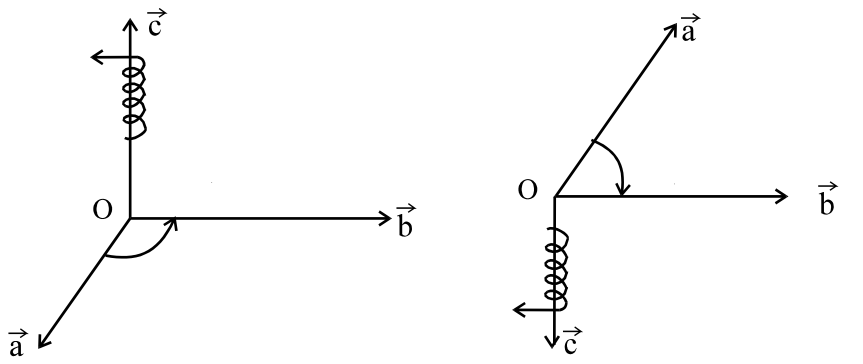
If a screw is taken and rotated in the anticlockwise direction, it translates towards the reader. It is called ***right handed screw***.

34.14.2 Left handed Screw

If a screw is taken and rotated in the clockwise direction, it translates away from the reader. It is called a left handed screw.

Now we associate a screw with given ordered vector triad.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors whose initial point is O.



(i) (ii)

Fig. 34.30

Now if a right handed screw at O is rotated from $\vec{a}$ towards $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (i)]

Similarly if a left handed screw at O is rotated from $\vec{a}$ to $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (ii)]. This time the direction of translation will be opposite to the first one.

Thus an ordered vector triad $\vec{a}$, $\vec{b}$, $\vec{c}$ is said to be right handed or left handed according as the right handed screw translated along $\vec{c}$ or opposite to $\vec{c}$ when it is rotated through an angle less than 180°.

34.14.3 VECTOR (CROSS) PRODUCT OF THE VECTORS :

If $\vec{a}$ and $\vec{b}$ are two non zero vectors then their cross product is denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta.\hat{n}$

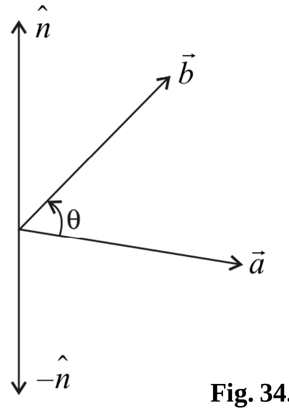


Fig. 34.31

Where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$; such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (see figure) i.e. the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

- $\vec{a} \times \vec{b}$ is a vector and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
- If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then θ is not defined and in that case we consider $\vec{a} \times \vec{b} = \vec{0}$.
- If $\vec{a}$ and $\vec{b}$ are non zero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors. i.e. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \,||\, \vec{b}$.
- In particular $\vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times (-\vec{b}) = \vec{0}$ because in the first situation θ = 0 and in 2nd case θ = π. Making the value of sinθ = 0 in both the cases.
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
- $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$.

- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$
- $\lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$.
- Angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}.$$

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Example 34.26 Using cross product find the angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{vmatrix} = \hat{i}(1-6) - \hat{j}(2+9) + \hat{k}(-4-3)$$

$$= -5\hat{i} - 11\hat{j} - 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{25+121+49} = \sqrt{195}$$

$$|\vec{a}| = \sqrt{4+1+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$$

$$\therefore \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{195}}{\sqrt{14}.\sqrt{14}} = \frac{\sqrt{195}}{14}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{195}}{14}\right)$$

Example 34.27 Find a unit vector perpendicular to each of the vectors $\vec{a} = 3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-2+3) - \hat{j}(-3+3) + \hat{k}(3-2)$$

$$\vec{a} \times \vec{b} = \hat{i} + \hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{1+1} = \sqrt{2}$$

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

Example 34.28 Find the area of the triangle having point A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution :

$$\overrightarrow{AB} = (1-1)\hat{i} + (2-1)\hat{j} + (3-1)\hat{k}$$

$$= \hat{j} + 2\hat{k}$$

$$\overrightarrow{AC} = (2-1)\hat{i} + (3-1)\hat{j} + (1-1)\hat{k}$$

$$= \hat{i} + 2\hat{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0-2) + \hat{k}(0-1)$$

$$= -4\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + (2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

Hence, Area of $\Delta ABC = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \dfrac{\sqrt{21}}{2}$ unit^2.

Example 34.29 Find the area of the parallelogram having A(5, –1, 1), B(–1, –3, 4), C(1, –6, 10) and D(7, –4, 7) as its vertices.

Solution :

$$\overrightarrow{AB} = (-1-5)\hat{i}+(-3+1)\hat{j}+(4-1)\hat{k}$$

$$= -6\hat{i}-2\hat{j}+3\hat{k}$$

$$\overrightarrow{AD} = (7-5)\hat{i}+(-4+1)\hat{j}+(7-1)\hat{k}$$

$$= 2\hat{i}-3\hat{j}+6\hat{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -2 & 3 \\ 2 & -3 & 6 \end{vmatrix} = \hat{i}(-12+9)-\hat{j}(-36-6)+\hat{k}(18+4)$$

$$= -3\hat{i}+42\hat{j}+22\hat{k}$$

$$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{9+1764+484} = \sqrt{2257}\ \text{unit}^2.$$

CHECK YOUR PROGRESS 34.7

1. (i) If $\vec{a}\times\vec{b}$ is a unit vector and $|\vec{a}|=3, |\vec{b}|=\frac{\sqrt{2}}{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is ...

   (ii) If $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$, then angle between $\vec{a}$ and $\vec{b}$ is ...

   (iii) The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is ...

2. Find a unit vector perpendicular to both the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

3. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

4. If $\vec{a}=2\hat{i}+2\hat{j}+2\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+j\vec{b}$ is perpendicular to $\vec{c}$, find the value of j.

34.15 SCALAR TRIPLE PRODUCT :

If $\vec{a},\vec{b}$ and $\vec{c}$ are any three vectors then the scalar product of $\vec{a}\times\vec{b}$ with $\vec{c}$ is called scalar triple product i.e. $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a},\vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}.\vec{b}\vec{c}]$

- $[\vec{a}\,\vec{b}\,\vec{c}]$ is a scalar quantity.
- $(\vec{a}\times\vec{b}).\vec{c}$ represents the volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as coterminous edges.
- $(\vec{a}\times\vec{b}).\vec{c}=0$ if $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors or any two of the three vectors are equal or parallel.
- In the scalar triple product the position of dot and cross can be interchanged provided the cyclic order of the vectors is maintained i.e.

$$(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$$

$$(\vec{b}\times\vec{c}).\vec{a} = \vec{b}.(\vec{c}\times\vec{a})$$

$$(\vec{c}\times\vec{a}).\vec{b} = \vec{c}.(\vec{a}\times\vec{b})$$

- $(\vec{a}\times\vec{b}).\vec{c}=(\vec{b}\times\vec{c}).\vec{a}=(\vec{c}\times\vec{a}).\vec{b}$
- $(\vec{a}\times\vec{b}).\vec{c}=-(\vec{b}\times\vec{a}).\vec{c}=-\vec{c}.(\vec{b}\times\vec{a})$

  $(\vec{b}\times\vec{c}).\vec{a}=-(\vec{c}\times\vec{b}).\vec{a}=-\vec{a}.(\vec{c}\times\vec{b})$

  $(\vec{c}\times\vec{a}).\vec{b}=-(\vec{a}\times\vec{c}).\vec{b}=-\vec{b}.(\vec{a}\times\vec{c})$

- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

  then $(\vec{a}\times\vec{b}).\vec{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- Four points A, B, C and D are coplanar if $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD}=0$

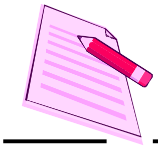
Example 34.30 Find the volume of the parallelepiped whose edges are represented by $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

Solution : Volume $=(\vec{a}\times\vec{b}).\vec{c}=\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(4-1)+3(2+3)+4(-1-6)$$

$$= 6+15-28=-7$$

Neglecting negative sign, required volume = 7 unit^3.



Example 34.31 Find the value of λ so that the vectors

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ b = \hat{i} + 2\hat{j} - 3\hat{k},\ c = 3\hat{i} + \lambda\hat{j} + 5\hat{k} \text{ are coplanar.}$$

Solution : The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ will be coplanar if $[\vec{a}\vec{b}\vec{c}] = 0$

i.e. $$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

i.e. $$2(10+3\lambda)+1(5+9)+1(\lambda-6) = 0$$

i.e. $$7\lambda + 28 = 0$$

$\Rightarrow$ $$\lambda = -4.$$

Example 34.32 Show that the four points A, B, C and D whose position vectors are $(4\hat{i}+5\hat{j}+\hat{k}), (-\hat{j}-\hat{k}),\ (3\hat{i}+9\hat{j}+4\hat{k})$ and $(-4\hat{i}+4\hat{j}+4\hat{k})$ respectively are coplanar.

Solution :
$$\overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$$

Now $$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3)+6(-3+24)-2(1+32)$$

$$= -60+126-66 = 0$$

Hence, A, B, C and D are coplanar.

Example 34.33 Prove that $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$

Solution :
$$\text{LHS} = (\vec{a}+\vec{b}) \cdot [(\vec{b}+\vec{c}) \times \vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b}) \cdot [(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b}) \cdot [(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}] \qquad \because \vec{c}\times\vec{c} = 0$$

$$= \vec{a} \cdot (\vec{b}\times\vec{c}) + \vec{a} \cdot (\vec{b}\times\vec{a}) + \vec{a} \cdot (\vec{c}\times\vec{a})$$

$$+ \vec{b} \cdot (\vec{b}\times\vec{c}) + \vec{b} \cdot (\vec{b}\times\vec{a}) + \vec{b} \cdot (\vec{c}\times\vec{a})$$

$$= \vec{a} \cdot (\vec{b}\times\vec{c}) + \vec{b} \cdot (\vec{c}\times\vec{a})$$ [$\because$ scalar triple product is zero when two vector are equal]

$$= 2[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= \text{RHS}$$

CHECK YOUR PROGRESS 34.8

1. Find the volume of the parallelopiped whose edges are represented by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = 3\hat{i} + 2\hat{j} + 5\hat{k}$.

2. Find the value of λ so that the vectors $\vec{a} = -4\hat{i} - 6\hat{j} + \lambda\hat{k}, \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$ are coplanar.

LET US SUM UP

- A physical quantity which can be represented by a number only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is 'a' and direction from A to B can be represented by $\overrightarrow{AB}$ and its magnitude is denoted by $|\overrightarrow{AB}| = a$.
- A vector whose magnitude is equal to the magnitude of another vector $\vec{a}$ but of opposite direction is called negative of the given vector and is denoted by $-\vec{a}$.
- A unit vector is of magnitude unity. Thus, a unit vector parallel to $\vec{a}$ is denoted by $\hat{a}$ and is equal to $\dfrac{\vec{a}}{|\vec{a}|}$.
- A zero vector, denoted by $\vec{0}$, is of magnitude 0 while it has no definite direction.
- Unlike addition of scalars, vectors are added in accordance with triangle law of addition of vectors and therefore, the magnitude of sum of two vectors is always less than or equal to sum of their magnitudes.
- Two or more vectors are said to be collinear if their supports are the same or parallel.
- Three or more vectors are said to be coplanar if their supports are parallel to the same plane or lie on the same plane.
- If $\vec{a}$ is a vector and x is a scalar, then $x\vec{a}$ is a vector whose magnitude is $|x|$ times the magnitude of $\vec{a}$ and whose direction is the same or opposite to that of $\vec{a}$ depending upon $x > 0$ or $x < 0$.
- Any vector co-planar with two given non-collinear vectors is expressible as their linear combination.
- Any vector in space is expressible as a linear combination of three given non-coplanar vectors.
- The position vector of a point that divides the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio of m : n internally/externally are given by
- $\dfrac{n\vec{a} + m\vec{b}}{m + n}$, $\dfrac{n\vec{a} - m\vec{b}}{n - m}$ respectively.

- The position vector of mid-point of the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}+\vec{b}}{2}$$

- The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$, $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.
- Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of *x*, *y* and *z* axes respectively.
- Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector.
- Usually, direction cosines of a vector are denoted by *l*, *m*, *n* and direction ratios by *a*, *b*, *c*.
- $l^2+m^2+n^2=1$ but $a^2+b^2+c^2\neq 1$, in general.
- If $\overrightarrow{AB}=x\hat{i}+y\hat{j}+z\hat{k}$, then direction ratios of $\overrightarrow{AB}$ are x, y, z and direction cosines are

$$\frac{\pm x}{\sqrt{x^2+y^2+z^2}},\frac{\pm y}{\sqrt{x^2+y^2+z^2}},\frac{\pm z}{\sqrt{x^2+y^2+z^2}}.$$

- Direction cosines of a vector are unique but direction ratios are infinite.
- Cross product of two non zero vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta.\hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$.
- $\vec{a}\times\vec{b}=\vec{0}$ if either $\vec{a}=0$ or $\vec{b}=0$ or $\vec{a}$ and $\vec{b}$ are parallel or $\vec{a}$ and $\vec{b}$ are collinear.
- $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$
- $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\vec{j}$.
- $\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\vec{j}$.
- $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
- $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

- $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

- Area of Δ = $\dfrac{1}{2}|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of a triangle.

- Area of || gm = $|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of the parallelogram.

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}\,\vec{b}\,\vec{c}]$

- Volume of parallelepiped $= (\vec{a}\times\vec{b}).\vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ represent coterminous edges of the parallelopiped.

- $(\vec{a}\times\vec{b}).\vec{c} = 0$, if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar or any two of the three vectors are equal or parallel.

- $(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$

- $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b}$

- $(\vec{a}\times\vec{b}).\vec{c} = -(\vec{b}\times\vec{a}).\vec{c}$

- Four points A, B, C and D are coplanar if $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD} = 0.$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$(\vec{a}\times\vec{b}).\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

SUPPORTIVE WEB SITES

www.youtube.com/watch?v=ihNZlp7iUHE

http://emweb.unl.edu/math/mathweb/vectors/vectors.html

http://www.mathtutor.ac.uk/geometry_vectors

www.khanacademy.org/.../introduction-to-**vectors-and-scalars**

TERMINAL EXERCISE

1. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that any two of them are non-collinear. Find their sum if the vector $\vec{a}+\vec{b}$ is collinear with the vector $\vec{c}$ and if the vector $\vec{b}+\vec{c}$ is collinear with $\vec{a}$.

2. Prove that any two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist numbers x and y, both not zero simultaneously, such that $x\vec{a}+y\vec{b}=\vec{0}$.

3. ABCD is a parallelogram in which M is the mid-point of side CD. Express the vectors $\overrightarrow{BD}$ and $\overrightarrow{AM}$ in terms of vectors $\overrightarrow{BM}$ and $\overrightarrow{MC}$.

4. Can the length of the vector $\vec{a}-\vec{b}$ be (i) less than, (ii) equal to or (iii) larger than the sum of the lengths of vectors $\vec{a}$ and $\vec{b}$?

5. Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors. Find the number x and y, if the vector $(2-x)\vec{a}+\vec{b}$ and $y\vec{a}+(x-3)\vec{b}$ are equal.

6. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find the number x if the vector $3\vec{a}+x\vec{b}$ and $(1-x)\vec{a}-\frac{2}{3}\vec{b}$ are parallel.

7. Determine x and y such that the vector $\vec{a}=-2\hat{i}+3\hat{j}+y\hat{k}$ is collinear with the vector $\vec{b}=x\hat{i}-6\hat{j}+2\hat{k}$. Find also the magnitudes of $\vec{a}$ and $\vec{b}$.

8. Determine the magnitudes of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ if $\vec{a}=3\hat{i}-5\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-4\hat{k}$.

9. Find a unit vector in the direction of $\vec{a}$ where $\vec{a}=-6\hat{i}+3\hat{j}-2\hat{k}$.

10. Find a unit vector parallel to the resultant of vectors $3\hat{i}-2\hat{j}+\hat{k}$ and $-2\hat{i}+4\hat{j}+\hat{k}$

11. The following forces act on a particle P :

    $\vec{F_1}=2\hat{i}+\hat{j}-3\hat{k}$, $\vec{F_2}=-3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{F_3}=3\hat{i}-2\hat{j}+\hat{k}$ measured in Newtons.

    Find (a) the resultant of the forces, (b) the magnitude of the resultant.

12. Show that the following vectors are co-planar :

$(\vec{a} - 2\vec{b} + \vec{c})$, $(2\vec{a} + \vec{b} - 3\vec{c})$ and $(-3\vec{a} + \vec{b} + 2\vec{c})$

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three non-coplanar vectors.

13. A vector makes angles $\frac{\pi}{3}, \frac{\pi}{3}$ with $\overline{OX}$ and $\overline{OY}$ respectively. Find the angle made by it with $\overline{OZ}$.

14. If $P(\sqrt{3}, 1, 2\sqrt{3})$ is a point in space, find direction cosines of $\overline{OP}$ where O is the origin.

15. Find the direction cosines of the vector joining the points $(-4, 1, 7)$ and (2, –3, 2).

16. Using the concept of direction ratios show that $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ where coordinates of P, Q, R and S are (0, 1, 2), (3, 4, 8), $\left(-2, \frac{3}{2}, -3\right)$ and $\left(\frac{5}{2}, 6, 6\right)$ respectively.

17. If the direction ratios of a vector are (3, 4, 0). Find its directions cosines.

18. Find the area of the parallelogram whose adjacent sides are represented by the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

19. Find the area of the ΔABC where coordinates of A, B, C are (3, –1, 2), (1, –1, –3) and (4, –3, 1) respectively.

20. Find a unit vector perpendicular to each of the vectors $2\hat{i} - 3\hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} - \hat{k}$.

21. If $\vec{A} = 2\hat{i} - 3\hat{j} - 6\hat{k}$ and $\vec{B} = \hat{i} + 4\hat{j} - 2\hat{k}$, then find $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B})$.

22. Prove that : $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

23. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$.

24. Find the volume of the parallelepiped whose edges are represented by $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = 3\hat{i} - 5\hat{j} + 2\hat{k}$.

25. Show that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ are coplanar.

26. Find the value of λ if the points A(3, 2, 1), B(4, λ, 5), C(4, 2, –2) and D(6, 5, –1) are coplanar.

MODULE - IX
Vectors and three dimensional Geometry

Notes

ANSWERS

CHECK YOUR PROGRESS 34.1

1. (d) 2. (b)

3.

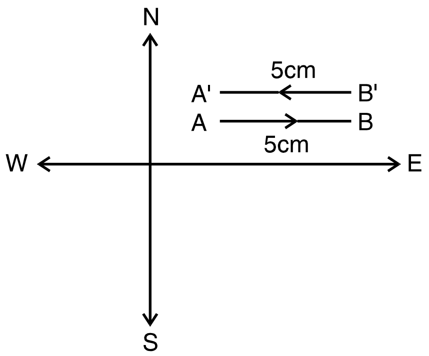


Fig. 34.32

4. Two vectors are said to be like if they have same direction what ever be their magnitudes.

   But in case of equal vectors magnitudes and directions both must be same.

5.

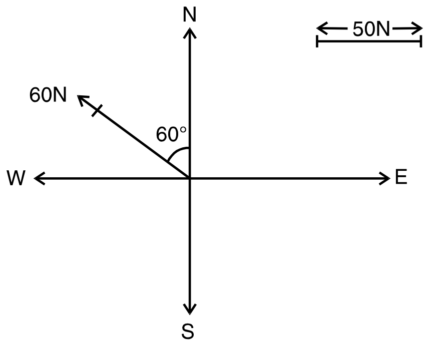


Fig. 34.33

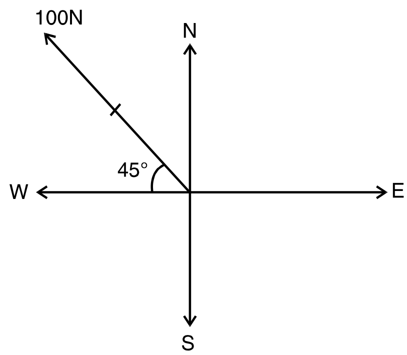


Fig. 34.34

CHECK YOUR PROGRESS 34.2

1. $\vec{0}$ 2. $\vec{0}$

CHECK YOUR PROGRESS 34.3

1. $\vec{b} - \vec{a}$

2. (i) It is a vector in the direction of $\vec{a}$ and whose magnitudes is 3 times that of $\vec{a}$.

   (ii) It is a vector in the direction opposite to that of $\vec{b}$ and with magnitude 5 times that of $\vec{b}$.

3. $\overrightarrow{DB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = 2\vec{a} + 3\vec{b}$.

4. $|y\vec{n}| = y|\vec{n}|$ if $y > 0$

5. Vector $= -y|\vec{n}|$ if $y < 0 = 0$ if $y = 0$

6. $\vec{p} = x\vec{q}$, x is a non-zero scalar.

CHECK YOUR PROGRESS 34.4

1. If there exist scalars x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

2. $\vec{r} = 3\hat{i} + 4\hat{j}$ 3. $\overrightarrow{OP} = 4\hat{i} + 3\hat{j} + 5\hat{k}$

5. $\frac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$ 6. $\frac{1}{\sqrt{51}}\hat{i} - \frac{5}{\sqrt{51}}\hat{j} - \frac{5}{\sqrt{51}}\hat{k}$

CHECK YOUR PROGRESS 34.5

1. (i) $\frac{1}{5}(2\vec{a} + 3\vec{b})$ (ii) $(3\vec{a} - 2\vec{b})$

2. $\frac{1}{7}(4\vec{p} + 3\vec{q})$ 3. $\frac{1}{3}(2\vec{c} + \vec{d}), \frac{1}{3}(\vec{c} + 2\vec{d})$

CHECK YOUR PROGRESS 34.6

1. (i) (0, 1, 0) (ii) 1 (iii) not equal (iv) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(v) proportional (vi) sum of their squares is not equal to 1 (vii) infinite

2. $\frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}, \frac{-1}{\sqrt{2}}$ 3. $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$ 4. 0, –1, 1

CHECK YOUR PROGRESS 34.7

1. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) 1

2. $\frac{-\hat{i}}{\sqrt{6}} + \frac{2\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$. 3. $\sqrt{42}$ unit^2

CHECK YOUR PROGRESS 34.8

1. 42 unit^3 2. $\lambda = -2$

TERMINAL EXERCISE

1. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

3. $\overrightarrow{BD} = \overrightarrow{BM} - \overrightarrow{MC}, \ \overrightarrow{AM} = \overrightarrow{BM} + 2\overrightarrow{MC}$

4. (i) Yes, $\vec{a}$ and $\vec{b}$ are either any non-collinear vectors or non-zero vectors of same direction.

   (ii) Yes, $\vec{a}$ and $\vec{b}$ are either in the opposite directions or at least one of them is a zero vector.

   (iii) Yes, $\vec{a}$ and $\vec{b}$ have opposite directions.

5. $x = 4, \ y = -2$ 6. $x = 2, \ -1$

7. $x = 4, \ y = -1$ $|\vec{a}| = \sqrt{14}, \ |\vec{b}| = 2\sqrt{14}$

8. $|\vec{a} + \vec{b}| = 6, \ |\vec{a} - \vec{b}| = 14$

9. $-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$ 10. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

11. $2\hat{i} + \hat{j}; \sqrt{5}$ 13. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$

14. $\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$ 15. $\frac{6}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}$

17. $\frac{3}{5}, \frac{4}{5}, 0$ 18. $8\sqrt{3}$ unit^2

19. $\frac{1}{2}\sqrt{165}$ unit^2 20. $\frac{7\hat{i} + 5\hat{j} + \hat{k}}{\sqrt{75}}$

21. $-60\hat{i} + 4\hat{j} - 22\hat{k}$ 24. 8 unit^3

26. $\lambda = 5$

35

PLANE

Look closely at a room in your house. It has four walls, a roof and a floor. The floor and roof are parts of two parallel planes extending infinitely beyond the boundary. You will also see two pairs of parallel walls which are also parts of parallel planes. Similarly, the tops of tables, doors of rooms etc. are examples of parts of planes.

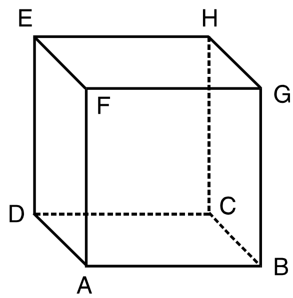


Fig. 35.1

If we consider any two points in a plane, the line joining these points will lie entirely in the same plane. This is the characteristic of a plane.

Look at Fig.35.1.You know that it is a representation of a rectangular box. This has six faces, eight vertices and twelve edges.

The pairs of opposite and parallel faces are

(i) ABCD and FGHE

(ii) AFED and BGHC

(iii) ABGF and DCHE

and the sets of parallel edges are given below :

(i) AB, DC EH and FG

(ii) AD, BC, GH and FE

(iii) AF, BG, CH and DE

Each of the six faces given above forms a part of the plane, and there are three pairs of parallel planes, denoted by the opposite faces.

In this lesson, we shall establish the general equation of a plane, the equation of a plane passing through three given points, the intercept form of the equation of a plane and the normal form of the equation of a plane. We shall show that a homogeneous equation of second degree in three variables x,y and z represents a pair of planes. We shall also find the equation of a plane bisecting the angle between two planes and area of a triangle in space.

OBJECTIVES

After studying this lesson, you will be able to :

- identify a plane;

- establish the equation of a plane in normal form;
- find the general equation of a plane passing through a given point;
- find the equation of a plane passing through three given points;
- find the equation of a plane in the intercept form and normal form;
- find the angle between two given planes;

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines and direction ratio of a line.
- Projection of a line segment on another line.
- Condition of perpendicularity and parallelism of two lines in space.

35.1 VECTOR EQUATION OF A PLANE

A plane is uniquely determined if any one of the following is known:

(i) Normal to the plane and its distance from the origin is given.

(ii) One point on the plane is given and normal to the plane is also given.

(iii) It passes through three given non collinear points.

35.2 EQUATION OF PLANE IN NORMAL FROM

Let the distance (OA) of the plane from origin O be d and let $\hat{n}$ be a unit vector normal to the plane. Consider $\vec{r}$ as position vector of an arbitarary point P on the plane.

Since OA is the perpendicular distance of the plane from the origin and $\hat{n}$ is a unit vector perpendicular to the plane.

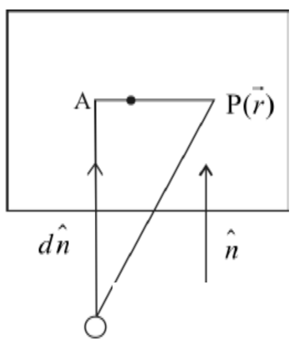


Fig. 35.2

$$\therefore \qquad \overrightarrow{OA} = d\hat{n}$$

$$\text{Now} \qquad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - d\hat{n}$$

$\overrightarrow{OA}$ is perpendicular to the plane and $\overrightarrow{AP}$ lies in the plane, therefore $\overrightarrow{OA} \perp \overrightarrow{AP}$

$$\Rightarrow \qquad \overrightarrow{AP}\,.\,\overrightarrow{OA} = 0$$

$$\text{i.e.} \qquad (\vec{r} - d\hat{n}).\hat{n} = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} - d = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} = d \qquad \text{...(3)}$$

which is the equation of plane in vector from.

35.3 CONVERSION OF VECTOR FORM INTO CARTESIAN FORM

Let (x, y, z) be the co-ordinates of the point P and l, m, n be the direction cosines of $\hat{n}$.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

Substituting these value in equation (3) we get

$$(x\hat{i} + y\hat{j} + z\hat{k}).(l\hat{i} + m\hat{y} + n\hat{k}) = d$$

$$\Rightarrow \qquad lx + my + nz = d$$

This is the corresponding Cartesian form of equation of plane in normal form.

Note : In equation (3), if $\vec{r}.\vec{n} = d$ is the equation of the plane then d is not the distance of the plane from origin. To find the distance of the plane from origin we have to convert $\vec{n}$ into $\hat{n}$ by dividing both sides by $|\vec{n}|$. Therefore $\dfrac{d}{|\vec{n}|}$ is the distance of the plane from the origin.

Example 15.1 Find the distance of the plane $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) - 1 = 0$ from the origin. Also find the direction cosines of the unit vector perpendicular to the plane.

Solution : The given equation can be written as

$$\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) = 1$$

$$|6\hat{i} - 3\hat{j} - 2\hat{k}| = \sqrt{36 + 9 + 4} = 7$$

Dividing both sides of given equation by 7 we get

$$\frac{\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k})}{7} = \frac{1}{7}$$

i.e. $$\vec{r}.\left(\frac{6}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}\right) = \frac{1}{7}$$

$\therefore$ d.c.'s of unit vector normal to the plane are $\dfrac{6}{7}, \dfrac{-3}{7}, \dfrac{-2}{7}$ and distance of plane from origin $= \dfrac{1}{7}$

35.4 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

Let $\vec{a}$ be the position vector of the given point A and $\vec{r}$ the position vector of an arbitrary point on the plane. $\vec{n}$ is a vector perpendicular to the plane.

Now $\quad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$= \vec{r} - \vec{a}$$

Now $\quad \vec{n} \perp (\vec{r} - \vec{a})$

$$\therefore \quad (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \qquad ...(4)$$

This is vector equation of plane in general form.

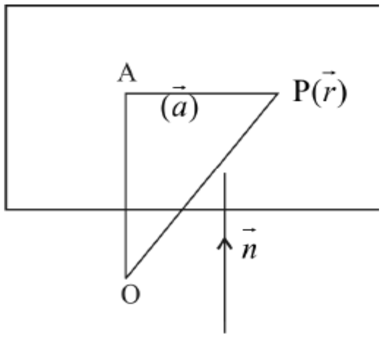


Fig. 35.3

35.5 CARTESIAN FORM

Let (x_1, y_1, z_1) be the coordinates of the given point A and (x, y, z) be the coordinates of point P. Again let a, b, c be the direction ratios of normal vector $\vec{n}$.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

Substituting these values in equation (4) we get

$$\{(x - x_1)i + (y - y_1)j + (z - z_1)k\} . \{ai + bj + ck\} = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

which is the corresponding Cartesian form of the equation of plane.

Example 35.2 Find the vector equation of a plane passing through the point (5, 5, –4) and perpendicular to the line with direction ratios 2, 3, –1.

Solution : Here $\quad \vec{a} = 5\hat{i} + 5\hat{j} - 4\hat{k}$

and $\quad \vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\therefore$ Equation of plane is $\left(\vec{r} - (5\hat{i} + 5\hat{j} - 4\hat{k})\right) . (2\hat{i} + 3\hat{j} - \hat{k}) = 0$

35.6 EQUATION OF A PLANE PASSING THROUGH THREE NON COLLINEAR POINTS

(a) Vector Form

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the given points Q, R and S respectively.

Let $\vec{r}$ be the position vector of an arbitrary point P on the plane.

Vectors $\overrightarrow{QR} = \vec{b} - \vec{a}, \overrightarrow{QS} = \vec{c} - \vec{a}$ and $\overrightarrow{QP} = \vec{r} - \vec{a}$ lie in the same plane and $\overrightarrow{QR} \times \overrightarrow{QS}$ is a vector perpendicular to both $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Therefore $\overrightarrow{QR} \times \overrightarrow{QS}$ is perpendicular to $\overrightarrow{QP}$ also.

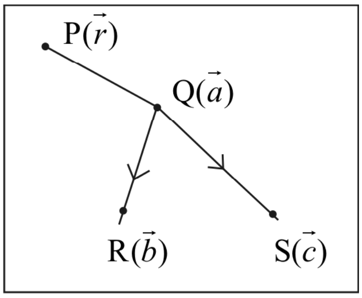


Fig. 35.4

$$\therefore \qquad \overrightarrow{QP}\,.\,(\overrightarrow{QR} \times \overrightarrow{QS}) = 0$$

$$(\vec{r}-\vec{a}).\left\{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})\right\} = 0 \qquad ...(5)$$

This is the equation of plane in vector form.

(b) Cartesian Form

Let (x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points P, Q, R and S respectively.

$$\therefore \quad \overrightarrow{QP} = \vec{r}-\vec{a} = (x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}$$

$$\overrightarrow{QR} = \vec{b}-\vec{a} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$$

$$\overrightarrow{QS} = \vec{c}-\vec{a} = (x_3-x_1)\hat{i} + (y_3-y_1)\hat{j} + (z_3-z_1)\hat{k}$$

Substituting these values in equation (5) we get.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is the equation of plane in Cartesian form.

Example 35.3 Find the vector equation of the plane passing through the points Q(2, 5, –3), R(–2, –3, 5) and S(5, 3, –3).

Solution : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of points Q, R and S respectively and $\vec{r}$ be the position vector of an arbitrary point on the plane.

Vector equation of plane is $\{\vec{r}-\vec{a}\}.\left\{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})\right\} = 0$

Here
$$\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$
$$\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$$
$$\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\vec{b}-\vec{a} = -4\hat{i}-8\hat{j}+8\hat{k}$$
$$\vec{c}-\vec{a} = 3\hat{i}-2\hat{j}$$

$\therefore$ Required equation is $\{\vec{r}-(2\hat{i}+5\hat{j}-3\hat{k})\}.\{(-4\hat{i}-8\hat{j}+8\hat{k})\times(3\hat{i}-2\hat{j})\} = 0$

35.7 EQUATION OF A PLANE IN THE INTERCEPT FORM

Let a, b,c be the lengths of the intercepts made by the plane on the x,y and z axes respectively. It implies that the plane passes through the points (a,0,0), (0,b,0) and (0,0,c)

Putting $x_1 = a$ $\quad y_1 = 0$ $\quad z_1 = 0$

$x_2 = 0$ $\quad y_2 = b$ $\quad z_2 = 0$

and $x_3 = 0$ $\quad y_3 = 0$ $\quad z_3 = c$ in (A),

we get the required equation of the plane as

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

which on expanding gives $bcx + acy + abz - abc = 0$

or $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad .....(B)$$

Equation (B) is called the ***Intercept form*** of the equation of the plane.

Example 35.4 Find the equation of the plane passing through the points (0,2,3), (2,0,3) and (2,3,0).

Solution : Using (A), we can write the equation of the plane as

$$\begin{vmatrix} x-0 & y-2 & z-3 \\ 2-0 & 0-2 & 3-3 \\ 2-0 & 3-2 & 0-3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y-2 & z-3 \\ 2 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0$$

or $\quad x(6-0)-(y-2)(-6)+(z-3)(2+4) = 0$

or $\quad 6x + 6(y-2)+6(z-3) = 0$

or $\quad x + y - 2 + z - 3 = 0 \quad$ or $\quad x + y + z = 5$

Example 35.5 Show that the equation of the plane passing through the points (2,2,0), (2,0,2) and (4,3,1) is $x = y + z$.

Solution : Equation of the plane passing through the point (2,2,0) is

$$a(x-2)+b(y-2)+cz=0 \qquad .....(i)$$

$\because$ (i) passes through the point (2, 0, 2)

$\therefore \qquad a(2-2)+b(0-2)+2c=0$

or $\qquad c = b \qquad .....(ii)$

Again (i) passes through the point (4, 3, 1)

$\therefore \qquad a(4-2)+b(3-2)+c=0$

or $\qquad 2a+b+c=0 \qquad ....(iii)$

From (ii) and (iii), we get $2a+2b=0$ or $a=-b$

$\therefore \qquad$ (i) becomes

$$-b(x-2)+b(y-2)+bz=0$$

or $\qquad -(x-2)+y-2+z=0$

or $\qquad y+z-x=0$

or $\qquad x=y+z$

Example 35.6 Reduce the equation of the plane $4x-5y+6z-60=0$ to the intercept form. Find its intercepts on the co-ordinate axes.

Solution : The equation of the plane is

$$4x-5y+6z-60=0 \qquad \text{or} \qquad 4x-5y+6z=60 \qquad ....(i)$$

The equation (i) can be written as $\frac{4x}{60}-\frac{5y}{60}+\frac{6z}{60}=1$ or $\frac{x}{15}+\frac{y}{(-12)}+\frac{z}{10}=1$

which is the interecept form of the equation of the plane and the intercepts on the co-ordinate axes are $15, -12$ and 10 respectively.

Example 35.7 Reduce each of the following equations of the plane to the normal form :

(i) $2x-3y+4z-5=0$ (ii) $2x+6y-3z+5=0$

Find the length of perpendicular from origin upon the plane in both the cases.

Solution : (i) The equation of the plane is $2x-3y+4z-5=0 \qquad ....(A)$

Dividing (A) by $\sqrt{2^2+(-3^2)+4^2}$ or, by $\sqrt{29}$

we get, $\qquad \frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}-\frac{5}{\sqrt{29}}=0$

or $\qquad \frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}=\frac{5}{\sqrt{29}}$

which is the equation of the plane in the normal form.

$\therefore$ Length of the perpendicular is $\frac{5}{\sqrt{29}}$

(ii) The equation of the plane is $2x+6y-3z+5=0 \qquad .....(B)$

Dividing (B) by $\sqrt{2^2 + 6^2 + (-3)^2}$

or by -7 we get, [refer to corollary 2]

$$-\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} - \frac{5}{7} = 0 \quad \text{or} \quad -\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} = \frac{5}{7}$$

which is the required equation of the plane in the normal form.

$\therefore$ Length of the perpendicular from the origin upon the plane is $\frac{5}{7}$

Example 35.8 The foot of the perpendicular drawn from the origin to the plane is (4, –2, –5). Find the equation of the plane.

Solution : Let P be the foot of perpendicular drawn from origin O to the plane.

Then P is the point $(4, -2, -5)$.

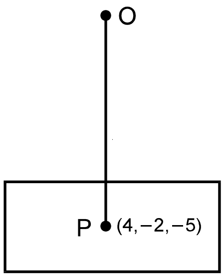


Fig. 35.5

The equation of a plane through the point $P(4, -2, -5)$ is

$$a(x-4) + b(y+2) + c(z+5) = 0 \qquad .....(i)$$

Now OP $\perp$ plane and direction cosines of OP are proportional to

$$4-0, -2-0, -5-0$$

i.e. $\qquad 4, -2, -5$.

Substituting $4, -2$ and -5 for a, b and c in (i), we get

$$4(x-4) - 2(y+2) - 5(z+5) = 0$$

or $\qquad 4x - 16 - 2y - 4 - 5z - 25 = 0$

or $\qquad 4x - 2y - 5z = 45$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.1

1. Reduce each of the following equations of the plane to the normal form :
   (i) $4x + 12y - 6z - 28 = 0$
   (ii) $3y + 4z + 3 = 0$

2. The foot of the perpendicular drawn from the origin to a plane is the point $(1, -3, 1)$. Find the equation of the plane.

3. The foot of the perpendicular drawn from the origin to a plane is the point (1, –2, 1). Find the equation of the plane.

4. Find the equation of the plane passing through the points

   (a) $\quad (2, 2, -1), (3, 4, 2)$ and $(7, 0, 6)$

(b) $(2, 3, -3)$, $(1, 1, -2)$ and $(-1, 1, 4)$

(c) $(2, 2, 2)$, $(3, 1, 1)$ and $(6, -4, -6)$

5. Show that the equation of the plane passing through the points $(3, 3, 1)$, $(-3, 2 -1)$ and $(8, 6, 3)$ is $4x + 2y - 13z = 5$

6. Find the equation of a plane whose intercepts on the coordinate axes are 2,3 and 4 respectively.

7. Find the intercepts made by the plane $2x + 3y + 4z = 24$ on the co-ordinate axes.

8. Show that the points $(-1, 4, -3)$, $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar.

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?

(ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

(iii) The planes $\vec{r}.(\hat{i} - \hat{j} + 3\hat{k}) = 7$ and $\vec{r}.(3\hat{i} - 12\hat{j} - 5\hat{k}) = 6$ are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$.

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i} - 2\hat{j} + \hat{k}$.

35.6 ANGLE BETWEEN TWO PLANES

Let the two planes p_1 and p_2 be given by

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{....(i)}$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{....(ii)}$$

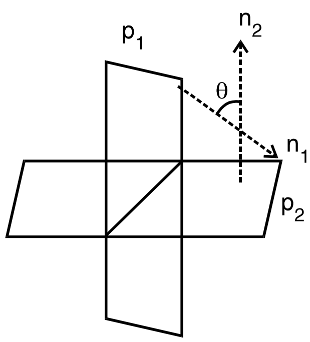


Fig. 35.5

Let the two planes intersect in the line l and let n_1 and n_2 be normals to the two planes. Let θ be the angle between two planes.

$\therefore$ The direction cosines of normals to the two planes are

$$\pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and $$\pm\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad \pm\frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad \pm\frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

$\therefore$ $\cos\theta$ is given by $\cos\theta = \pm \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$

where the sign is so chosen that $\cos\ \theta$ is positive

Corollary 1 :

When the two planes are perpendicular to each other then $\theta = 90°$ i.e., $\cos\theta = 0$

$\therefore$ The condition for two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ to be perpendicular to each other is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2 :

If the two planes are parallel, then the normals to the two planes are also parallel

$$\therefore \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition of parallelism of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

This implies that the equations of two parallel planes differ only by a constant. Therefore, any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant.

Example 35.9 Find the angle between the planes

$$3x + 2y - 6z + 7 = 0 \qquad \text{......(i)}$$

and

$$2x + 3y + 2z - 5 = 0 \qquad \text{......(ii)}$$

Solution : Here $a_1 = 3, b_1 = 2, c_1 = -6$

and $a_2 = 2, b_2 = 3, c_2 = 2$

$\therefore$ If θ is the angle between the planes (i) and (ii), then

$$\cos\theta = \frac{3.2 + 2.3 + (-6).2}{\sqrt{3^2 + 2^2 + (-6)^2}\ \sqrt{2^2 + 3^2 + 2^2}} = 0$$

$\therefore\ \theta = 90°$

Thus the two planes given by (i) and (ii) are perpendicular to each other.

Example 35.10 Find the equation of the plane parallel to the plane $x - 3y + 4z - 1 = 0$ and passing through the point $(3, 1, -2)$.

Solution : Let the equation of the plane parallel to the plane

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?

(ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

(iii) The planes $\vec{r}.(\hat{i}-\hat{j}+3\hat{k})=7$ and $\vec{r}.(3\hat{i}-12\hat{j}-5\hat{k})=6$ are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i}+3\hat{j}-4\hat{k})=1.$

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i}-2\hat{j}+\hat{k}.$

$$x-3y+4z-1=0 \text{ be } x-3y+4z+k=0 \qquad \text{..... (i)}$$

Since (i) passes through the point $(3, 1, -2)$, it should satisfy it

$\therefore \qquad 3-3-8+k=0 \quad \text{or} \quad k=8$

$\therefore$ The required equation of the plane is $x-3y+4z+8=0$

Example 35.11 Find the equation of the plane passing through the points $(-1, 2, 3)$ and $(2, -3, 4)$ and which is perpendicular to the plane $3x+y-z+5=0$

Solution : The equation of any plane passing through the point $(-1, 2, 3)$ is

$$a(x+1)+b(y-2)+c(z-3)=0 \qquad \text{.....(i)}$$

Since the point $(2, -3, 4)$ lies on the plane (i)

$\therefore \qquad 3a-5b+c=0 \qquad \text{....(ii)}$

Again the plane (i) is perpendicular to the plane $3x+y-z+5=0$

$\therefore \qquad 3a+b-c=0 \qquad \text{....(iii)}$

From (ii) and (iii), by cross multiplication method, we get,

$$\frac{a}{4}=\frac{b}{6}=\frac{c}{18} \quad \text{or} \quad \frac{a}{2}=\frac{b}{3}=\frac{c}{9}$$

Hence the required equation of the plane is

$$2(x+1)+3(y-2)+9(z-3)=0 \quad \text{....[From (i)]}$$

or $\qquad 2x+3y+9z=31$

Example 35.12 Find the equation of the plane passing through the point $(2, -1, 5)$ and perpendicular to each of the planes

$$x+2y-z=1 \qquad \text{and } 3x-4y+z=5$$

Solution : Equation of a plane passing through the point $(2, -1, 5)$ is

$$a(x-2)+b(y+1)+c(z-5)=0 \qquad \text{.....(i)}$$

As this plane is perpendicular to each of the planes

$$x+2y-z=1 \qquad \text{and} \qquad 3x-4y+z=5$$

We have $\qquad a.1+b.2+c.(-1)=0$

and $\qquad a.3+b.(-4)+c.(1)=0$

or $$a + 2b - c = 0 \qquad \text{.........(ii)}$$

$$3a - 4b + c = 0 \qquad \text{........(iii)}$$

From (ii) and (iii), we get

$$\frac{a}{2-4} = \frac{b}{-3-1} = \frac{c}{-4-6}$$

or $$\frac{a}{-2} = \frac{b}{-4} = \frac{c}{-10} \quad \text{or} \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{5} = \lambda \text{ (say)}$$

$\therefore$ $$a = \lambda,\ b = 2\lambda \text{ and } c = 5\lambda$$

Substituting for a, b and c in (i), we get

$$\lambda(x-2) + 2\lambda(y+1) + 5\lambda(z-5) = 0$$

or $$x - 2 + 2y + 2 + 5z - 25 = 0$$

or $$x + 2y + 5z - 25 = 0$$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.2

1. Find the angle between the planes

   (i) $2x - y + z = 6$ and $x + y + 2z = 3$

   (ii) $3x - 2y + z + 17 = 0$ and $4x + 3y - 6z + 25 = 0$

2. Prove that the following planes are perpendicular to each other.

   (i) $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

   (ii) $3x + 4y - 5z = 9$ and $2x + 6y + 6z = 7$

3. Find the equation of the plane passing through the point $(2, 3, -1)$ and parallel to the plane $2x + 3y + 6z + 7 = 0$

4. Find the equation of the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$

5. Find the equation of the plane which passes through the origin and is perpendicular to each of the planes $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

35.9 DISTANCE OF A POINT FROM A PLANE

Let the equation of the plane in normal form be

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p \text{ where } p>0 \qquad \text{.......(i)}$$

Case I : Let the point $P(x', y', z')$ lie on the same side of the plane in which the origin lies.

Let us draw a plane through point P parallel to plane (i). Its equation is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p' \qquad \text{.....(ii)}$$

where p' is the length of the perpendicular drawn from origin upon the plane given by (ii). Hence the perpendicular distance of P from plane (i) is $p - p'$

As the plane (ii) passes through the point (x', y', z'),

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = p'$$

$\therefore$ The distance of P from the given plane is

$$p - p' = p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)$$

Case II : If the point P lies on the other side of the plane in which the origin lies, then the distance of P from the plane (i) is,

$$p' - p = x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p$$

Note : If the equation of the plane be given as $ax + by + cz + d = 0$, we have to first convert it into the normal form, as discussed before, and then use the above formula.

Example 35.13 Find the distance of the point (1,2,3) from the plane $3x - 2y + 5z + 17 = 0$

Solution : Required distance $= \dfrac{3.1 - 2.2 + 5.3 + 17}{\sqrt{3^2 + (-2)^2 + 5^2}} = \dfrac{31}{\sqrt{38}}$ units.

Example 35.14 Find the distance between the planes

$$x - 2y + 3z - 6 = 0$$

and

$$2x - 4y + 6z + 17 = 0$$

Solution : The equations of the planes are

$$x - 2y + 3z - 6 = 0 \qquad \text{....(i)}$$

$$2x - 4y + 6z + 17 = 0 \qquad \text{....(ii)}$$

Here

$$\frac{1}{2} = \frac{(-2)}{(-4)} = \frac{3}{6}$$

$\therefore$ Planes (i) and (ii) are parallel

Any point on plane (i) is (6, 0, 0)

$\therefore$ Distance between planes (i) and (ii) = Distance of point (6,0,0) from (ii)

$$= \frac{2 \times 6 - 4.0 + 6.0 + 17}{\sqrt{(2)^2 + (-4)^2 + 6^2}}$$

$$= \frac{29}{\sqrt{56}} \text{ units} = \frac{29}{2\sqrt{14}} \text{ units}$$

CHECK YOUR PROGRESS 35.3

1. Find the distance of the point

   (i) $(2, -3, 1)$ from the plane $5x - 2y + 3z + 11 = 0$

   (ii) $(3, 4, -5)$ from the plane $2x - 3y + 3z + 27 = 0$

2. Find the distance between planes

$3x + y - z - 7 = 0$ and $6x + 2y - 2z + 11 = 0$

LET US SUM UP

- A plane is a surface such that if any two points are taken on it, the line joining these two points lies wholly in the plane.
- $\vec{r}.\hat{n} = d$ is the vector equation of a plane where $\hat{n}$ is a unit vector normal to the plane and d is the distance of the plane from origin.
- Corresponding cartesian form of the equation is $lx + my + nz = d$, where l,m,n are the direction cosines of the normal vector to the plane and d is the distance of the plane from origin.
- $\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ is another vecter equation of a plane where $\vec{a}$ is position vecter of a given point on the plane and $\vec{n}$ is a vecter normal to the plane.
- Corresponding cartesion form of this equation is $a(x-x_1) + b(y-y_1) + c(z-z_1)=0$, where a,b,c are the direction ratios of normal to the plane and (x_1,y_1,z_1) are coordinates of given point on plane.
- $\left(\vec{r} - \vec{a}\right).\{\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\} = 0$ is the equation of a plane possing through three points with position vecter $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.
- Its corresponding cartesian equation is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Equation of a plane in the intercept from is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

  where a,b and c are intercepts made by the plane on x,y and z axes respectively.
- Angle θ between two planes $a_1x + b_1y + c_1z + d_1 = 0$

  and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- Two planes are perpendicular to each other if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- Two planes are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- Distance of a point (x', y', z') from a plane

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p \text{ is}$$

$|p - (x' \cos\alpha + y' \cos\beta + z' \cos\gamma)|$, where the point (x', y', z') lies on the same side of the plane in which the origin lies.

SUPPORTIVE WEB SITES

http://www.mathopenref.com/plane.html

http://en.wikipedia.org/wiki/Plane_(geometry)

TERMINAL EXERCISE

1. Find the equation of a plane passing through the point $(-2, 5, 4)$

2. Find the equation of a plane which divides the line segment joining the points $(2, 1, 4)$ and $(2, 6, 4)$ internally in the ratio of 2 : 3.

3. Find the equation of the plane through the points $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.

4. Show that the four points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. Also find the equation of the palne in which they lie.

5. The foot of the perpendicular drawn from $(1, -2, -3)$ to a plane is $(3, 2, -1)$. Find the equation of the plane.

6. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$

7. Prove that the planes $3x - 5y + 8z - 2 = 0$ and $12x - 20y + 32z + 9 = 0$ are parallel.

8. Determine the value of k for which the planes $3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ may be perpendicular to each other.

9. Find the distance of the point $(3, 2, -5)$ from the plane $2x - 3y - 5z = 7$

10. Find the vector equation of a plane possing through the point (3,-1,5) and perpendicular to the line with direction ratios (2,-3,1).

11. Find the vector equation of a plane perpendicular to the vecter $3\hat{i}+5\hat{j}-6\hat{k}$ and at a distance of 7 units from origin.

12. Find the vector equation of a plane passing through the points A(-2,6,-6), B(-3,10,-9), and C(-5,0,-6).

ANSWERS

CHECK YOUR PROGRESS 35.1

1. (i) $\frac{4x}{14}+\frac{12y}{14}-\frac{6z}{14}=2$ (ii) $-\frac{3}{5}y-\frac{4}{5}z=\frac{3}{5}$

2. $x-3y+z-11=0$ 3. $x-2y+z-6=0$ 4. (a) $5x+2y-3z-17=0$

(b) $3x-y+2=0$ (c) $x+2y-2=4$ 6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

7. Intercepts on x,y & z axes are 12,8,6 respectively.

9. (i) $\frac{1}{\sqrt{26}},\frac{-4}{\sqrt{26}},\frac{3}{\sqrt{26}}$ (ii) $\frac{17}{\sqrt{14}}$ units (iii) perpendicular

10. $2x+3y-4z=1$ 11. $\vec{x}.\left(2\hat{x}+3\hat{y}-3\hat{k}\right)=5$ 12. $\vec{x}.\left(\hat{x}-2\hat{y}+\hat{k}\right)+1=0$

CHECK YOUR PROGRESS 35.2

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ 3. $2x+3y+6z=7$ 4. $2x+2y-3z+3=0$

5. $2x-2y+z=0$

CHECK YOUR PROGRESS 35.3

1. (i) $\frac{30}{\sqrt{38}}$ units (ii) $\frac{6}{\sqrt{22}}$ units. 2. $\frac{25}{2\sqrt{11}}$ units.

TERMINAL EXERCISE

1. $a(x+2)+b(y-5)+c(z-4)=0$

2. $a(x-2)+b(y-3)+c(z-4)=0$

3. $2x+3y-3z-5=0$ 4. $5x-7y+11z+4=0$ 5. $x+2y+z=6$

6. $\frac{\pi}{3}$ 8. $k=-1$ 9. $\frac{18}{\sqrt{38}}$ 10. $\{\vec{r}-\left(-3\hat{i}+\hat{j}+5\hat{k}\right)\}.\left(2\hat{i}-3\hat{j}+\hat{k}\right)=0$

11. $\vec{r}.\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\}=7$

12. $\{\vec{x}.\left(-2\hat{i}+6\hat{j}-6\hat{k}\right)\}.\{\left(-\hat{i}+4\hat{j}-3\hat{k}\right)\times\left(-3\hat{i}-6\hat{j}\right)\}=0$

37

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LINEAR PROGRAMMING

37.1 INTRODUTION TO LINEAR PROGRAMMING PROBLEMS

A toy dealer goes to the wholesale market with Rs. 1500.00 to purchase toys for selling. In the market there are various types of toys available. From quality point of view, he finds that the toy of type 'A' and type 'B' are suitable. The cost price of type 'A' toy is Rs. 300 each and that of type 'B' is Rs. 250 each. He knows that the type 'A' toy can be sold for Rs. 325 each, while the type 'B' toy can be sold for Rs. 265 each. Within the amount available to him he would like to make maximum profit. His problem is to find out how many type 'A' and type 'B' toys should be purchased so to get the maximum profit.

He can prepare the following table taking into account all possible combinations of type 'A' and type 'B' toys subject to the limitation on the investment.

'A' type	'B' type	Investment	Amount after sale (including the unutilised amount if any)	Profit on the investment
0	6	1500.00	1590.00	90.00
1	4	1300.00	1585.00	85.00
2	3	1350.00	1595.00	95.00
3	2	1400.00	1605.00	105.00
4	1	1450.00	1615.00	115.00
5	0	1500.00	1625.00	125.00

Now, the decision leading to maximum profit is clear. Five type A toys should be purchased.

The above problem was easy to handle because the choice was limited to two types, and the number of items to be purchased was small. Here, all possible combinations were thought of and the corresponding gain calculated. But one must make sure that he has taken all possibilities into account.

A situation faced by a retailer of radio sets similar to the one given above is described below.

A retailer of radio sets wishes to buy a number of transistor radio sets from the wholesaler. There are two types (type A and type B) of radio sets which he can buy. Type A costs Rs.360 each and type *B* costs Rs. 240 each. The retailer can invest up to Rs. 5760. By selling the radio sets, he can make a profit of Rs. 50 on each set of type A and of Rs. 40 on each set of type B. How many of each type should he buy to maximize his total profit?

Here we have to maximize the profit. Sometimes we come across a problem in which the costs are to be minimized. Consider the following problem :

Two tailors A and B earn Rs.150 and Rs.200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 4 shirts and 7 pants per day. How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost?

In this problem we have to minimise the labour cost.

These types of problems of maximisation and minimisation are called optimisation problems.

The technique followed by mathematicians to solve such problems is called **'Linear Programming'.**

OBJECTIVES

After studying this lesson, you will be able to :

- undertstand the terminology used in linear programming;
- convert different type of problems into a linear programming problem;
- use graphical mehtod to find solution of the linear programming problems

EXPECTED BACKGROUND KNOWLEDGE

- good idea of converting a mathematical information into a in equality
- to be able to solve system of on equalities using graphical method.

37.2 DEFINITIONS OF VARIOUS TERMS INVOLVED IN LINEAR PROGRAMMING

A close examination of the examples cited in the introduction points out one basic property that all these problems have in common, i.e., in each example, we were concerned with maximising or minimising some quantity.

In first two examples, we wanted to maximise the return on the investment. In third example, we wanted to minimise the labour cost. In linear programming terminology the maximization or minimization of a quantity is referred to as the objective of the problem.

37.2.1 OBJECTIVE FUNCTION

In a linear programming problem. z, the linear function of the variables which is to be optimized is called **objective function.**

Here, a linear form means a mathematical expression of the type

$$a_1x_1 + a_2\,x_2 + ... + a_nx_n\,,$$

where $a_1,\ a_{2,\,.....,}\ a_n$ are **constants** and $x_1,\ x_{2,}\ _{.....,}\ x_n$ are **variables.**

In linear programming problems, the products, services, projects etc. that are competing with each other for sharing the given limited resources are called the variables or decision variables.

37.2.2 CONSTRAINTS

The limitations on resources (like cash in hand, production capacity, man power, time, machines, etc.) which are to be allocated among various competing variables are in the form of linear equations or inequations (inequalities) and are called **constraints** or **restrictions.**

37.2.3 NON-NEGATIVE RESTRICTIONS

All decision variables must assume non-negative values, as negative values of physical quantities is an impossible situation.

37.3 FORMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of a linear programming problem as a mathematical model involves the following key steps.

Step 1 : Identify the decision variables to be determined and express them in terms of algebraic symbols such as $x_{1,} x_{2,} x_{3,}$ ....

Step 2 : Identify all the limitations in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 3 : Identify the objective which is to be optimised (maximised or minimised) and express it as a linear function of the above defined decision variables.

Example 37.1 A retailer wishes to buy a number of transistor radio sets of types *A* and *B*. Type *A* cost Rs.360 each and type *B* cost Rs. 240 each. The retailer knows that he cannot sell more than 20 sets, so he does not want to buy more than 20 sets and he cannot afford to pay more than Rs.5760. His expectation is that he would get a profit of Rs.50 for each set of type *A* and Rs.40 for each set of type *B*. Form a mathematical model to find how many of each type should be purchased in order to make his total profit as large as possible?

Solution : Suppose the retailer purchases x_1 sets of type *A* and x_2 sets of type *B*. Since the number of sets of each type is non-negative, so we have

$$x_1 \geq 0, \qquad \cdots(1)$$

$$x_2 \geq 0, \qquad \cdots(2)$$

Also the cost of x_1 sets of type *A* and x_2 sets of type *B* is 360 x_1 + 240 x_2 and it should be equal to or less than Rs.5760, that is,

$$360\,x_1 + 240\,x_2 \leq 5760$$

or $$3x_1 + 2x_2 \leq 48 \qquad \cdots(3)$$

Further, the number of sets of both types should not exceed 20, so

$$x_1 + x_2 \leq 20 \qquad \cdots (4)$$

Since the total profit consists of profit derived from selling the x_1 type *A* sets and x_2 type *B* sets, therefore, the retailer earns a profit of Rs.50 x_1 on type *A* sets and Rs.40 x_2 on type *B* sets. So the total profit is given by :

$$z = 50x_1 + 40x_2 \qquad \cdots (5)$$

Hence, the mathematical formulation of the given linear programming problem is as follows :

Find x_1, x_2 which

Maximise z = $50x_1 + 40x_2$ **(Objective function) subject to the conditions**

$$\left.\begin{array}{l} 3x_1 + 2x_2 \leq 48 \\ x_1 + x_2 \leq 20 \\ x_1 \geq 0, x_2 \geq 0 \end{array}\right\} \textbf{Constraints}$$

Example 37.2 A soft drink company has two bottling plants, one located at *P* and the other at *Q*. Each plant produ ces three different soft drinks *A*, *B*, and *C*. The capacities of the two plants in terms of number of bottles per day, are as follows :

Products ↓ \ Plants →	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of May, there will be a demand for 24000 bottles of *A*, 16000 bottles of *B* and 48000 bottles of *C*. The operating cost per day of running plants *P* and *Q* are respectively Rs.6000 and Rs.4000. How many days should the firm run each plant in the month of May so that the production cost is minimised while still meeting the market demand.

Solution : Suppose that the firm runs the plant *P* for x_1 days and plant Q for x_2 days in the month of May in order to meet the market demand.

The per day operating cost of plant *P* is Rs.6000. Therefore, for x_1 days the operating cost will be Rs.6000 x_1.

The per day operating cost of plant *Q* is Rs.4000. Therefore, for x_2 days the operating cost will be Rs.4000 x_2.

Thus the total operating cost of two plants is given by :

$$z = 6000x_1 + 4000x_2 \qquad \cdots (1)$$

Plant P produces 3000 bottles of soft drink A per day. Therefore, in x_1 days plant P will produce 3000 x_1 bottles of soft drink A.

Plant Q produces 1000 bottles of soft drink A per day.

Therefore, in x_2 days plant Q will produce 1000 x_2 bottles of soft drink A.

Total production of soft drink A in the supposed period is 3000 x_1 + 1000 x_2

But there will be a demand for 24000 bottles of this soft drink, so the total production of this soft drink must be greater than or equal to this demand.

$\therefore \quad 3000x_1 + 1000x_2 \geq 24000$

or $\quad 3x_1 + x_2 \geq 24 \qquad \cdots (2)$

Similarly, for the other two soft drinks, we have the constraints

$1000x_1 + 1000x_2 \geq 16000$

or $\quad x_1 + x_2 \geq 16 \qquad \cdots (3)$

and

$2000x_1 + 6000x_2 \geq 48000$

or $\quad x_1 + 3x_2 \geq 24 \qquad \cdots (4)$

x_1 and x_2 are non-negative being the number of days, so

$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots (5)$

Thus our problem is to find x_1 and x_2 which

Minimize $z = 6000x_1 + 4000x_2$ ***(objective function)***

subject to the conditions

$$\left.\begin{aligned} 3x_1 + x_2 &\geq 24 \\ x_1 + x_2 &\geq 16 \\ x_1 + 3x_2 &\geq 24 \end{aligned}\right\} \qquad \textit{(constraints)}$$

and $x_1 \geq 0,\ x_2 \geq 0$

Example 37.3 A firm manufactures two types of products *A* and *B* and sells them at a profit of Rs.2 on type *A* and Rs.3 on type *B*. Each product is processed on two machines *G* and *H*. Type A requires one minute of processing time on *G* and 2 minutes on H, type *B* requires one minute on *G* and one minute on *H*. The machine *G* is available for not more than 6 hours and 40 minutes while machine *H* is available for 10 hours during one working day. Formulate the problem as a linear programming problem so as to maximise profit.

Solution : Let x_1 be the number of products of type *A* and x_2 be the number of products of type *B*.

The given information in the problem can systematically be arranged in the form of following table :

Machine	Processing time of the products (in minute)		Available time (in minute)
	Type *A* (x_1 units)	Type *B* (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs.2	Rs.3	

Since the profit on type *A* is Rs.2 per product, so the profit on selling x_1 units of type *A* will be $2x_1$. Similarly, the profit on selling x_2 units of type *B* will be $3x_2$. Therefore, total profit on selling x_1 units of type *A* and x_2 units of type *B* is given by

$$z = 2x_1 + 3x_2 \qquad \text{(objective function)} \qquad \cdots(1)$$

Since machine G takes 1 minute time on type *A* and 1 minute time on type *B*, therefore, the total number of minutes required on machine *G* is given by

$$x_1 + x_2$$

But the machine *G* is not available for more than 6 hours and 40 minutes (i.e., 400 minutes). Therefore,

$$x_1 + x_2 \leq 400 \qquad \cdots(2)$$

Similarly, the total number of minutes required on machine *H* is given by

$$2x_1 + x_2$$

Also, the machine *H* is available for 10 hours (i.e., 600 minutes). Therefore,

$$2x_1 + x_2 \leq 600 \qquad \cdots(3)$$

Since, it is not possible to produce negative quantities, so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots (4)$$

Thus, the problem is to find x_1 and x_2 which

Maximize $\quad z = 2x_1 + 3x_2 \qquad$ (objective function)

subject to the conditions

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Example 37.4 A furniture manufacturer makes two types of sofas – sofa of type *A* and sofa of type *B*. For simplicity, divide the production process into three distinct operations, say carpentary, finishing and upholstery. The amount of labour required for each operation varies. Manufacture of a sofa of type *A* requires 6 hours of carpentary, 1 hour of finishing and 2 hours of upholstery. Manufacture of a sofa of type *B* requires 3 hours of carpentary, 1 hour of finishing and 6 hours of upholstery. Owing to limited availability of skilled labour as well as of tools and equipment, the factory has available each day 96 man hours of carpentary, 18 man hours for finishing and 72 man hours for upholstery. The profit per sofa of type *A* is Rs.80 and the profit per sofa of type *B* is Rs. 70. How many sofas of type *A* and type *B* should be produced each day in order to maximise the profit? Formulate the problems as linear programming problem.

Solution : The different operations and the availability of man hours for each operation can be put in the following tabular form :

Operations	**Sofa of type *A***	**Sofa of type *B***	**Available labour**
Carpentary	6 hours	3 hours	96 man hours
Finishing	1 hour	1 hour	18 man hours
Upholstery	2 hours	6 hours	72 man hours
Profit	Rs. 80	Rs. 70	

Let x_1 be the number of sofas of type *A* and x_2 be the number of sofas of type *B*.

Each row of the chart gives one restriction. The first row says that the amount of carpentary required is 6 hours for each sofa of type *A* and 3 hours for each sofa of type *B*. Further, only 96 man hours of carpentary are available per day. We can compute the total number of man hours of carpentary required per day to produce x_1 sofas of type *A* and x_2 sofas of type *B* as follows:

Number of man - hours per day of carpentary

= { (Number of hours carpentary per sofa of type *A*) $\times$ (Number of sofas of type A) }

+ { (Number of hours carpentary per sofa of type B)$\times$

(Number of sofas of type B) }

$$= 6x_1 + 3x_2$$

The requirement that at most 96 man hours of carpentary per day means

$$6x_1 + 3x_2 \leq 96$$

or $\quad \mathbf{2x_1 + x_2 \leq 32} \qquad \cdots \mathbf{(1)}$

Similarly, second and third row of the chart give the restrictions on finishing and upholstery respectively as

$$\mathbf{x_1 + x_2 \leq 18} \qquad \cdots \mathbf{(2)}$$

and

$$2x_1 + 6x_2 \leq 72$$

or $\quad \mathbf{x_1 + 3x_2 \leq 36} \qquad \cdots \mathbf{(3)}$

Since, the number of the sofas cannot be negative, therefore

$$\mathbf{x_1 \geq 0, x_2 \geq 0} \qquad \cdots \mathbf{(4)}$$

Now, the profit comes from two sources, that is, sofas of type A and sofas of type B. Therefore,

Profit = (Profit from sofas of type A) + (Profit from sofas of type B)

= { (Profit per sofa of type A) $\times$ (Number of sofas of type A) }

+ { (Profit per sofa of type B) $\times$ (Number of sofas of type B) }

$$z = 80x_1 + 70x_2 \ \textbf{(objective function)} \cdots \mathbf{(5)}$$

Thus, the problem is to find x_1 and x_2 which

Maximize $z = 80\,x_1 + 70\,x_2$ (objective function)

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\leq 32 \\ x_1 + x_2 &\leq 18 \\ x_1 + 3x_2 &\leq 36 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}\right\} \qquad \textbf{(Constraints)}$$

CHECK YOUR PROGRESS 37.1

1. A company is producing two products *A* and *B*. Each product is processed on two machines *G* and *H*. Type *A* requires 3 hours of processing time on *G* and 4 hours on *H*; type *B* requires 4 hours of processing time time on G and 5 hours on *H*. The available time is 18 hours and 21 hours for operations on *G* and *H* respectively. The products *A* and *B* can be sold at the profit of Rs. 3 and Rs. 8 per unit respectively. Formulate the problem as a linear programming problem.

2. A furniture dealer deals in only two items, tables and chairs. He has Rs. 5000 to invest and a space to store at most 60 pieces. A table costs him Rs. 250 and a chair Rs. 50. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assuming, he can sell all the items that he buys, how should he invest his money in order that may maximize his profit? Formulate a linear programming problem.

3. A dairy has its two plants one located at *P* and the other at *Q*. Each plant produces two types of products *A* and *B* in 1 kg packets. The capacity of two plants in number of packets per day are as follows:

Products ↓ \ Plants →	P	Q
A	2000	1500
B	4000	6000

   A market survey indicates that during the month of April, there will be a demand for 20000 packets of *A* and 16000 packets of *B*. The operating cost per day of running plants *P* and *Q* are respectively Rs.4000 and Rs.7500. How many days should the firm run each plant in the month of April so that the production cost is minimized while still meeting the market demand? Formulate a Linear programming problem.

4. A factory manufactures two articles *A* and *B*. To manufacture the article *A*, a certain machine has to be worked for 1 hour and 30 minutes and in addition a craftsman has to work for 2 hours. To manufacture the article *B*, the machine has to be worked for 2 hours and 30 minutes and in addition the craftsman has to work for 1 hour and 30 minutes. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article A is Rs.5 and that on each article *B* is Rs.4. If all the articles produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as a linear programming problem.

37.4 GEOMETRIC APPORACH OF LINEAR PROGRAMMING PROBLEM

Let us consider a simple problem in two variables *x* and *y*. Find *x* and *y* which satisfy the following equations

$$x + y = 4$$
$$3x + 4y = 14$$

Solving these equations, we get $x = 2$ and $y = 2$. What happens when the number of equations and variables are more?

Can we find a unique solution for such system of equations?

However, a unique solution for a set of simultaneous equations in n-variables can be obtained if there are exactly n-relations. What will happen when the number of relations is greater than or less then n ?

A unique solution will not exist, but a number of trial solutions can be found. Again, if the number of relations are greater than or less than the number of variables involved and the relation are in the form of inequalities.

Can we find a solution for such a system?

Whenever the analysis of a problem leads to minimising or maximising a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution of the linear programming problem.**

37.5 SOLUTION OF LINEAR PROGRAMMING PROBLEMS

In the previous section we have seen the problems in which the number of relations are not equal to the number of variables and many of the relations are in the form of inequation (i.e., $\leq$ or $\geq$) to maximise (or minimise) a linear function of the variables subject to such conditions.

Now the question is how one can find a solution for such problems?

To answer this questions, let us consider the system of equations and inequations (or inequalities).

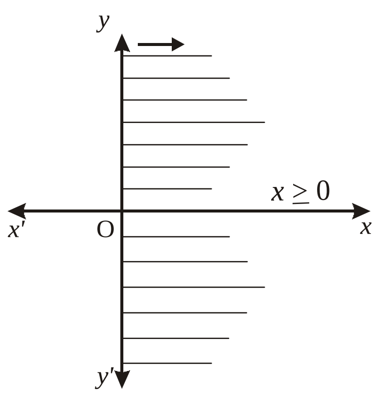

Fig. 37.1

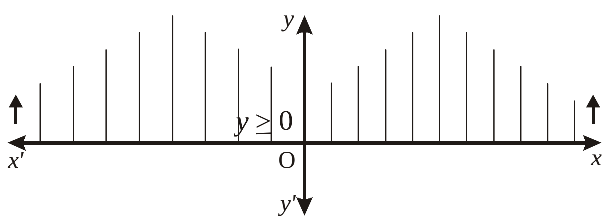

Fig. 37.2

We know that $x \geq 0$ represents a region lying towards the right of y - axis including the y - axis.

Similarly, the region represented by $y \geq 0$, lies above the x - axis including the x-axis.

The question arises: what region will be represented by $x \geq 0$ and $y \geq 0$ simultaneously.

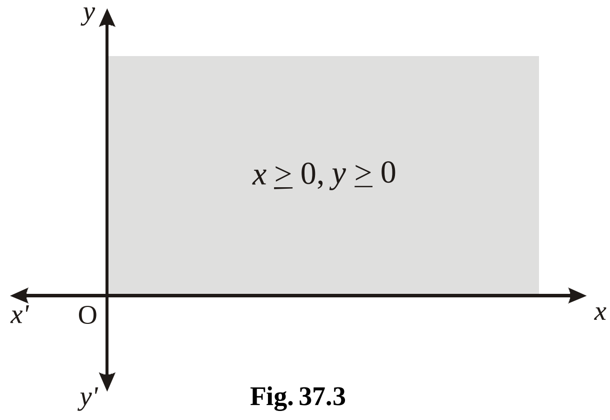

Fig. 37.3

Obviously, the region given by $x \geq 0$, $y \geq 0$ will consist of those points which are common to both $x \geq 0$ and $y \geq 0$. It is the first quadrant of the plane.

Next, we consider the graph of the equation $x + 2y \leq 8$. For this, first we draw the line $x + 2y = 8$ and then find the region satisfying $x + 2y \leq 8$.

Usually we choose x = 0 and calculate the corresponding value of y and choose y = 0 and calculate the corresponding value of x to obtain two sets of values (This method fails, if the line is parallel to either of the axes or passes through the origin. In that case, we choose any arbitrary value for x and choose y so as to satisfy the equation).

Plotting the points (0,4) and (8,0) and joining them by a straight line, we obtain the graph of the line as given in the Fig. 37.4 below.

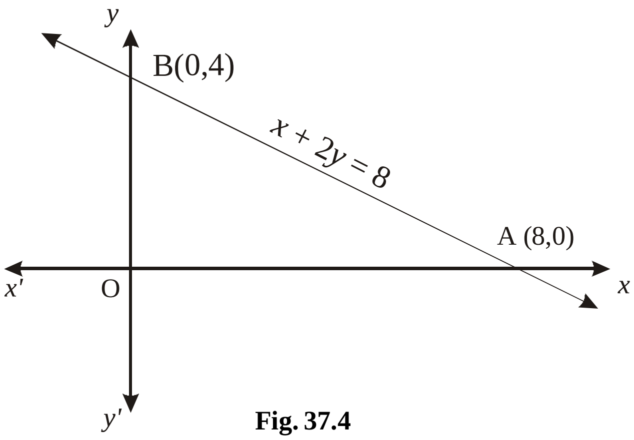

Fig. 37.4

We have already seen that $x \geq 0$ and $y \geq 0$ represents the first quadrant. The graph given by

$x + 2y < 8$ lies towards that side of the line x + 2y = 8 in which the origin is situated because any point in this region will satisfy the inequality. Hence the shaded region in the Fig. 37.5 represents $x \geq 0$, $y \geq 0$ and $x + 2y \leq 8$ simultaneously.

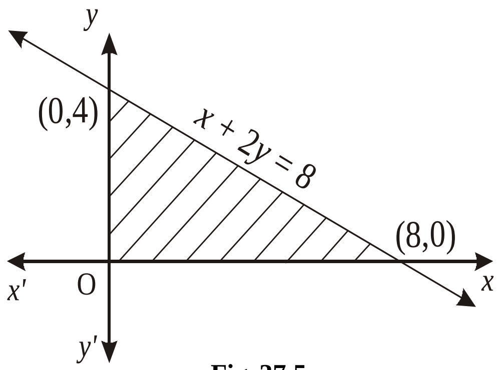

Fig. 37.5

Similarly, if we have to consider the regions bounded by $x \geq 0$, $y \geq 0$ and $x + 2y \geq 8$, then it will lie in the first quadrant and on that side of the line $x + 2y = 8$ in which the origin is not located. The graph is shown by the shaded region, in Fig. 37.6

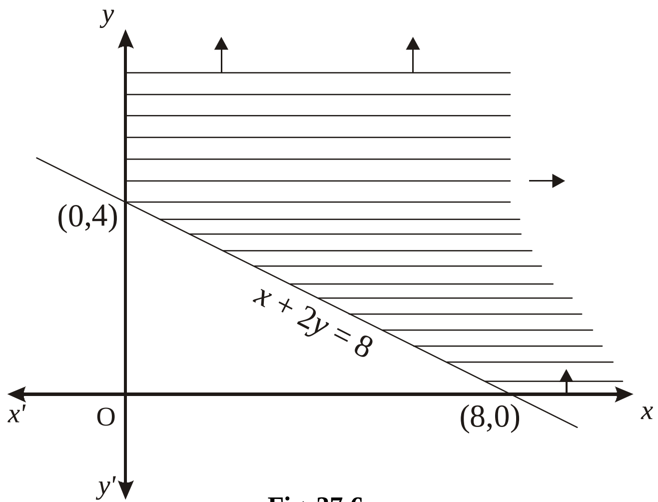

Fig. 37.6

The shaded region in which all the given constraints are satisfied is called the **feasible region.**

37.5.1 Feasible Solution

A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution** of the problem.

37.5.2 Optimal Solution

A feasible solution of a linear programming problem which optimises its objective functions is called the **optimal solution** of the problem.

Note : *If none of the feasible solutions maximise (or minimise) the objective function, or if there are no feasible solutions, then the linear programming problem has* ***no solution.***

In order to find a graphical solution of the linear programming problem, following steps be employed.

Step 1 : Formulate the linear programming problem.

Step 2 : Graph the constraints (inequalities), by the method discussed above.

Step 3 : Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to' constraints the region is generally below the lines and 'for greater than or equal to' constraints, the region is above the lines.

Step 4 : Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.

Step 5 : Evaluate the objective function at each of the vertex (corner point)

Step 6 : Identify the optimum value of the objective function.

Example 37.5 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities, which is as follows :

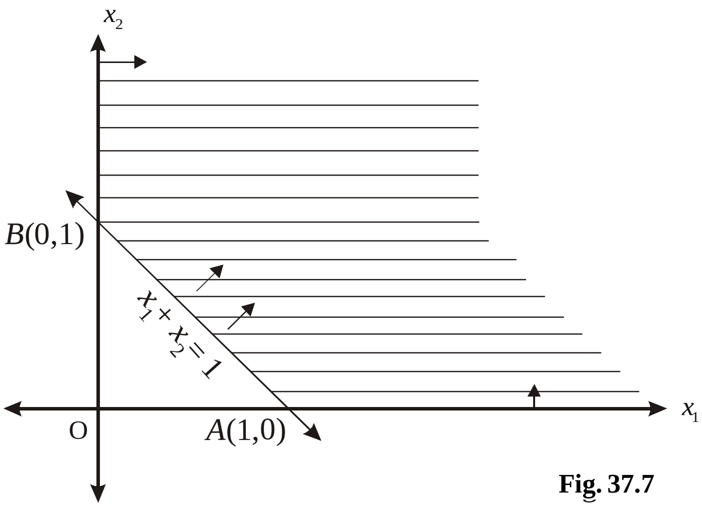

Fig. 37.7

As we have discussed earlier that the region satisfied by $x_1 \geq 0$ and $x_2 \geq 0$ is the first quadrant and the region satisfied by the line $x_1 + x_2 \geq 1$ along with $x_1 \geq 0,\ x_2 \geq 0$ will be on that side of the line $x_1 + x_2 = 1$ in which the origin is not located. Hence, the shaded region is our feasible solution because every point in this region satisfies all the constraints. Now, we have to find optimal solution. The vertex of the feasible region are A (1,0) and B (0,1).

The value of z at A = 1

The value of z at B = 2

Take any other point in the feasible region say (1,1), (2,0), (0,2) etc. We see that the value of z is minimum at A (1,0).

Example 37.6 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities (as discussed earlier) which is as follows :

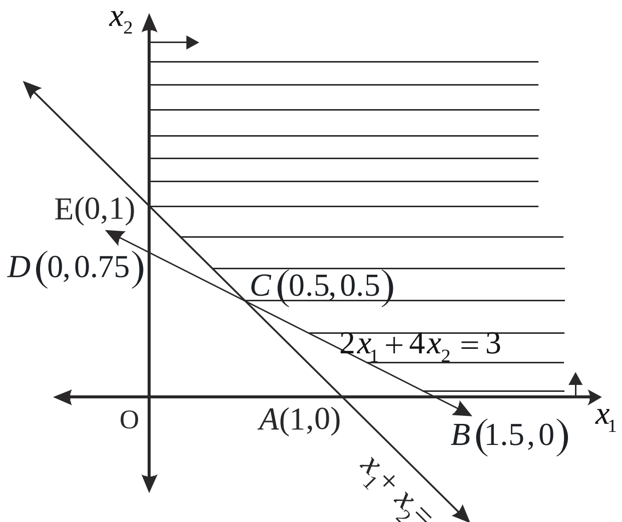

Fig. 37.8

The shaded region is the feasible region. Every point in the region satisfies all the mathematical inequalities and hence the feasible solution.

Now, we have to find the optimal solution.

The value of z at B $(1.5, 0)$ is 1.5

The value of z at $C\ (0.5, 0.5)$ is 1.5

The value of z at E (0,1) is 2

If we take any point on the line $2x_1 + 4x_2 = 3$ between B and C we will get $\frac{3}{2}$ and elsewhere in the feasible region greater than $\frac{3}{2}$. Of course, the reason any feasible point (between B and C) on $2x_1 + 4x_2 = 3$ minimizes the objective function (equation) $z = x_1 + 2x_2$ is that the two lines are parallel (both have slope $-\frac{1}{2}$). Thus this linear programming problem has infinitely many solutions and two of them occur at the vertices.

Example 37.7 Maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Solution : The objective function is to maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

First of all we draw the graphs of these inequalities, which is as follows :

The shaded region $OABC$ is the feasible region. Every point in the region satisfies all the mathematical inequations and hence the feasible solutions.

Now, we have to find the optimal solution.

The value of z at A (160, 0) is 40.00

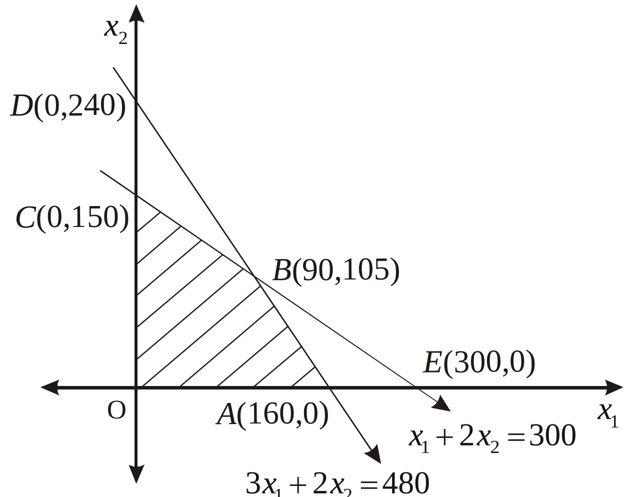

Fig. 37.9

The value of z at B (90, 105) is 69.75.

The value of z at C (0, 150) is 67.50

The value of z at O (0, 0) is 0.

If we take any other value from the feasible region say (60, 120), (80, 80) etc. we see that still the maximum value is 69.75 obtained at the vertex B (90, 105) of the feasible region.

Note : *For any linear programming problem that has a solution, the following general rule is true.*

If a linear programming problem has a solution it is located at a vertex of the feasible region. If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the feasible region. In either case, the value of the objective function is unique.

Example 37.8 In a small scale industry a manufacturer produces two types of book cases. The first type of book case requires 3 hours on machine A and 2 hours on machines B for completion, whereas the second type of book case requires 3 hours on machine A and 3 hours on machine B. The machine A can run at the most for 18 hours while the machine B for at the most 14 hours per day. He earns a profit of Rs. 30 on each book case of the first type and Rs. 40 on each book case of the second type.

How many book cases of each type should he make each day so as to have a maximum porfit?

Solution : Let x_1 be the number of first type book cases and x_2 be the number of second type book cases that the manufacturer will produce each day.

Since x_1 and x_2 are the number of book cases so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots \textbf{(1)}$$

Since the first type of book case requires 3 hours on machine A, therefore, x_1 book cases of first type will require $3x_1$ hours on machine A. second type of book case also requires 3 hours on machine A, therefore, x_2 book cases of second type will require $3x_2$ hours on machine A. But the working capacity of machine A is at most 18 hours per day, so we have

$$3x_1 + 3x_2 \leq 18$$

or $\quad x_1 + x_2 \leq 6 \qquad \cdots$ **(2)**

Similarly, on the machine B, first type of book case takes 2 hours and second type of book case takes 3 hours for completion and the machine has the working capacity of 14 hours per day, so we have

$$\mathbf{2x_1 + 3x_2 \le 14} \qquad \cdots \textbf{(3)}$$

Profit per day is given by

$$z = 30x_1 + 40x_2 \qquad \cdots \textbf{(4)}$$

Now, we have to determine x_1 and x_2 such that

Maximize $z = 30x_1 + 40x_2$ (objective function) subject to the conditions

$$\left.\begin{array}{l} x_1 + x_2 \le 6 \\ 2x_1 + 3x_2 \le 14 \\ x_1 \ge 0,\ x_2 \ge 0 \end{array}\right\} \qquad \textbf{constraints}$$

We use the graphical method to find the solution of the problem. First of all we draw the graphs of these inequalities, which is as follows :

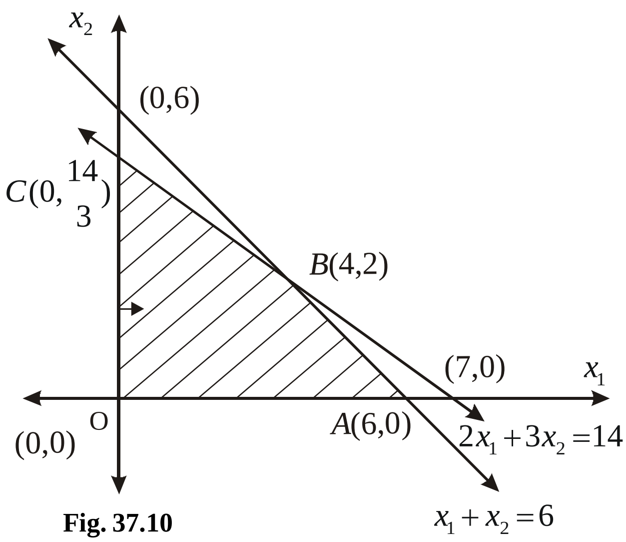

Fig. 37.10

The shaded region OABC is the feasible region. Every point in the region satisfies all the mathematical inequations and hence known as feasible solution.

We know that the optimal solution will be obtained at the vertices O (0, 0), A (6, 0). B (4,2). Since the co-ordinates of C are not integers so we don't consider this point. Co-ordinates of B are calculated as the intersection of the two lines.

Now the profit at O is zero.

Profit at A = 30 × 6 + 40 × 0

= 180

Profit at B = 30 × 4 + 40 × 2

$= 120 + 80$

$= 200$

Thus the small scale manufacturer gains the maximum profit of Rs.200 if he prepares 4 first type book cases and 2 second type book cases.

Example 37.9 Maximize the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

Solutions : First we graph the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

The shaded portion is the set of feasible solution.

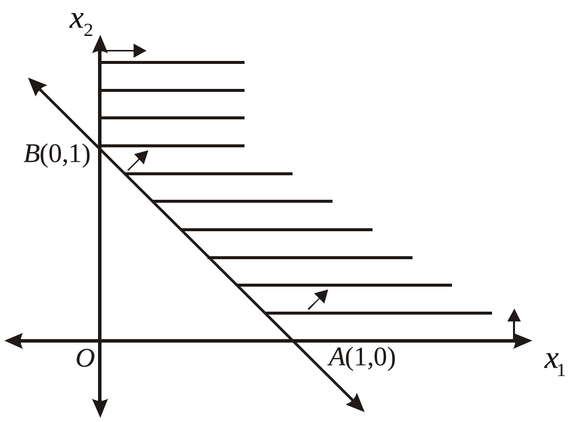

Fig. 37.11

Now, we have to maximize the objective function.

The value of z at $A(1, 0)$ is 1.

The value of z at $B(0, 1)$ is 2.

If we take the value of z at any other point from the feasible region, say (1, 1) or (2, 3) or (5, 4) etc, then we notice that every time we can find another point which gives the larger value than the previous one. Hence, there is no feasible point that will make z largest. Since there is no feasible point that makes z largest, we conclude that this linear programming problem has no solution.

Example 37.10 Solve the following problem graphically.

Minimize $z = 2x_1 - 10x_2$

subject to the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : First we graph the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

$$x_2 - x_1 \leq 0,$$

or $$5x_2 - x_1 \geq 5$$

$$x_1 - x_2 = 0$$

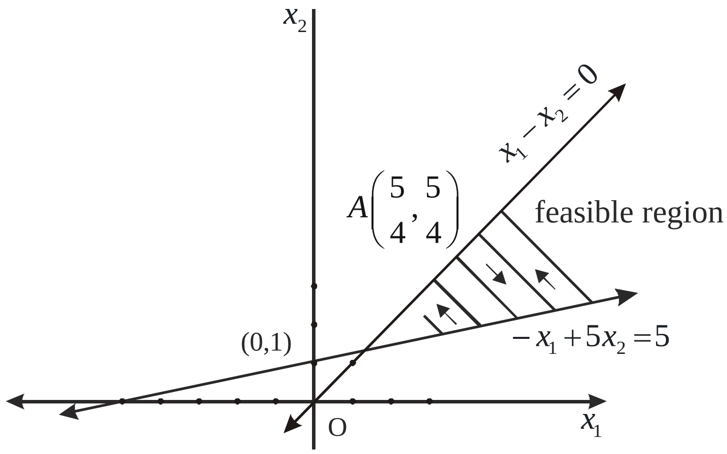

Fig. 37.12

The shaded region is the feasible region.

Here, we see that the feasible region is unbounded from one side.

But it is clear from Fig. 37.26 that the objective function attains its minimum value at the point *A* which is the point of intersection of the two lines $x_1 - x_2 = 0$ and $-x_1 + 5x_2 = 5$.

Solving these we get $x_1 = x_2 = \frac{5}{4}$

Hence, z is minimum when $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{4}$, and its minimum value is

$$2 \times \frac{5}{4} - 10 \times \frac{5}{4} = -10.$$

***Note :** If we want to find max. z with these constraints then it is not possible in this case because the feasible region is unbounded from one side.*

CHECK YOUR PROGRESS 37.2

Solve the following problems graphically

1. **Maximize** $z = 3x_1 + 4x_2$

   subject to the conditions

2. **Maximize** $\neq = 2x_1 + 3x_2$

   subject to the conditions

$$x_1 + x_2 \le 40$$
$$x_1 + 2x_2 \le 60$$
$$x_1 \ge 0,\ x_2 \ge 0$$

$$x_1 + x_2 \le 400$$
$$2x_1 + x_2 \le 600$$
$$x_1 \ge 0, x_2 \ge 0$$

3. **Minimize** $z = 60x_1 + 40x_2$

subject to the conditions

$$3x_1 + x_2 \ge 24$$
$$x_1 + x_2 \ge 16$$
$$x_1 + 3x_2 \ge 24$$
$$x_1 \ge 0,\ x_2 \ge 0$$

4. **Maximize** $z = 20x_1 + 30x_2$

subject to the conditions

$$x_1 + x_2 \le 12,$$
$$5x_1 + 2x_2 \le 50$$
$$x_1 + 3x_2 \le 30,$$
$$x_1 \ge 0,\ x_2 \ge 0$$

5. **Maximize** $z = 50x_1 + 15x_2$

subject to the conditions

$$5x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 60$$
$$x_1 \ge 0, x_2 \ge 0$$

6. **Minimize** $z = 4000x_1 + 7500x_2$

subject to the conditions

$$4x_1 + 3x_2 \ge 40$$
$$2x_1 + 3x_2 \ge 8$$
$$x_1 \ge 0, x_2 \ge 0$$

LET US SUM UP

- Linear programming is a technique followed by mathematicians to solve the optimisation problems.
- A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution**.
- A feasible solution of a linear programming problem which optimises its objective function is called the **Optimal solution** of the problem.
- The optimal solution of a linear programming problem is located at a vertex of the set of feasible region.
- If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the set of feasible region. But in all the cases the value of the objective function remains the same.

SUPPORTIVE WEB SITES

http://people.brunel.ac.uk/~mastjjb/jeb/or/morelp.html

http://en.wikipedia.org/wiki/Simplex_algorithm

http://www.youtube.com/watch?v=XbGM4LjM52k

TERMINAL EXERCISE

1. A dealer has ₹ 1500 only for a purchase of rice and wheat. A bag of rice costs ₹ 1500 and a bag of wheat costs ₹ 1200. He has a storage capacity of ten bags only and the dealer gets a profit of ₹ 100 and ₹ 80 per bag of rice and wheat respectively. Formulate the problem as a linear programming problem to get the maximum profit.

2. A business man has ₹ 600000 at his disposal and wants to purchase cows and buffaloes to take up a business. The cost price of a cow is ₹ 20,000 and that of a buffalo is ₹ 60000. The man can store fodder for the live stock to the extent of 40 quintals per week. A cow gives 10 litres of milk and buffalo gives 20 litres of milk per day. Profit per litre of milk of cow is ₹ 5 and per litre of the milk of a buffalo is ₹ 7. If the consumption of fodder per cow is 1 quintal and per buffalo is 2 quintals a week, formulate the problem as a linear programming problem to find the number of live stock of each kind the man has to purchase so as to get maximum profit (assuming that he can sell all the quantity of milk, he gets from the livestock)

3. A factory manufactures two types of soaps each with the help of two machines *A* and *B*. *A* is operated for two minutes and *B* for 3 minutes to manufacture the first type, while the second type is manufactured by operating A for 3 minutes and B for 5 minutes. Each machine can be used for at most 8 hours on any day. The two types of soaps are sold at a profit of 25 paise and 50 paise each respectively. How many soaps of each type should the factory produce in a day so as to maximize the profit (assuming that the manufacturer can sell all the soaps he can manufacture). Formulate the problem as a linear programming problem.

4. Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and three times the first number plus the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

5. Vitamins *A* and *B* are found in two different foods *E* and *F*. One unit of food *E* contains 2 units of vitamin *A* and 3 units of vitamin *B*. One unit of food *F* contains 4 units of vitamin *A* and 2 units of vitamin *B*. One unit of food *E* and *F* costs Rs.5 and Rs.2.50 respectively. The minimum daily requirements for a person of vitamin *A* and *B* is 40 units and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamin *A* and *B* is not harmful, find out the optimal mixture of food *E* and *F* at the minimum cost which meets the daily minimum requirement of vitamin *A* and *B*. Formulate this as a linear programming problem.

6. A machine producing either product *A* or *B* can produce *A* by using 2 units of chemicals and 1 unit of a compound and can produce *B* by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1000 units of the compound are available. The profits available per unit of *A* and *B* are respectively Rs. 30 and Rs.20. Find the optimum allocation of units between *A* and *B* to maximise the total profit. Find the maximum profit.

7. Solve the following Linear programming problem graphically.

(a) Maximize $z = 25x_1 + 20x_2$

subject to the constraints

$$3x_1 + 6x_2 \leq 50$$
$$x_1 + 2x_2 \leq 10$$
$$x_1 \geq 0,\ x_2 \geq 0$$

(b) Maximize $z = 9x_1 + 10x_2$

subject to the constraints

$$11x_1 + 9x_2 \leq 9900$$
$$7x_1 + 12x_2 \leq 8400$$
$$3x_1 + 8x_2 \leq 4800$$
$$x_1 \geq 0,\ x_2 \geq 0$$

(c) Maximise $z = 22x_1 + 18x_2$

subject to the constraints

$$x_1 + x_2 \leq 20,$$
$$3x_1 + 2x_2 \leq 48$$
$$x_1 \geq 0,\ x_2 \geq 0$$

ANSWERS

CHECK YOUR PROGRESS 37.1

1. Maximize $z = 3x_1 + 8x_2$

subject to the constraints

$$3x_1 + 4x_2 \le 18$$
$$4x_1 + 5x_2 \le 21$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

2. Maximize $z = 50x_1 + 15x_2$

subject to the constraints

$$5x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 60$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

3. Minimize $z = 4000x_1 + 7500x_2$

subject to the constraints

$$4x_1 + 3x_2 \ge 40$$
$$2x_1 + 3x_2 \ge 8$$
$$x_1 \ge 0,\ x_2 \ge 0$$

4. Maximize $z = 5x_1 + 4x_2$

subject to the constraints

$$1.5x_1 + 2.5x_2 \le 80$$
$$2x_1 + 1.5\ x_2 \le 70$$
$$x_1 \ge 0,\ x_2 \ge 0$$

CHECK YOUR PROGRESS 37.2

1.

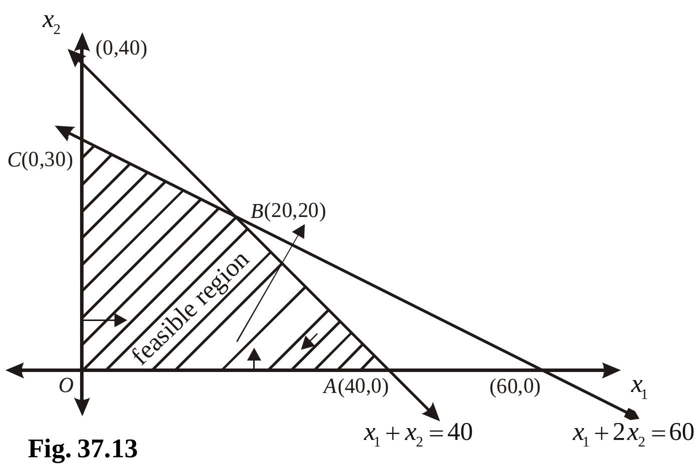

Fig. 37.13

Maximum $z = 140$ at $B(20,20)$

2.

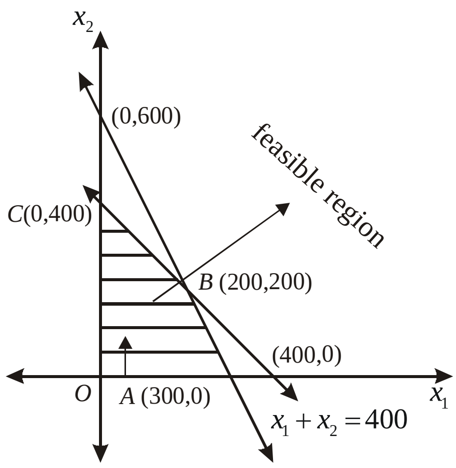

Fig. 37.14

Maximize $z = 1200$ at $C(0,400)$

3.

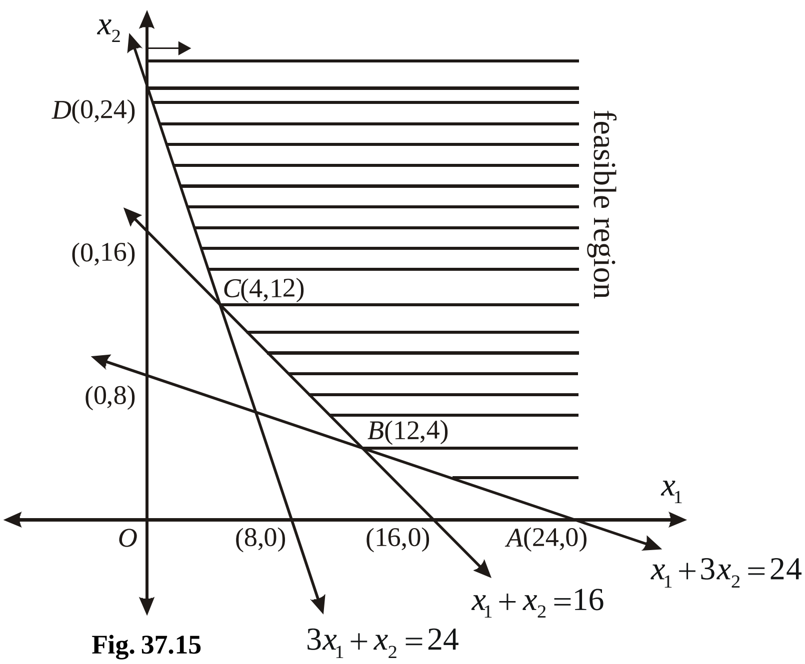

Fig. 37.15

Minimize $z = 720$ *at* $C(4,12)$, $x_1 = 4$, $x_2 = 12$

4.

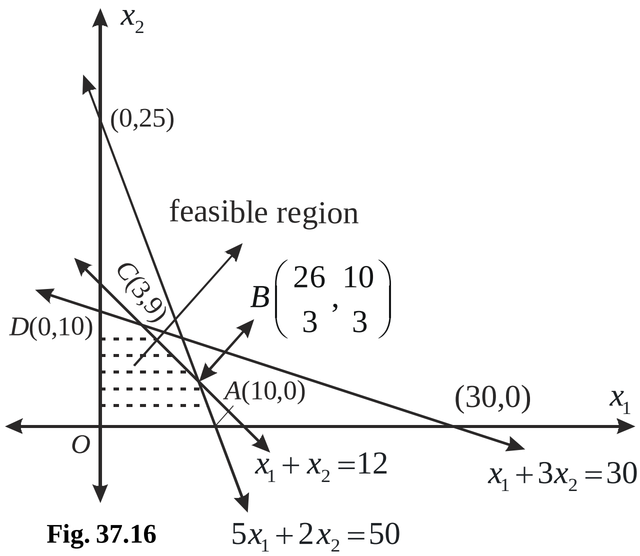

Fig. 37.16

Maximum $z = 330$ *at* $C(3,9)$, $x_1 = 3$, $x_2 = 9$

5.

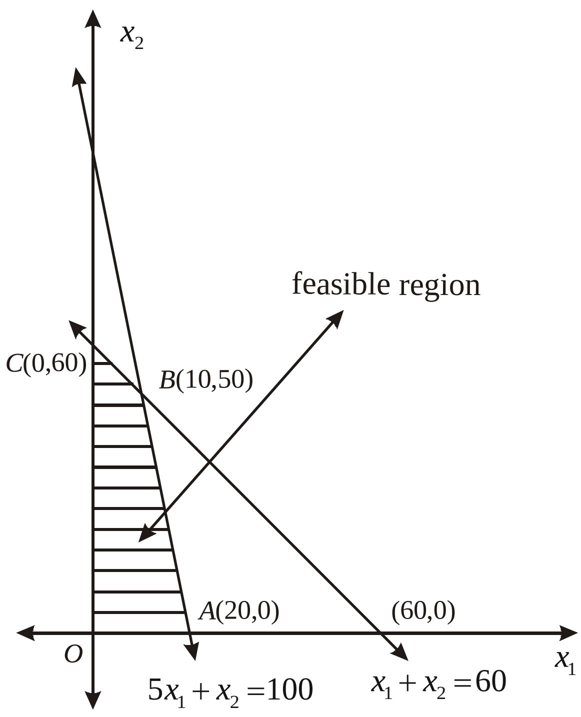

Fig. 37.17

Maximum $z = 1250$ *at* $B(10,50)$, $x_1 = 10$, $x_2 = 50$

6.

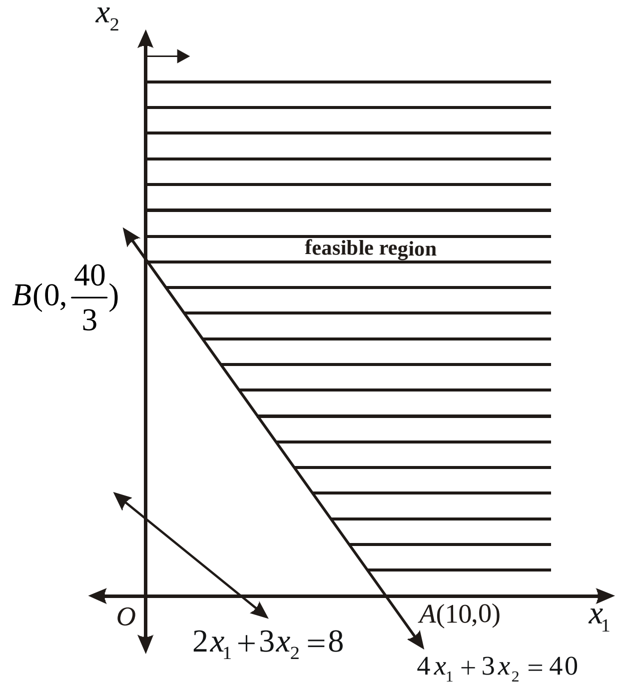

Fig. 37.18

Maximum z $= 40{,}000$ at A(10,0), $x_1 = 10,\ x_2 = 0$

TERMINAL EXERCISE

1. Maximize $z = 100x_1 + 80x_2$

   subject to the conditions

   $5x_1 + 4x_2 \le 50$
   $x_1 + x_2 \le 10$
   $x_1 \ge 0,\ x_2 \ge 0$

2. Maximize $z = 150x_1 + 980x_2$

   subject to the conditions

   $x_1 + 3x_2 \le 30$
   $7x_1 + 14x_2 \le 40$
   $x_1 \ge 0,\ x_2 \ge 0$

3. Maximize $z = 25x_1 + 50x_2$

   subject to the conditions

   $2x_1 + 3x_2 \le 480$
   $3x_1 + 5x_2 \le 480$
   $x_1 \ge 0,\ x_2 \ge 0$

4. Maximize $z = x_1 + x_2$

   subject to the conditions

   $x_1 - x_2 \ge 4$
   $3x_1 + x_2 \le 9$
   $x_1 \ge 0,\ x_2 \ge 0$

5. Minimize $z = 5x_1 + 2.5x_2$

   subject to the conditions

$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_1 \geq 0,\ x_2 \geq 0$

6. Maximize $z = 30x_1 + 20x_2$
subject to the constraints

$2x_1 + x_2 \leq 800$
$x_1 + 2x_2 \leq 1000$
$x_1 \geq 0,\ x_2 \geq 0$

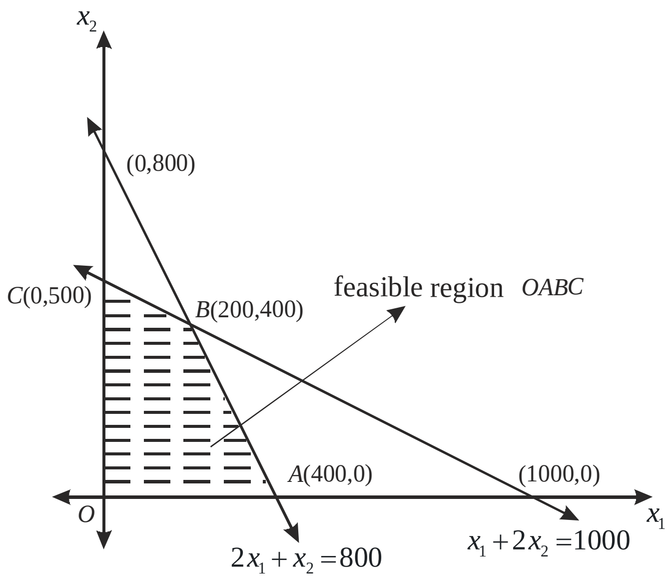

Fig. 37.19

Maximum z = 14000 at
B (200,400). 1 4 0 0 0 = z

8. (a)

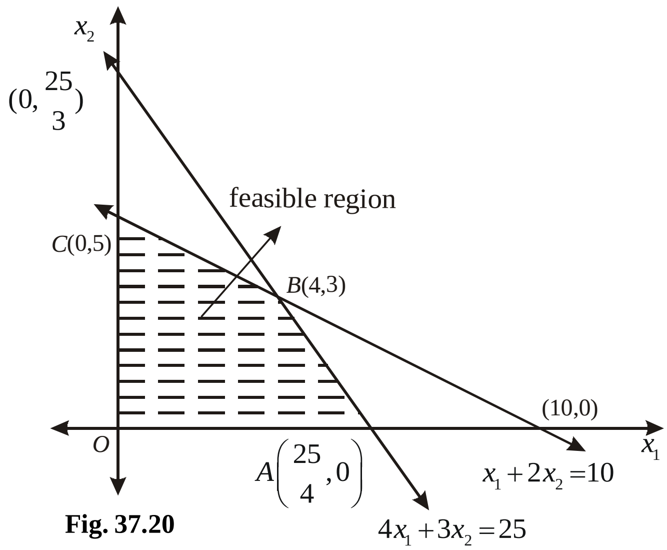

Fig. 37.20

Maximum $z = 160\, at\, B(4,3), x_1 = 4 x_2 = 3$

MODULE - X
Linear Programming and Mathematical Reasoning

Notes

(b)

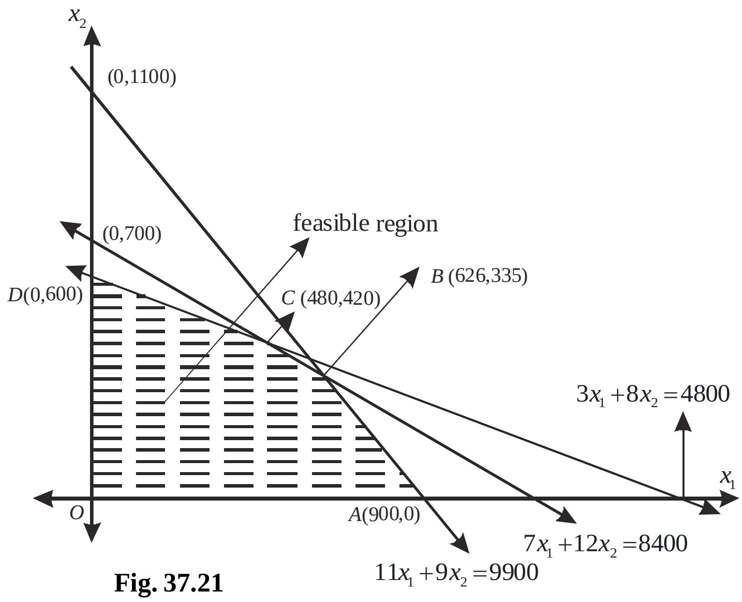

Fig. 37.21

A (900,0)   D(0,600)   B (626, 335), O(0, 0) and C (480, 420)

Maximum $z = 8984$ *at* $B(626, 335)$ $x_1 = 626$, $x_2 = 335$

(c)

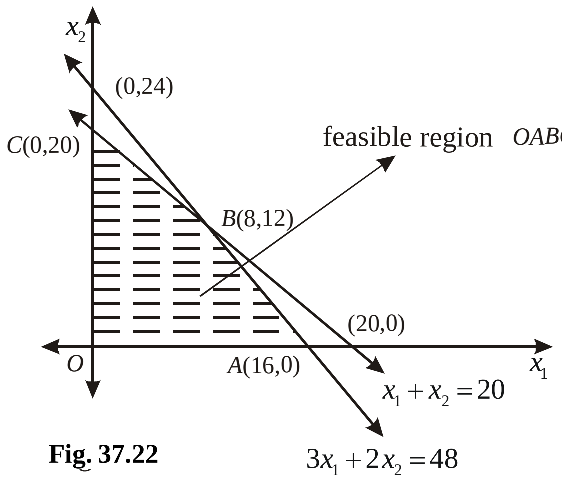

Fig. 37.22

Maximum $z = 392$ *at* $B(8,12)$ $x_1 = 8$ $x_2 = 12$

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